

Maths for Signals and Systems

Linear Algebra in Engineering

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Mathematics for Signals and Systems

Problems

1. Consider u, v, w to be non-zero vectors in R^7 . These vectors span a vector space. What are the possible dimensions of that space?

Answer: 1, 2, or 3.

2. Consider a 5×3 matrix R which is in echelon form and has 3 pivots, i.e., $r = 3$. What is the null space of this matrix?

Answer: It is the zero space. Since the rank is 3, the rows (and columns) form R^3 . The rows are however 3-dimensional vectors and therefore, there isn't any 3-dimensional vector that is perpendicular to all the rows of this matrix. Therefore, the only vector which satisfies $Rx = 0$ is $x = 0$.

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3. Consider matrix R of the previous question and the 10×3 matrix $B = \begin{bmatrix} R \\ 2R \end{bmatrix}$. What is the rank and echelon form of matrix B ?

Answer: Row $i, i = 6, \dots, 10$ which belongs to the bottom half matrix $2R$ of B can be fully eliminated by doubling row $i - 5$ of R and subtracting it from row i . Therefore, the echelon form of B is $\begin{bmatrix} R \\ 0 \end{bmatrix}$.

The rank doesn't change since the rows of $2R$ are dependent on the rows of R .

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4. Consider a 5×3 matrix R which is in echelon form, with rank 3 and the 10×6 matrix $C = \begin{bmatrix} R & R \\ R & 0 \end{bmatrix}$. What is the rank and echelon form of matrix C ?

Answer: By subtracting rows $i - 5, i = 6, \dots, 10$ of the top half extended matrix $\begin{bmatrix} R & R \end{bmatrix}$ of C from rows $i, i = 6, \dots, 10$ which belong to the bottom half extended matrix $\begin{bmatrix} R & 0 \end{bmatrix}$ of C we get $\begin{bmatrix} R & R \\ 0 & -R \end{bmatrix}$. By adding now rows $i, i = 6, \dots, 10$ of the bottom half extended matrix $\begin{bmatrix} 0 & -R \end{bmatrix}$ to the rows $i - 5, i = 6, \dots, 10$ of the top half extended matrix $\begin{bmatrix} R & R \end{bmatrix}$ we get $\begin{bmatrix} R & 0 \\ 0 & -R \end{bmatrix}$. By multiplying rows $i, i = 6, \dots, 10$ of the bottom half extended matrix $\begin{bmatrix} 0 & -R \end{bmatrix}$ with -1 we get $\begin{bmatrix} R & 0 \\ 0 & R \end{bmatrix}$.

The rank of $\begin{bmatrix} R & 0 \\ 0 & R \end{bmatrix}$ is $2 \times 3 = 6$ (observe the form of $\begin{bmatrix} R & 0 \\ 0 & R \end{bmatrix}$).

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Problem

5. Consider a 5×3 matrix R which is in echelon form, with rank 3 and the 10×6 matrix $C = \begin{bmatrix} R & R \\ R & 0 \end{bmatrix}$. What is the dimension of the null space of C^T ?

Answer: C^T is of dimension 6×10 . Therefore, the vectors of the null space of C^T are of dimension 10×1 . Since the rank of C^T is 6, and we need 10 independent vectors in order to form the 10-dimensional space, we can find 4 independent vectors which are perpendicular to all the rows of C^T . Therefore, the dimension of the null space of C^T is 4.

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6. Consider the system $Ax = b$. The complete solution of that system is $x = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} + c \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + d \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ with c, d scalars and $b = \begin{bmatrix} 2 \\ 4 \\ 2 \end{bmatrix}$. Find the rank of A .

Answer: The vectors $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ are independent. Since we are allowed to “throw” to the solution of the above system any amount of these vectors, the vectors must belong to the null space of A (what is the dimension of A ?). Therefore, the rank of A is 1.

For $c = d = 0$, $A \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 2 \end{bmatrix}$ and therefore the first column of A is $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$.

By choosing $c = 0, d = 1$ and $c = 1, d = 0$ we find $A = \begin{bmatrix} 1 & -1 & 0 \\ 2 & -2 & 0 \\ 1 & -1 & 0 \end{bmatrix}$.

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Problem

7. Consider the previous system $Ax = b$. For what values of b does the system has a solution?

Answer: We found that $A = \begin{bmatrix} 1 & -1 & 0 \\ 2 & -2 & 0 \\ 1 & -1 & 0 \end{bmatrix}$. The column space of A is all

vectors which are multiples of $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$. The system has a solution if b belongs to

the column space of A , i.e., if it is a multiple of $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$.

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Problems

8. Consider a square matrix A with null space 0. What can you say about the null space of A^T ?

Answer: In that case the matrix is full rank and therefore, the null space of A^T is also 0.

9. Consider the space of 5×5 matrices and consider the subset of these which contains only the invertible 5×5 matrices. Do they form a subspace?

Answer: NO, since if I add two invertible matrices the result might not be an invertible matrix. There are alternative ways to answer this problem.

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Problems

8. Consider a matrix A . If $A^2 = 0$, is $A = 0$?

Answer: NO, for example consider $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$.

9. Is a system of n equations and n unknowns solvable for any right hand side if the columns of A are independent?

Answer: YES, since any vector can be written as a linear combination of the columns of A .

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Problem

10. Consider a matrix $B = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. Find a basis for the null space without carrying out the above matrix multiplication.

Answer: The null space of B is a subspace of R^4 . The matrix $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$ is invertible and therefore, it doesn't have any impact in the null space of

B . Matrix $\begin{bmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ has two pivot columns. The two special

solutions are $\begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -2 \\ 1 \\ 0 \\ 1 \end{bmatrix}$ and form a basis of the null space.

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Problem

11. For the previous example find a complete solution to the system $Bx =$

$$\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}. \text{ Answer: } B = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \text{ If we write } B = CD \text{ we see}$$

that the first column of D is $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and this means that the first column of B is

equal to the first column of C , i.e., $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ and also the same as the right hand

side. A vector which will multiply B from the right and give the first column of

$$B \text{ is } \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}. \text{ Therefore, a complete solution is } x_p + x_n = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + c \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + d \begin{bmatrix} -2 \\ 1 \\ 0 \\ 1 \end{bmatrix}.$$

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Problems

12. In a square matrix is the row space the same as the column space?

Answer: NO, consider again the matrix $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$.

13. Do the matrices A and $-A$ share the same four subspaces?

Answer: YES

14. If two matrices A and B have the same four subspaces, is A a multiple of B ?

Answer: NO - consider all invertible matrices of the same size.

15. If I exchange two rows of A which subspaces remain the same?

Answer: The row space and the null space.