Imperial College London

Maths for Signals and Systems

Linear Algebra in Engineering

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Problems

1. Consider u, v, w to be non-zero vectors in R^7. These vectors span a vector space. What are the possible dimensions of that space?
   Answer: 1,2, or 3.

2. Consider a 5×3 matrix R which is in echelon form and has 3 pivots, i.e., $r = 3$. What is the null space of this matrix?

   Answer: It is the zero space. Since the rank is 3, the rows (and columns) form R^3. The rows are however 3-dimensional vectors and therefore, there isn't any 3-dimensional vector that is perpendicular to all the rows of this matrix. Therefore, the only vector which satisfies $Rx = 0$ is $x = 0$.

Problems

3. Consider matrix R of the previous question and the 10×3 matrix $B = \begin{bmatrix} R \\ 2R \end{bmatrix}$. What is the rank and echelon form of matrix B?

   Answer: Row $i, i = 6, \dots, 10$ which belongs to the bottom half matrix $2R$ of B can be fully eliminated by doubling row $i - 5$ of R and subtracting it from row i. Therefore, the echelon form of B is $\begin{bmatrix} R \\ 0 \end{bmatrix}$.

   The rank doesn't change since the rows of $2R$ are dependent on the rows of R.

Problems

4. Consider a 5×3 matrix R which is in echelon form, with rank 3 and the 10×6 matrix $C = \begin{bmatrix} R & R \\ R & 0 \end{bmatrix}$. What is the rank and echelon form of matrix C?

Answer: By subtracting rows $i-5$, $i = 6, \dots, 10$ of the top half extended matrix $[R \quad R]$ of C from rows $i, i = 6, \dots, 10$ which belong to the bottom half extended matrix $[R \quad 0]$ of C we get $\begin{bmatrix} R & R \\ 0 & -R \end{bmatrix}$. By adding now rows i, $i = 6, \dots, 10$ of the bottom half extended matrix $[0 \quad -R]$ to the rows $i-5, i = 6, \dots, 10$ of the top half extended matrix $[R \quad R]$ we get $\begin{bmatrix} R & 0 \\ 0 & -R \end{bmatrix}$. By multiplying rows i, $i = 6, \dots, 10$ of the bottom half extended matrix $[0 \quad -R]$ with -1 we get $\begin{bmatrix} R & 0 \\ 0 & R \end{bmatrix}$. In general, if some of the last rows of R are 0 we must move them to the bottom of $\begin{bmatrix} R & 0 \\ 0 & R \end{bmatrix}$ (this is not the case here). The rank of $\begin{bmatrix} R & 0 \\ 0 & R \end{bmatrix}$ is $2 \times 3 = 6$ (observe the form of $\begin{bmatrix} R & 0 \\ 0 & R \end{bmatrix}$).

Problems

5. Consider a 5×3 matrix R which is in echelon form, with rank 3 and the 10×6 matrix $C = \begin{bmatrix} R & R \\ R & 0 \end{bmatrix}$. What is the dimension of the null space of C^T?

   Answer: C^T is of dimension 6×10. Therefore, the vectors of the null space of C^T are of dimension 10×1. Since the rank of C^T is 6, and we need 10 independent vectors in order to form the 10-dimensional space, we can find 4 independent vectors which are perpendicular to all the rows of C^T. Therefore, the dimension of the null space of C^T is 4.

Problems

6. Consider the system $Ax = b$. The complete solution of that system is $x = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} + c \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + d \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ with c, d scalars and $b = \begin{bmatrix} 2 \\ 4 \\ 2 \end{bmatrix}$. Find the rank of A.

Answer: The vectors $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ are independent. Since we are allowed to "throw" to the solution of the above system any amount of these vectors, the vectors must belong to the null space of A. Therefore, the rank of A is 1.

For $c = d = 0$, $A \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 2 \end{bmatrix}$ and therefore the first column of A is $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$.

By choosing $c = 0, d = 1$ and $c = 1, d = 0$ we find $A = \begin{bmatrix} 1 & -1 & 0 \\ 2 & -2 & 0 \\ 1 & -1 & 0 \end{bmatrix}$.

Problems

7. Consider the previous system $Ax = b$. For what values of b does the system has a solution?

Answer: We found that $A = \begin{bmatrix} 1 & -1 & 0 \\ 2 & -2 & 0 \\ 1 & -1 & 0 \end{bmatrix}$. The column space of A is all vectors which are multiples of $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$. The system has a solution if b belongs to the column space of A, i.e., if it is a multiple of $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$.

Problems

8. Consider a square matrix A with null space 0. What can you say about the null space of A^T?

   Answer: In that case the matrix is full rank and therefore, the null space of A^T is also 0.

9. Consider the space of 5×5 matrices and consider the subset of these which contains only the invertible 5×5 matrices. Do they form a subspace?

   Answer: NO, since if I add two invertible matrices the result might not be an invertible matrix. There are alternative ways to answer this problem.

Problems

8. Consider a matrix A. If $A^2 = 0$, is A 0?

   Answer: NO, for example consider $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$.

9. Is a system of n equations and n unknowns solvable for any right hand side if the columns of A are independent?

   Answer: YES, since any vector can be written as a linear combination of the columns of A.

Problems

10. Consider a matrix $B=\begin{bmatrix}1 & 1 & 0\\ 0 & 1 & 0\\ 1 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & -1 & 2\\ 0 & 1 & 1 & -1\\ 0 & 0 & 0 & 0\end{bmatrix}$. Find a basis for the null space without carrying out the above matrix multiplication.

    Answer: The null space of B is a subspace of R^4. The matrix $\begin{bmatrix}1 & 1 & 0\\ 0 & 1 & 0\\ 1 & 0 & 1\end{bmatrix}$ is invertible and therefore, it doesn't have any impact in the null space of B. Matrix $\begin{bmatrix}1 & 0 & -1 & 2\\ 0 & 1 & 1 & -1\\ 0 & 0 & 0 & 0\end{bmatrix}$ has two pivot columns. The two special solutions are $\begin{bmatrix}1\\ -1\\ 1\\ 0\end{bmatrix}$ and $\begin{bmatrix}-2\\ 2\\ 0\\ 1\end{bmatrix}$ and form a basis of the null space.

Problems

11. For the previous example find a complete solution to the system $Bx = \begin{bmatrix}1\\0\\1\end{bmatrix}$. **Answer:** $B = \begin{bmatrix}1 & 1 & 0\\0 & 1 & 0\\1 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & -1 & 2\\0 & 1 & 1 & -1\\0 & 0 & 0 & 0\end{bmatrix}$. If we write $B = CD$ we see that the first column of D is $\begin{bmatrix}1\\0\\0\end{bmatrix}$ and this means that the first column of B is equal to the first column of C, i.e., $\begin{bmatrix}1\\0\\1\end{bmatrix}$ and also the same as the right hand side. The vector which will multiply B from the right and give the first column of B is $\begin{bmatrix}1\\0\\0\\0\end{bmatrix}$. A complete solution is $x_p + x_n = \begin{bmatrix}1\\0\\0\\0\end{bmatrix} + c\begin{bmatrix}1\\-1\\1\\0\end{bmatrix} + d\begin{bmatrix}-2\\2\\0\\1\end{bmatrix}$.

Problems

12. In a square matrix is the row space the same as the column space?

    Answer: NO, consider again the matrix $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$.

13. Do the matrices A and $-A$ share the same four subspaces?

    Answer: YES

14. If two matrices A and B have the same four subspaces, is A a multiple of B?

    Answer: NO - consider all invertible matrices of the same size.

15. If I exchange two rows of A which subspaces remain the same?

    Answer: The row space and the null space.