Imperial College London

Maths for Signals and Systems

Linear Algebra in Engineering

Some problems by Gilbert Strang

Problems

1. Consider u, v, w to be non-zero vectors in R^7. These vectors span a vector space. What are the possible dimensions of that space?
   Answer: 1,2, or 3.

2. Consider a 5×3 matrix R which is in echelon form and has 3 pivots, i.e., $r = 3$. What is the null space of this matrix?

   Answer: It is the zero space. Since the rank is 3, the rows (and columns) form R^3. The rows are however 3-dimensional vectors and therefore, there isn't any 3-dimensional vector that is perpendicular to all the rows of this matrix. Therefore, the only vector which satisfies $Rx = 0$ is $x = 0$.

Problem

3. Consider matrix R of the previous question and the 10×3 matrix $B = \begin{bmatrix} R \\ 2R \end{bmatrix}$. What is the rank and echelon form of matrix B?

   Answer: Row $i, i = 6, \dots, 10$ which belongs to the bottom half matrix $2R$ of B can be fully eliminated by doubling row $i - 5$ of R and subtracting it from row i. Therefore, the echelon form of B is $\begin{bmatrix} R \\ 0 \end{bmatrix}$.

   The rank doesn't change since the rows of $2R$ are dependent on the rows of R.

Problem

4. Consider a 5×3 matrix R which is in echelon form, with rank 3 and the 10×6 matrix $C = \begin{bmatrix} R & R \\ R & 0 \end{bmatrix}$. What is the rank and echelon form of matrix C?

Answer: By subtracting rows $i-5, i=6,\dots,10$ of the top half extended matrix $[R \quad R]$ of C from rows $i, i=6,\dots,10$ which belong to the bottom half extended matrix $[R \quad 0]$ of C we get $\begin{bmatrix} R & R \\ 0 & -R \end{bmatrix}$. By adding now rows i, $i=6,\dots,10$ of the bottom half extended matrix $[0 \quad -R]$ to the rows $i-5, i=6,\dots,10$ of the top half extended matrix $[R \quad R]$ we get $\begin{bmatrix} R & 0 \\ 0 & -R \end{bmatrix}$. By multiplying rows i, $i=6,\dots,10$ of the bottom half extended matrix $[0 \quad -R]$ with -1 we get $\begin{bmatrix} R & 0 \\ 0 & R \end{bmatrix}$.

The rank of $\begin{bmatrix} R & 0 \\ 0 & R \end{bmatrix}$ is $2 \times 3 = 6$ (observe the form of $\begin{bmatrix} R & 0 \\ 0 & R \end{bmatrix}$).

Problem

5. Consider a 5×3 matrix R which is in echelon form, with rank 3 and the 10×6 matrix $C = \begin{bmatrix} R & R \\ R & 0 \end{bmatrix}$. What is the dimension of the null space of C^T?

   Answer: C^T is of dimension 6×10. Therefore, the vectors of the null space of C^T are of dimension 10×1. Since the rank of C^T is 6, and we need 10 independent vectors in order to form the 10-dimensional space, we can find 4 independent vectors which are perpendicular to all the rows of C^T. Therefore, the dimension of the null space of C^T is 4.

Problem

6. Consider the system $Ax = b$. The complete solution of that system is $x = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} + c\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + d\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ with c, d scalars and $b = \begin{bmatrix} 2 \\ 4 \\ 2 \end{bmatrix}$. Find the rank of A.

Answer: The vectors $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ are independent. Since we are allowed to "throw" to the solution of the above system any amount of these vectors, the vectors must belong to the null space of A (what is the dimension of A?). Therefore, the rank of A is 1.

For $c = d = 0$, $A\begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 2 \end{bmatrix}$ and therefore the first column of A is $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$.

By choosing $c = 0, d = 1$ and $c = 1, d = 0$ we find $A = \begin{bmatrix} 1 & -1 & 0 \\ 2 & -2 & 0 \\ 1 & -1 & 0 \end{bmatrix}$.

Problem

7. Consider the previous system $Ax = b$. For what values of b does the system has a solution?

Answer: We found that $A = \begin{bmatrix} 1 & -1 & 0 \\ 2 & -2 & 0 \\ 1 & -1 & 0 \end{bmatrix}$. The column space of A is all vectors which are multiples of $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$. The system has a solution if b belongs to the column space of A, i.e., if it is a multiple of $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$.

Problems

8. Consider a square matrix A with null space 0. What can you say about the null space of A^T?

   Answer: In that case the matrix is full rank and therefore, the null space of A^T is also 0.

9. Consider the space of 5×5 matrices and consider the subset of these which contains only the invertible 5×5 matrices. Do they form a subspace?

   Answer: NO, since if I add two invertible matrices the result might not be an invertible matrix. There are alternative ways to answer this problem.

Problems

8. Consider a matrix A. If $A^2 = 0$, is A 0?

   Answer: NO, for example consider $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$.

9. Is a system of n equations and n unknowns solvable for any right hand side if the columns of A are independent?

   Answer: YES, since any vector can be written as a linear combination of the columns of A.

Problem

10. Consider a matrix $B = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. Find a basis for the null space without carrying out the above matrix multiplication.

Answer: The null space of B is a subspace of R^4. The matrix $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$ is invertible and therefore, it doesn't have any impact in the null space of B. Matrix $\begin{bmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ has two pivot columns. The two special solutions are $\begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -2 \\ 1 \\ 0 \\ 1 \end{bmatrix}$ and form a basis of the null space.

Problem

11. For the previous example find a complete solution to the system $Bx = \begin{bmatrix}1\\0\\1\end{bmatrix}$. **Answer:** $B = \begin{bmatrix}1 & 1 & 0\\0 & 1 & 0\\1 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & -1 & 2\\0 & 1 & 1 & -1\\0 & 0 & 0 & 0\end{bmatrix}$. If we write $B = CD$ we see that the first column of D is $\begin{bmatrix}1\\0\\0\end{bmatrix}$ and this means that the first column of B is equal to the first column of C, i.e., $\begin{bmatrix}1\\0\\1\end{bmatrix}$ and also the same as the right hand side. A vector which will multiply B from the right and give the first column of B is $\begin{bmatrix}1\\0\\0\\0\end{bmatrix}$. Therefore, a complete solution is $x_p + x_n = \begin{bmatrix}1\\0\\0\\0\end{bmatrix} + c\begin{bmatrix}1\\-1\\1\\0\end{bmatrix} + d\begin{bmatrix}-2\\1\\0\\1\end{bmatrix}$.

Problems

12. In a square matrix is the row space the same as the column space?

    Answer: NO, consider again the matrix $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$.

13. Do the matrices A and $-A$ share the same four subspaces?

    Answer: YES

14. If two matrices A and B have the same four subspaces, is A a multiple of B?

    Answer: NO - consider all invertible matrices of the same size.

15. If I exchange two rows of A which subspaces remain the same?

    Answer: The row space and the null space.