Imperial College London

DSP & Digital Filters

Lectures 2-3 Three Different Fourier Transforms

DR TANIA STATHAKI

READER (ASSOCIATE PROFESSOR) IN SIGNAL PROCESSING

IMPERIAL COLLEGE LONDON

Three different Fourier Transforms

There are three useful representations of signals in frequency domain.

- **Continuous Time Fourier Transform (CTFT)**
  - Continuous aperiodic signals. Continuous time and continuous frequency.
- **Discrete Time Fourier Transform (DTFT)**
  - Discrete aperiodic signals. Discrete time and continuous frequency.
- **Discrete Fourier Transform (DFT)**
  - Discrete periodic signals. Discrete Time and discrete frequency.

	Forward Transform	Inverse Transform
CTFT	$X(j\Omega) = \int_{-\infty}^{\infty} x(t)e^{-j\Omega t}dt$ Ω: “real” frequency	$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\Omega)e^{j\Omega t}d\Omega$
DTFT	$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$ $\omega = \Omega T$: "normalised" angular frequency	$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega$
DFT	$X[k] = \sum_{n=0}^{N-1} x[n]e^{-j2\pi\frac{kn}{N}}$	$x[n] = \frac{1}{N}\sum_{n=0}^{N-1} X[k]e^{j2\pi\frac{kn}{N}}$

Discrete Time Fourier Transform

- The **discrete-time Fourier transform (DTFT)** $X(e^{j\omega})$ of a sequence $x[n]$ is given by

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

- In general $X(e^{j\omega})$ is a complex function of the real variable ω and can be written as

$$X(e^{j\omega}) = X_{\text{re}}(e^{j\omega}) + jX_{\text{im}}(e^{j\omega})$$

where $X_{\text{re}}(e^{j\omega})$ and $X_{\text{im}}(e^{j\omega})$ are the real and imaginary parts of $X(e^{j\omega})$ and are real functions of ω.

- $X(e^{j\omega})$ can alternatively be expressed as

$$X(e^{j\omega}) = \left|X(e^{j\omega})\right|e^{j\theta(\omega)}$$

where $\left|X(e^{j\omega})\right|$ and $\theta(\omega)$ are the amplitude and phase of $X(e^{j\omega})$ and are real functions of ω as well.

Imperial College London

Discrete Time Fourier Transform

- For a real sequence $x[n]$, $\left|X(e^{j\omega})\right|$ and $X_{\mathrm{re}}(e^{j\omega})$ are even functions of ω, whereas, $\theta(\omega)$ and $X_{\mathrm{im}}(e^{j\omega})$ are odd functions of ω.

- Note that for any integer k

$$X\left(e^{j\omega}\right) = \left|X\left(e^{j\omega}\right)\right| e^{j[\theta(\omega)+2\pi k]} = \left|X(e^{j\omega})\right| e^{j\theta(\omega)}$$

- The above property indicates that the phase function $\theta(\omega)$ cannot be uniquely specified for the DTFT. Recall that the same observation holds for the CTFT.

- Unless otherwise stated, we shall assume that the phase function $\theta(\omega)$ is restricted to the following range of values:

$$-\pi \leq \theta(\omega) < \pi$$

called the ***principal value***.

Discrete Time Fourier Transform

- The phase response of DTFT might exhibit discontinuities of 2π radians in the plot.
  - [In numerical computations, when the computed phase function is outside the range $[-\pi, \pi]$, the phase is computed modulo 2π to bring the computed value to the above range.]
- An alternate type of phase function that is a continuous function of ω is often used in that case.
- It is derived from the original phase function by removing the discontinuities of 2π.
- The process of removing the discontinuities is called ***phase unwrapping***.
- Sometimes the continuous phase function generated by unwrapping is denoted as $\theta_c(\omega)$.

Discrete Time Fourier Transform Periodicity

- Unlike the Continuous Time Fourier Transform, the DTFT is a periodic function in ω with period 2π.
$X\left(e^{j(\omega_o+2\pi k)}\right)=\sum_{n=-\infty}^{\infty}x[n]e^{-j(\omega_o+2\pi k)n}=\sum_{n=-\infty}^{\infty}x[n]e^{-j\omega_o n}\,e^{-j2\pi kn}=\sum_{n=-\infty}^{\infty}x[n]e^{-j\omega_o n}=X(e^{j\omega_o})$, for any integer k.
- Therefore, $X(e^{j\omega})=\sum_{n=-\infty}^{\infty}x[n]e^{-j\omega n}$ imitates a Fourier Series representation of the periodic function $X(e^{j\omega})$.
- As a result, the Fourier Series coefficients $x[n]$ can be derived from $X(e^{j\omega})$ using the Fourier integral

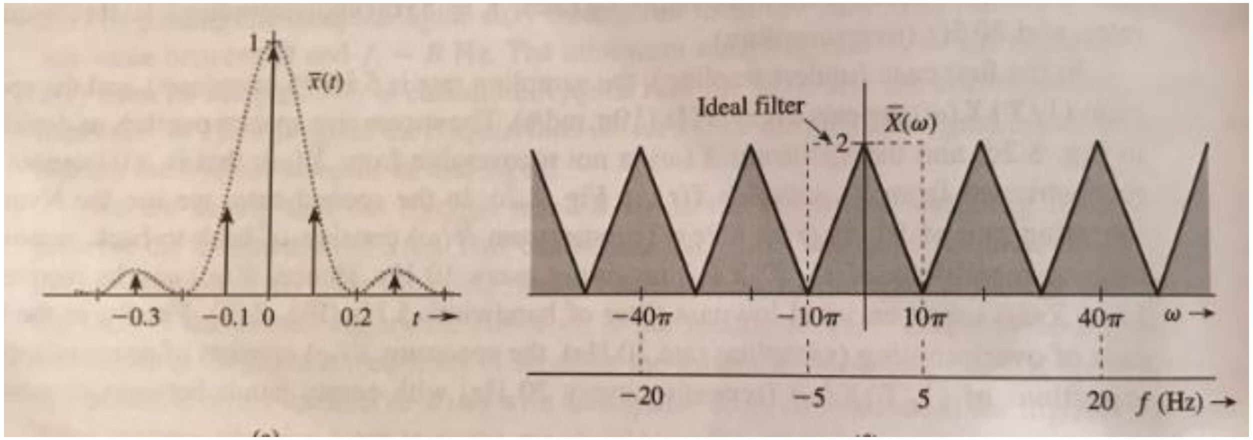
$$x[n]=\frac{1}{2\pi}\int_{-\pi}^{\pi}X(e^{j\omega})e^{j\omega n}d\omega$$

called the Inverse DTFT (IDTFT).
- **Periodicity of DTFT is not a new concept; we know from sampling theory, that sampling a continuous signal results in a periodic repetition of its CTFT.**

Revision

Nyquist sampling: Just about the correct sampling rate

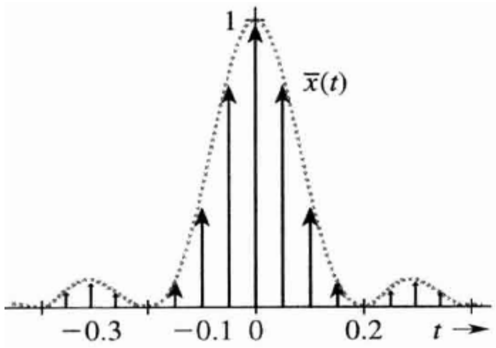
- In that case we use the Nyquist sampling rate of $10Hz$.
- The spectrum $\bar{X}(\omega)$ consists of back-to-back, non-overlapping repetitions of $\frac{1}{T_s}X(\omega)$ repeating every $10Hz$.
- In order to recover $X(\omega)$ from $\bar{X}(\omega)$ we must use an ideal lowpass filter of bandwidth $5Hz$. This is shown in the right figure below with the dotted line.

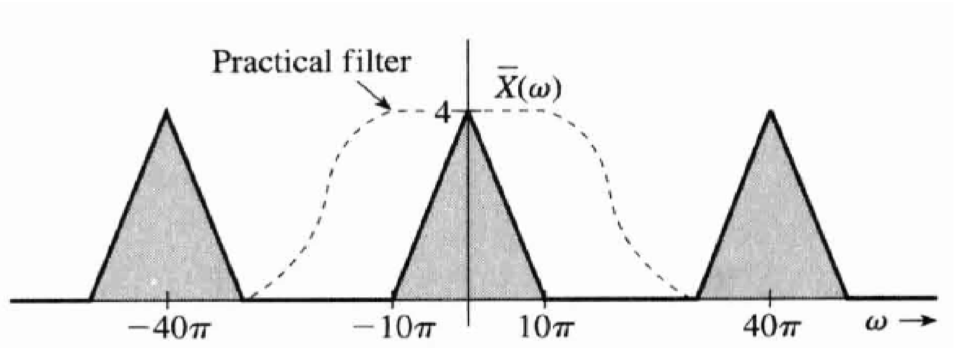


Revision

Oversampling: What happens if we sample too quickly?

- Sampling at higher than the Nyquist rate (in this case $20Hz$) makes reconstruction easier.
- The spectrum $\bar{X}(\omega)$ consists of non-overlapping repetitions of $\frac{1}{T_s}X(\omega)$, repeating every $20Hz$ with empty bands between successive cycles.
- In order to recover $X(\omega)$ from $\bar{X}(\omega)$ we can use a practical lowpass filter and not necessarily an ideal one. This is shown in the right figure below with the dotted line.



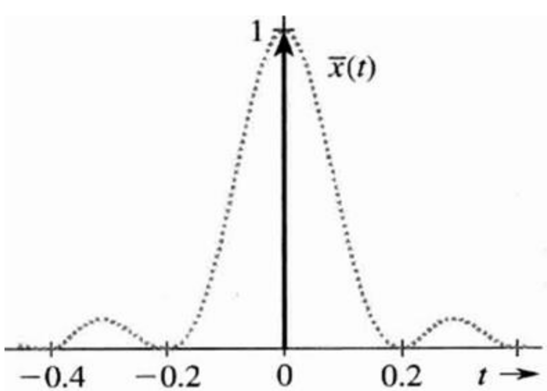


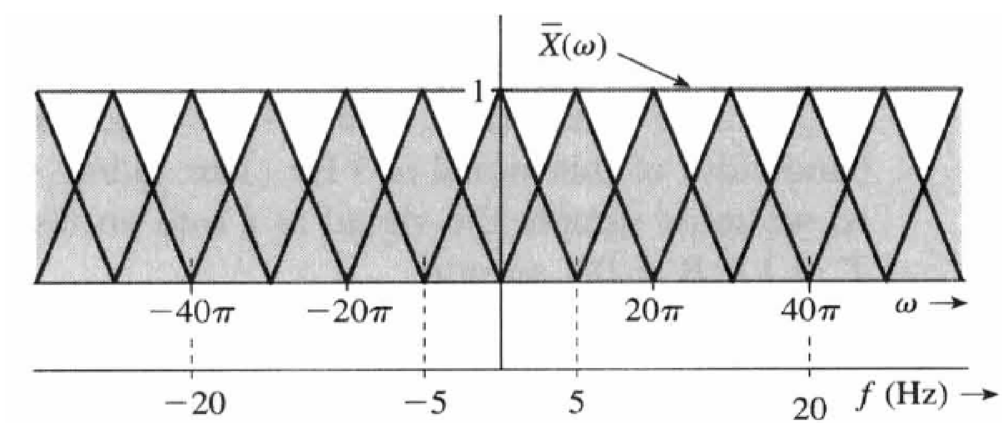
- The filter we use for reconstruction must have gain T_s and bandwidth of any value between B and $(f_s - B)Hz$.

Revision

Undersampling: What happens if we sample too slowly?

- Sampling at lower than the Nyquist rate (in this case $5Hz$) makes reconstruction impossible.
- The spectrum $\bar{X}(\omega)$ consists of overlapping repetitions of $\frac{1}{T_s}X(\omega)$ repeating every $5Hz$.
- $X(\omega)$ is not recoverable from $\bar{X}(\omega)$.
- Sampling below the Nyquist rate corrupts the signal. This type of distortion is called **aliasing**.





More DTFT Properties

- The DTFT is the z –transform evaluated at $z = e^{j\omega}$.

  [Recall that $X(z) = \sum_{-\infty}^{\infty} x[n]\, z^{-n}$].

  Therefore, the DTFT converges if the ROC includes $|z| = 1$ ($z = e^{j\omega}$).
- The DTFT is the same as the CTFT of a signal comprising impulses of appropriate heights at the sample instances.

  $x_\delta(t) = \sum_n x[n]\delta(t - nT) = x(t)\sum_{-\infty}^{\infty} \delta(t - nT)$

  - Recall that $x[n] = x(nT)$

  $$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} = \sum_{n=-\infty}^{\infty} x[n] \int_{-\infty}^{\infty} \delta(t - nT)e^{-j\omega\frac{t}{T}}dt$$

  $$= \int_{-\infty}^{\infty}\left[\sum_{n=-\infty}^{\infty} x[n]\, \delta(t - nT)\right]e^{-j\omega\frac{t}{T}}dt = \int_{-\infty}^{\infty} x_\delta(t)e^{-j\Omega t}dt$$

  - For the above the condition $\sum_{n=-\infty}^{\infty}|x[n]| < \infty$ must hold.
  - $\omega = \Omega T$

Examples

- The DTFT of a shifted discrete Dirac function $\delta[n-k]$ is given by:

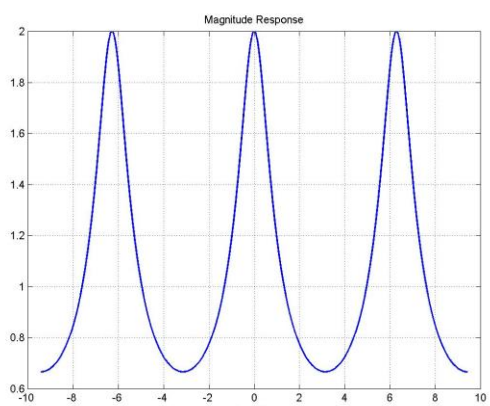
$$\Delta(\omega) = \sum_{n=-\infty}^{\infty} \delta\,[n-k]e^{-j\omega n} = e^{-j\omega k}$$

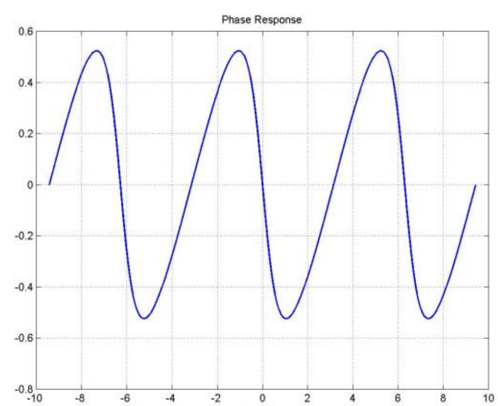
- The DTFT of the causal sequence $x[n] = \alpha^n u[n], |\alpha| < 1$ is given by:

$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} \alpha^n u[n]e^{-j\omega n} = \sum_{n=0}^{\infty} \alpha^n e^{-j\omega n} = \sum_{n=0}^{\infty}(\alpha e^{-j\omega})^n = \frac{1}{1-\alpha e^{-j\omega}}$

if $\left|\alpha e^{-j\omega}\right| = |\alpha| < 1$

- For $\alpha = 0.5$, the magnitude and phase of $X(e^{j\omega}) = 1/(1-0.5e^{-j\omega})$ are shown below.





Inverse Discrete Time Fourier Transform (IDTFT)

- Lets us prove the previous statement that the IDTFT is defined as:

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega$$

Proof

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\sum_{\ell=-\infty}^{\infty} x[\ell]e^{-j\omega\ell}\right)e^{j\omega n}d\omega$$

$$= \sum_{\ell=-\infty}^{\infty} x[\ell]\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{j\omega(n-\ell)}d\omega\right) = \sum_{\ell=-\infty}^{\infty} x[\ell]\frac{\sin\pi\,(n-\ell)}{\pi(n-\ell)}$$

(Note that the order of integration and summation can be interchanged if the summation inside the top brackets converges uniformly, i.e., if $X(e^{j\omega})$ exists.)

Imperial College London

Inverse Discrete Time Fourier Transform cont.

$$x[\ell]\frac{\sin\pi(n-\ell)}{\pi(n-\ell)}=\begin{cases}1 & n=\ell\\ 0 & n\neq\ell\end{cases}$$

Hence,

$$\sum_{\ell=-\infty}^{\infty} x[\ell]\frac{\sin\pi(n-\ell)}{\pi(n-\ell)}=\sum_{\ell=-\infty}^{\infty} x[\ell]\delta[n-\ell]=x[n]$$

Discrete Time Fourier Transform: uniform convergence

- An infinite series of the form $X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$ may or may not converge.
- Let $X_K(e^{j\omega}) = \sum_{n=-K}^{K} x[n]e^{-j\omega n}$
- For ***uniform convergence*** (strong convergence) of $X(e^{j\omega})$ we require:

$$\lim_{K\to\infty} X_K\left(e^{j\omega}\right) = X(e^{j\omega})$$

- If $x[n]$ is an ***absolutely summable*** sequence, i.e., if $\sum_{n=-\infty}^{\infty}|x[n]| < \infty$, then

$$\left|X(e^{j\omega})\right| = \left|\sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}\right| \le \sum_{n=-\infty}^{\infty} |x[n]| \left|e^{-j\omega n}\right| \le \sum_{n=-\infty}^{\infty} |x[n]| < \infty$$

  for all values of ω

- Thus, the absolute summability of $x[n]$ is a sufficient condition for the existence of the DTFT $X\left(e^{j\omega}\right)$.

Examples

- The sequence $x[n] = \alpha^n u[n]$ is absolutely summable for $|\alpha| < 1$ since

$$\sum_{n=-\infty}^{\infty} |\alpha^n| u[n] = \sum_{n=0}^{\infty} |\alpha^n| = \frac{1}{1-|\alpha|} < \infty$$

and its DTFT converges uniformly to $1/(1-\alpha e^{-j\omega})$.

- Note that:
  - Since $\sum_{n=-\infty}^{\infty} |x[n]|^2 \leq (\sum_{n=-\infty}^{\infty} |x[n]|)^2$, an absolutely summable sequence has always finite energy.
  - However, a finite energy sequence is not necessarily absolutely summable.
    - The sequence $x[n] = \begin{cases} 1/n & n \geq 1 \\ 0 & n \leq 0 \end{cases}$

      has finite energy equal to $\sum_{n=1}^{\infty} (\frac{1}{n})^2 = \pi^2/6$ but is not absolutely summable.

Discrete Time Fourier Transform: mean square convergence

- To represent a finite energy sequence $x[n]$ that is not absolutely summable by DTFT, it is necessary to consider the so called ***mean-square convergence*** (weak convergence) of $X(e^{j\omega})$:

$$\lim_{K\to\infty} \int_{-\pi}^{\pi} \left|X(e^{j\omega}) - X_K(e^{j\omega})\right|^2 d\omega = 0$$

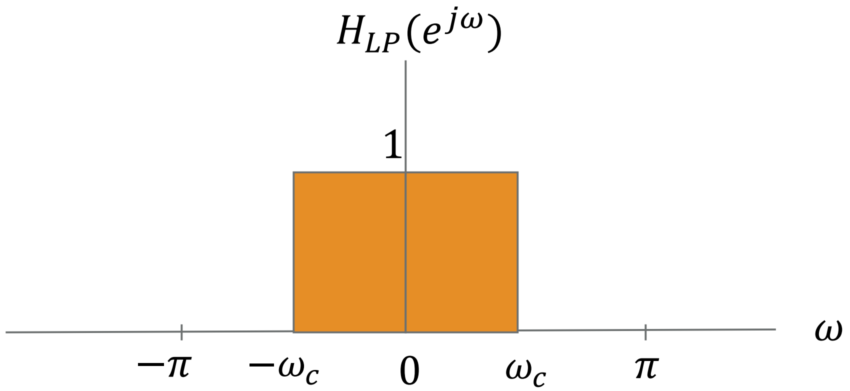
where $X_K(e^{j\omega}) = \sum_{n=-K}^{K} x[n]e^{-j\omega n}$.

- Here, the total energy of the error $X(e^{j\omega}) - X_K(e^{j\omega})$ must approach zero at each value of ω as K goes to ∞.
- In such a case, the absolute value of the error may not go to zero as K goes to ∞ and the DTFT is no longer bounded.

Example

- Consider the DTFT:

$$H_{LP}(e^{j\omega}) = \begin{cases} 1, & 0 \le |\omega| \le \omega_c \\ 0, & \omega_c < |\omega| \le \pi \end{cases}$$



- The inverse DTFT is given by

$$h_{LP}[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_{LP}(e^{j\omega})e^{j\omega n}\,d\omega = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{j\omega n}\,d\omega$$

$$= \frac{1}{2\pi}\left(\frac{e^{j\omega_c n}}{jn} - \frac{e^{-j\omega_c n}}{jn}\right) = \frac{\sin\omega_c n}{\pi n}, \; -\infty < n < \infty$$

- The energy of $h_{LP}[n]$ is given by $E_h = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left|H_{LP}\left(e^{j\omega}\right)\right|^2 d\omega = \frac{\omega_c}{\pi}$.
- $h_{LP}[n]$ is a finite-energy sequence, but it is not absolutely summable.

Example cont.

- As a result

$$\sum_{n=-K}^{K} h_{LP}[n]e^{-j\omega n} = \sum_{n=-K}^{K} \frac{\sin \omega_c n}{\pi n} e^{-j\omega n}$$

does not uniformly converge to

$$H_{LP}(e^{j\omega}) = \begin{cases} 1, & 0 \le |\omega| \le \omega_c \\ 0, & \omega_c < |\omega| \le \pi \end{cases}$$

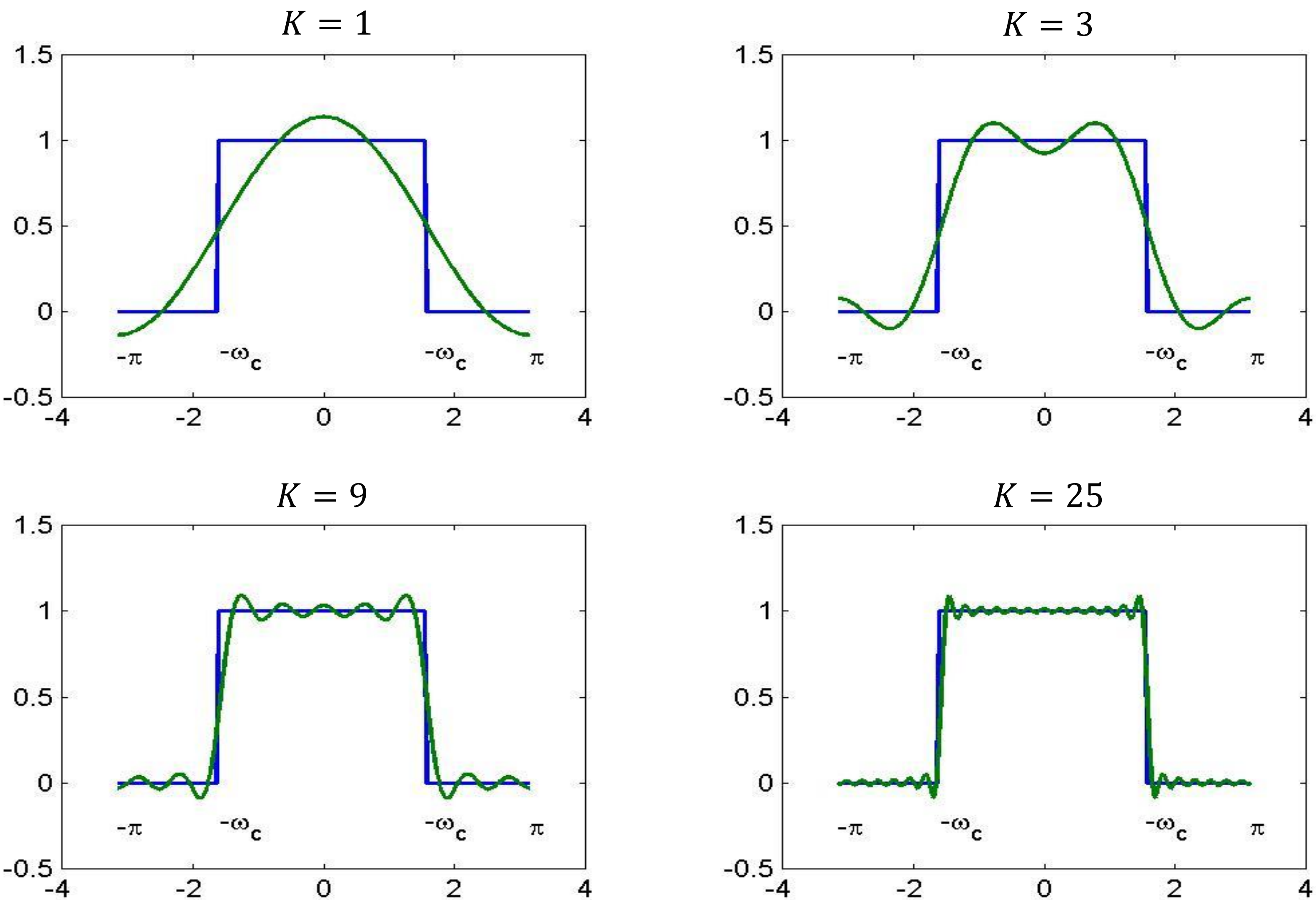
for all values of ω, but converges to $H_{LP}(e^{j\omega})$ in the mean-square sense.

- The mean-square convergence property of the sequence $h_{LP}[n]$ can be further illustrated by examining the plot of the function

$$H_{LP,K}(e^{j\omega}) = \sum_{n=-K}^{K} \frac{\sin \omega_c n}{\pi n} e^{-j\omega n}$$

for various values of K as shown next.

Example cont.



Example cont.

- As it can be seen from these plots, independent of the value of K there are ripples in the plot of $H_{LP,K}(e^{j\omega})$ around both sides of the point $\omega = \omega_c$.
- The number of ripples increases as K increases with the height of the largest ripple remaining the same for all values of K.
- As K goes to infinity, the condition

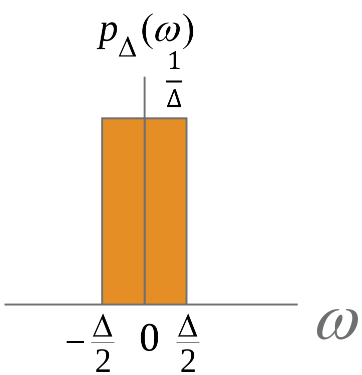
$$\lim_{K\to\infty} \int_{-\pi}^{\pi} \left|H_{LP}(e^{j\omega}) - H_{LP,K}(e^{j\omega})\right|^2 d\omega = 0$$

holds, indicating the convergence of $H_{LP,K}(e^{j\omega})$ to $H_{LP}(e^{j\omega})$.
- The oscillatory behavior observed in $H_{LP,K}(e^{j\omega})$ is known as the ***Gibbs phenomenon***.

Neither absolutely- nor square- summable

- The DTFT can also be defined for a certain class of sequences which are neither absolutely summable nor square summable.
- Examples of such sequences are the unit step sequence $u[n]$, the sinusoidal sequence $\cos(\omega_o n + \varphi)$ and the complex exponential sequence $A\alpha^n$. These are neither absolutely summable nor square summable.
- For this type of sequences, a DTFT representation is possible using Dirac delta functions.
- A ***Dirac delta function*** $\delta(\omega)$ is a "function" of ω with infinite height, zero width, and unit area.
- It is the limiting form of a unit area pulse function $p_\Delta(\omega)$ as Δ goes to zero

$$\delta(\omega) = \lim_{\Delta \to 0} p_\Delta(\omega)$$



satisfying

$$\int_{-\infty}^{\infty} p_\Delta(\omega) d\omega = 1,\ p_\Delta(\omega) = 0,\ \omega \neq 0$$

Example

- Consider the complex exponential sequence $x[n] = e^{j\omega_o n}$, ω_o real. Its DTFT is given by
$$X(e^{j\omega}) = \sum_{k=-\infty}^{\infty} 2\pi\delta(\omega - \omega_o + 2\pi k)$$
where $\delta(\omega)$ is an impulse function of ω and $-\pi \leq \omega_o \leq \pi$.
- To verify the above we can take the IDTFT of $X(e^{j\omega})$ above:

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi}\sum_{k=-\infty}^{\infty} 2\pi\delta(\omega - \omega_o + 2\pi k)e^{j\omega n}d\omega = \int_{-\pi}^{\pi}\delta(\omega - \omega_o)e^{j\omega n}\,d\omega = e^{j\omega_o n}$$

DTFT properties (listed without proof)

Type of Property	Sequence	Discrete-Time Fourier Transform
	$g[n]$	$G(e^{j\omega})$
	$h[n]$	$H(e^{j\omega})$
Linearity	$\alpha g[n] + \beta h[n]$	$\alpha G(e^{j\omega}) + \beta H(e^{j\omega})$
Time-shifting	$g[n - n_o]$	$e^{-j\omega n_o} G(e^{j\omega})$
Frequency-shifting	$e^{j\omega_o n} g[n]$	$G\left(e^{j(\omega-\omega_o)}\right)$
Differentiation in frequency	$ng[n]$	$j\dfrac{dG(e^{j\omega})}{d\omega}$
Convolution	$g[n] \circledast h[n]$	$G(e^{j\omega})H(e^{j\omega})$
Modulation	$g[n]h[n]$	$\frac{1}{2\pi}\int_{-\pi}^{\pi} G(e^{j\theta})H(e^{j(\omega-\theta)})\,d\theta$
Parseval's relation	$\sum_{n=-\infty}^{\infty} g[n]h^*[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} G(e^{j\omega})H^*(e^{j\omega})\,d\omega$	

Imperial College London

DTFT properties (listed without proof)

Sequence	Discrete-Time Fourier Transform
$x[n]$	$X(e^{j\omega})$
$x[-n]$	$X(e^{-j\omega})$
$x^*[-n]$	$X^*(e^{j\omega})$
$\mathrm{Re}\{x[n]\}$	$X_{cs}(e^{j\omega}) = \frac{1}{2}\{X(e^{j\omega}) + X^*(e^{-j\omega})\}$
$j\mathrm{Im}\{x[n]\}$	$X_{ca}(e^{j\omega}) = \frac{1}{2}\{X(e^{j\omega}) - X^*(e^{-j\omega})\}$
$x_{cs}[n]$	$X_{re}(e^{j\omega})$
$x_{ca}[n]$	$jX_{im}(e^{j\omega})$

$x[n]$: A complex sequence

Note: $X_{cs}(e^{j\omega})$ and $X_{ca}(e^{j\omega})$ are the conjugate-symmetric and conjugate-antisymmetric parts of $X(e^{j\omega})$, respectively. Likewise, $x_{cs}[n]$ and $x_{ca}[n]$ are the conjugate-symmetric and conjugate-antisymmetric parts of $x[n]$, respectively.

DTFT properties (listed without proof)

Sequence	Discrete-Time Fourier Transform
$x[n]$	$X(e^{j\omega}) = X_{re}(e^{j\omega}) + jX_{im}(e^{j\omega})$
$x_{ev}[n]$	$X_{re}(e^{j\omega})$
$x_{od}[n]$	$jX_{im}(e^{j\omega})$
Symmetry relations	$X(e^{j\omega}) = X^*(e^{-j\omega})$ $X_{re}(e^{j\omega}) = X_{re}(e^{-j\omega})$ $X_{im}(e^{j\omega}) = -X_{im}(e^{-j\omega})$ $\lvert X(e^{j\omega})\rvert = \lvert X(e^{-j\omega})\rvert$ $\arg\{X(e^{j\omega})\} = -\arg\{X(e^{-j\omega})\}$

$x[n]$: A real sequence

Note: $x_{ev}[n]$ and $x_{od}[n]$ denote the even and odd parts of $x[n]$, respectively.

Imperial College London

Common DTFT pairs

$$\delta[n] \leftrightarrow 1$$

$$1 \leftrightarrow \sum_{k=-\infty}^{\infty} 2\pi\delta(\omega + 2\pi k)$$

$$u[n] \leftrightarrow \frac{1}{1 - e^{-j\omega}} + \sum_{k=-\infty}^{\infty} \pi\delta(\omega + 2\pi k)$$

$$e^{j\omega_o n} \leftrightarrow \sum_{k=-\infty}^{\infty} 2\pi\delta(\omega - \omega_o + 2\pi k)$$

$$\alpha^n u[n], (|\alpha| < 1) \leftrightarrow \frac{1}{1 - \alpha e^{-j\omega}}$$

Example

- Determine the DTFT of the sequence
$$y[n] = (n+1)\alpha^n u[n], |\alpha| < 1$$
- Let $x[n] = \alpha^n u[n], |\alpha| < 1$. We can, therefore, write
$$y[n] = nx[n] + x[n]$$
- From tables, the DTFT of $x[n]$ is given by
$$X(e^{j\omega}) = \frac{1}{1 - \alpha e^{-j\omega}}$$
- Using the differentiation property of the DTFT given in previous tables, we observe that the DTFT of $nx[n]$ is given by
$$j\frac{dX(e^{j\omega})}{d\omega} = j\frac{d}{d\omega}\left(\frac{1}{1 - \alpha e^{-j\omega}}\right) = \frac{\alpha e^{-j\omega}}{(1 - \alpha e^{-j\omega})^2}$$
- Next using the linearity property of the DTFT given in previous tables we arrive at
$$Y(e^{j\omega}) = \frac{\alpha e^{-j\omega}}{(1 - \alpha e^{-j\omega})^2} + \frac{1}{1 - \alpha e^{-j\omega}} = \frac{1}{(1 - \alpha e^{-j\omega})^2}$$

Example

- Determine the DTFT of the sequence $v[n]$ defined by
$$d_0 v[n] + d_1 v[n-1] = p_0 \delta[n] + p_1 \delta[n-1], \ |d_1/d_0| < 1$$
- From previous tables, we see that the DTFT of $\delta[n]$ is 1.
- Using the time-shifting property of the DTFT given in previous tables, we observe that the DTFT of $\delta[n-1]$ is $e^{-j\omega}$ and the DTFT of $v[n-1]$ is $e^{-j\omega} V(e^{j\omega})$.
- Using the linearity property of previous tables we then obtain the frequency-domain representation of $d_0 v[n] + d_1 v[n-1] = p_0 \delta[n] + p_1 \delta[n-1]$ as
$$d_0 V(e^{j\omega}) + d_1 e^{-j\omega} V(e^{j\omega}) = p_0 + p_1 e^{-j\omega}$$
- Solving the above equation we get
$$V(e^{j\omega}) = \frac{p_0 + p_1 e^{-j\omega}}{d_0 + d_1 e^{-j\omega}}$$

Energy Density Spectrum

- The total energy of a finite-energy sequence $g[n]$ is given by

$$\varepsilon_g = \sum_{n=-\infty}^{\infty} |g[n]|^2$$

- From Parseval's Theorem we know that

$$\varepsilon_g = \sum_{n=-\infty}^{\infty} |g[n]|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left|G(e^{j\omega})\right|^2 d\omega$$

- The quantity $S_{gg}(\omega) = \left|G(e^{j\omega})\right|^2$ is called the ***energy density spectrum***.
- The area under this curve in the range $-\pi \leq \omega \leq \pi$ divided by 2π is the energy of the sequence.

Example

- Compute the energy of the sequence

$$h_{LP}[n] = \frac{\sin\omega_c n}{\pi n}, \ -\infty < n < \infty$$

- Here,

$$\sum_{n=-\infty}^{\infty} |h_{LP}[n]|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left|H_{LP}(e^{j\omega})\right|^2 d\omega$$

$$H_{LP}(e^{j\omega}) = \begin{cases} 1, & 0 \le |\omega| \le \omega_c \\ 0, & \omega_c < |\omega| \le \pi \end{cases}$$

- Therefore,

$$\sum_{n=-\infty}^{\infty} |h_{LP}[n]|^2 = \frac{1}{2\pi} \int_{-\omega_c}^{\omega_c} d\omega = \frac{\omega_c}{\pi} < \infty$$

- Hence, $h_{LP}[n]$ is a finite energy sequence.

Introduction. Time sampling theorem resume.

- We wish to perform spectral analysis using digital computers.
- Therefore, we must somehow sample the Discrete Time Fourier Transform of the signal!
- We will compute a discrete version of the DTFT of a sampled, finite-duration signal. This transform is known as the Discrete Fourier Transform (DFT).
- The goal is to understand how DFT is related to the original Fourier transform.
- We showed that a signal bandlimited to BHz can be reconstructed from signal samples if they are obtained at a rate of $f_s > 2B$ samples per second.
- Not that the signal spectrum exists over the frequency range (in Hz) from $-B$ to B.
- The interval $2B$ is called ***spectral width***.

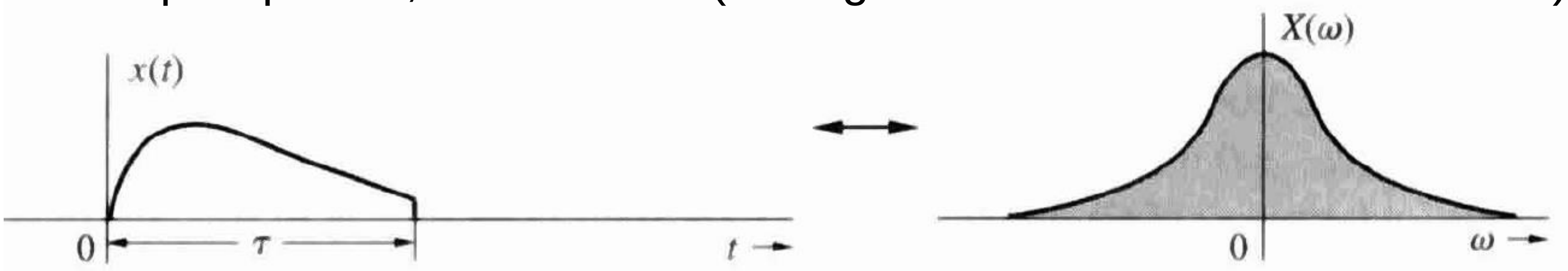
  Note the difference between spectral width ($2B$) and bandwidth (B).
- In time sampling theorem: $f_s > 2B$ or $f_s >$(spectral width).

Time sampling theorem has a dual: Spectral sampling theorem

- Consider a time-limited signal $x(t)$ with a spectrum $X(\omega)$.
- In general, a time-limited signal is 0 for $t < T_1$ and $t > T_2$. The duration of the signal is $\tau = T_2 - T_1$. Below we assume that $T_1 = 0$.
- Recall that $X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt = \int_{0}^{\tau} x(t)e^{-j\omega t}dt$.
- The Fourier transform $X(\omega)$ is assumed real for simplicity.

Spectral sampling theorem

The spectrum $X(\omega)$ of a signal $x(t)$, time-limited to a duration of τ seconds, can be reconstructed from the samples of $X(\omega)$ taken at a rate R samples per Hz, where $R > \tau$ (the signal width or duration in seconds).



Spectral sampling theorem

- We now construct the periodic signal $x_{T_0}(t)$. This is a periodic extension of $x(t)$ with period $T_0 > \tau$.
- This periodic signal can be expressed using Fourier series.

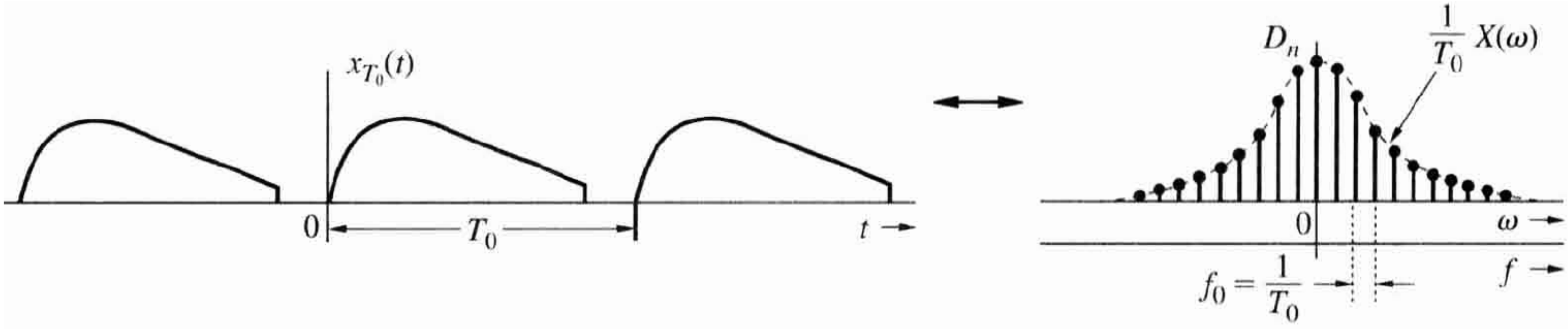
$$x_{T_0}(t) = \sum_{n=-\infty}^{n=\infty} D_n e^{jn\omega_0 t}, \ \omega_0 = \frac{2\pi}{T_0}$$

$$D_n = \frac{1}{T_0}\int_0^{T_0} x(t)\, e^{-jn\omega_0 t} dt = \frac{1}{T_0}\int_0^{\tau} x(t)\, e^{-jn\omega_0 t} dt = \frac{1}{T_0} X(n\omega_0)$$

- The result indicates that the coefficients of the Fourier series for $x_{T_0}(t)$ are the values of $X(\omega)$ taken at integer multiples of ω_0 and scaled by $\frac{1}{T_0}$.
- We call ***spectrum of a periodic signal*** the weights of the exponential terms in its Fourier series representation.
- The above implies that the spectrum of the periodic signal $x_{T_0}(t)$ is the sampled version of spectrum $X(\omega)$.

Spectral sampling theorem cont.

- The spectrum of the periodic signal $x_{T_0}(t)$ is the sampled version of spectrum $X(\omega)$ (see figure below).



  - If successive cycles of $x_{T_0}(t)$ do not overlap, $x(t)$ can be recovered from $x_{T_0}(t)$.
  - If we know $x(t)$ we can find $X(\omega)$.
  - The above imply that $X(\omega)$ can be reconstructed from its samples.
- These samples are separated by the so called fundamental frequency $f_0 = \frac{1}{T_0} Hz$ or $\omega_0 = 2\pi f_0 rads/s$ of the periodic signal $x_{T_0}(t)$.
- Therefore, the condition for recovery is $T_0 > \tau \Rightarrow f_0 < \frac{1}{\tau} Hz$.

Spectral interpolation formula

- To reconstruct the spectrum $X(\omega)$ from the samples of $X(\omega)$, the samples should be taken at frequency intervals $f_0 < \frac{1}{\tau} Hz$. If the sampling rate is R frequency samples/Hz we have:

$$R = \frac{1}{f_0} > \tau \text{ samples}/Hz$$

- We know that the continuous version of a signal can be recovered from its sampled version through the so called ***signal interpolation formula***:

(refer to a Signals and Systems book for the proof of it)

$$x(t) = \sum_n x(nT_s)h(t - nT_s) = \sum_n x(nT_s)\operatorname{sinc}\left(\frac{\pi t}{T_s} - n\pi\right)$$

We use the dual of the approach employed to derive the signal interpolation formula above, to obtain the ***spectral interpolation formula*** as follows. We assume that $x(t)$ is time-limited to τ and centred at T_c. We can prove that:

$$X(\omega) = \sum_{n=-\infty} X(n\omega_0)\operatorname{sinc}\left(\frac{\omega T_0}{2} - n\pi\right) e^{-j(\omega - n\omega_0)T_c},\ \omega_0 = \frac{2\pi}{T_0},\ T_0 > \tau$$

Spectral interpolation formula: Proof.

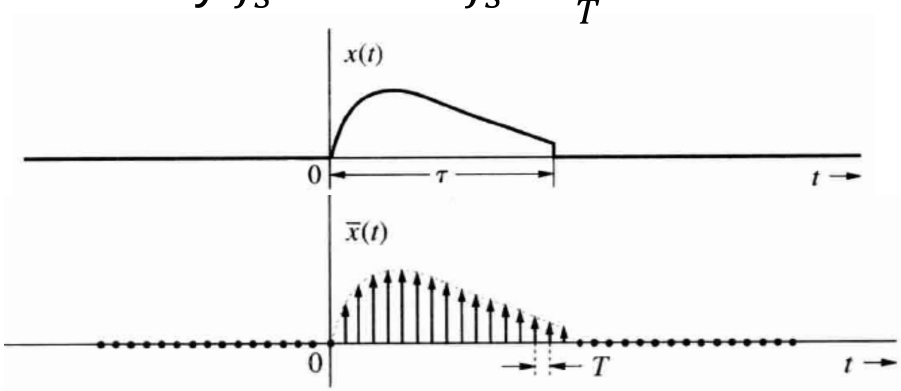
- We know that $x_{T_0}(t) = \sum_{n=-\infty}^{n=\infty} D_n e^{jn\omega_0 t}$, $\omega_0 = \frac{2\pi}{T_0}$
- Therefore, $\mathcal{F}\{x_{T_0}(t)\} = 2\pi \sum_{n=-\infty}^{n=\infty} D_n \,\delta(\omega - n\omega_0)$
  [It is easier to prove that $\mathcal{F}^{-1}\{\sum_{n=-\infty}^{n=\infty} D_n \,\delta(\omega - n\omega_0)\} = x_{T_0}(t)$]
- We can write $x(t) = x_{T_0}(t) \cdot \mathrm{rect}\left(\frac{t-T_c}{T_0}\right)$ (1)
  [We were given that $x(t)$ is centred at T_c]
- We know that $\mathcal{F}\left\{\mathrm{rect}\left(\frac{t}{T_0}\right)\right\} = T_0 \mathrm{sinc}\left(\frac{\omega T_0}{2}\right)$.
- Therefore, $\mathcal{F}\left\{\mathrm{rect}\left(\frac{t-T_c}{T_0}\right)\right\} = T_0 \mathrm{sinc}\left(\frac{\omega T_0}{2}\right) e^{-j\omega T_c}$.
- From (1) we see that $X(\omega) = \frac{1}{2\pi} \mathcal{F}\{x_{T_0}(t)\} * \mathcal{F}\left\{\mathrm{rect}\left(\frac{t-T_c}{T_0}\right)\right\}$
- $X(\omega) = \frac{1}{2\pi} 2\pi [\sum_{n=-\infty}^{n=\infty} D_n \,\delta(\omega - n\omega_0)] * T_0 \mathrm{sinc}\left(\frac{\omega T_0}{2}\right) e^{-j\omega T_c}$

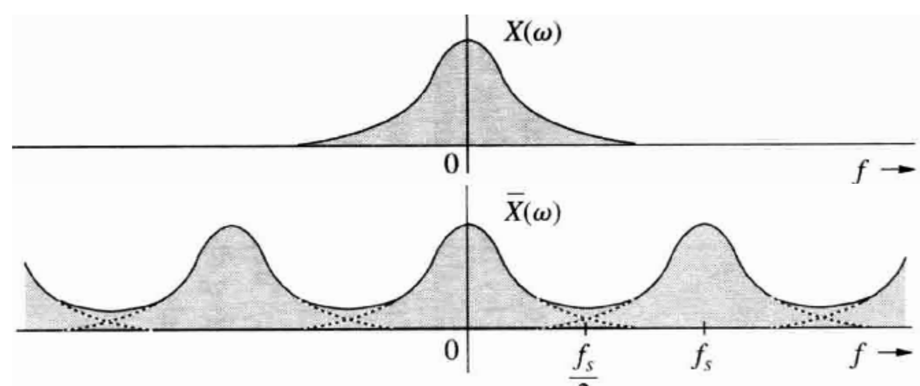
$$X(\omega) = \sum_{n=-\infty} D_n T_0 \mathrm{sinc}\left[\frac{(\omega - n\omega_0)T_0}{2}\right] e^{-j(\omega - n\omega_0)T_c},\ \omega_0 = \frac{2\pi}{T_0},\ T_0 > \tau$$

$$X(\omega) = \sum_{n=-\infty} X(n\omega_0) \mathrm{sinc}\left(\frac{\omega T_0}{2} - n\pi\right) e^{-j(\omega - n\omega_0)T_c}$$

Discrete Fourier Transform DFT

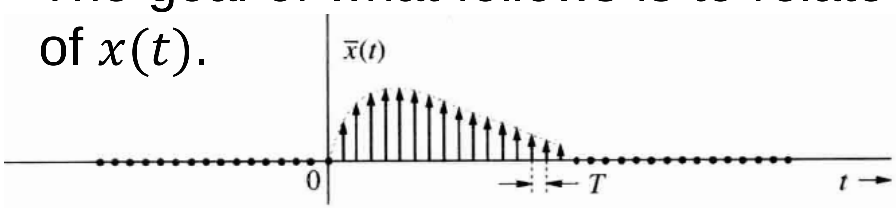
- The numerical computation of the Fourier transform requires samples of $x(t)$ since computers can work only with discrete values.
- Furthermore, the Fourier transform can only be computed at some discrete values of ω.
- The goal of what follows is to relate the samples of $X(\omega)$ with the samples of $x(t)$.
- Consider a time-limited signal $x(t)$. Its spectrum $X(\omega)$ will not be bandlimited (try to think why). In other words aliasing after sampling cannot be avoided.
- The spectrum $\bar{X}(\omega)$ of the sampled signal $\bar{x}(t)$ consist of $X(\omega)$ repeating every $f_s Hz$ with $f_s = \frac{1}{T}$.

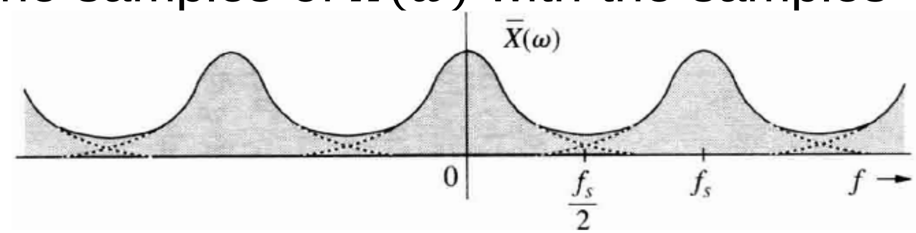


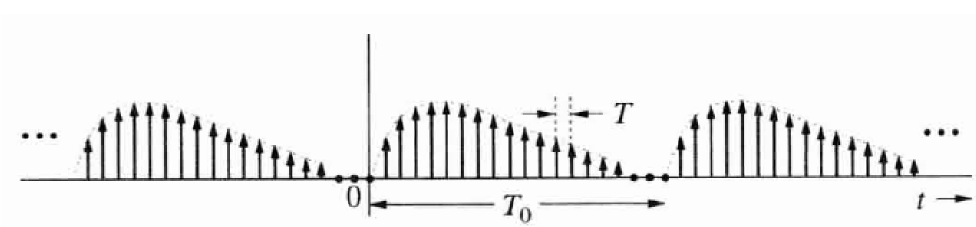


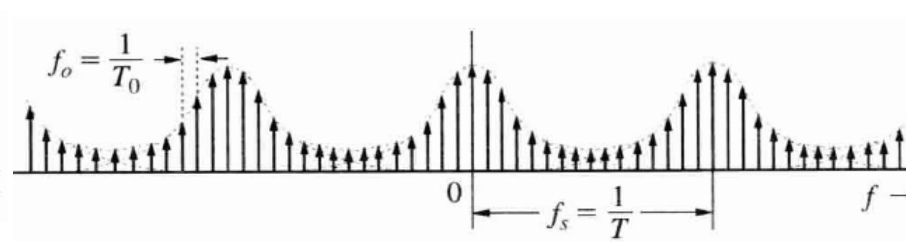
Discrete Fourier Transform DFT cont.

- Suppose now that the sampled signal $\bar{x}(t)$ is repeated periodically every T_0 seconds.
- According to the spectral sampling theorem, this operation results in sampling the spectrum at a rate of T_0 samples/Hz. This means that the samples are spaced at $f_0 = \frac{1}{T_0} Hz$.
- **Therefore, when a signal is sampled and periodically repeated, its spectrum is also sampled and periodically repeated.**
- The goal of what follows is to relate the samples of $X(\omega)$ with the samples of $x(t)$.



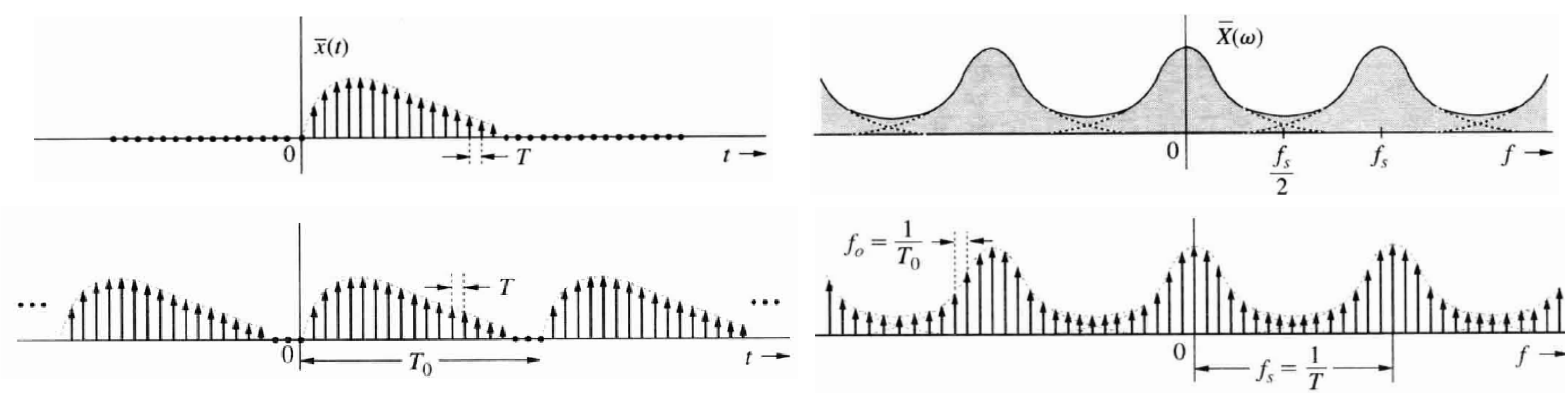






Discrete Fourier Transform DFT cont.

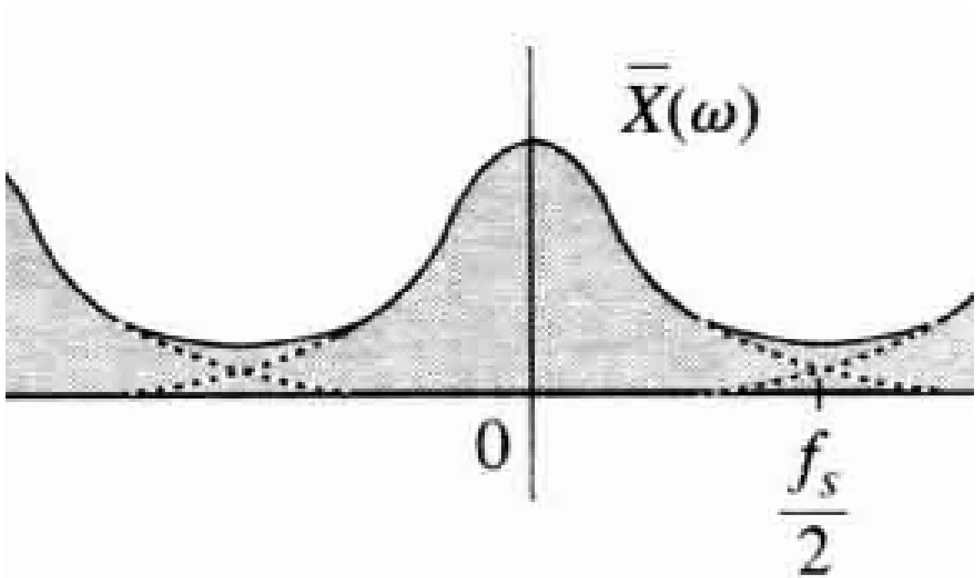
- The number of samples of the discrete signal in one period T_0 is $N_0 = \frac{T_0}{T}$ (figure below left).
- The number of samples of the discrete spectrum in one period is $N_0' = \frac{f_s}{f_0}$.
- We see that $N_0' = \frac{f_s}{f_0} = \frac{\frac{1}{T}}{\frac{1}{T_0}} = \frac{T_0}{T} = N_0$.
- **This is an interesting observation: the number of samples in a period of time is identical to the number of samples in a period of frequency.**



Imperial College London

Aliasing and leakage effects

- Since $X(\omega)$ is not bandlimited, we will get some aliasing effect:



- Furthermore, if $x(t)$ is not time limited, we need to truncate $x(t)$ with a window function. This leads to a “leakage” effect (refer to a Signals and Systems book for the demonstration of it).

Formal definition of DFT

- If $x(nT)$ and $X(k\omega_0)$ are the n^{th} and k^{th} samples of $x(t)$ and $X(\omega)$ respectively, we define:

$$x[n] = Tx(nT) = \frac{T_0}{N_0}x(nT)$$

$$X[k] = X(k\omega_0),\ \omega_0 = 2\pi f_0 = \frac{2\pi}{T_0}$$

- It can be shown that $x[n]$ and $X[k]$ are related by the following equations:

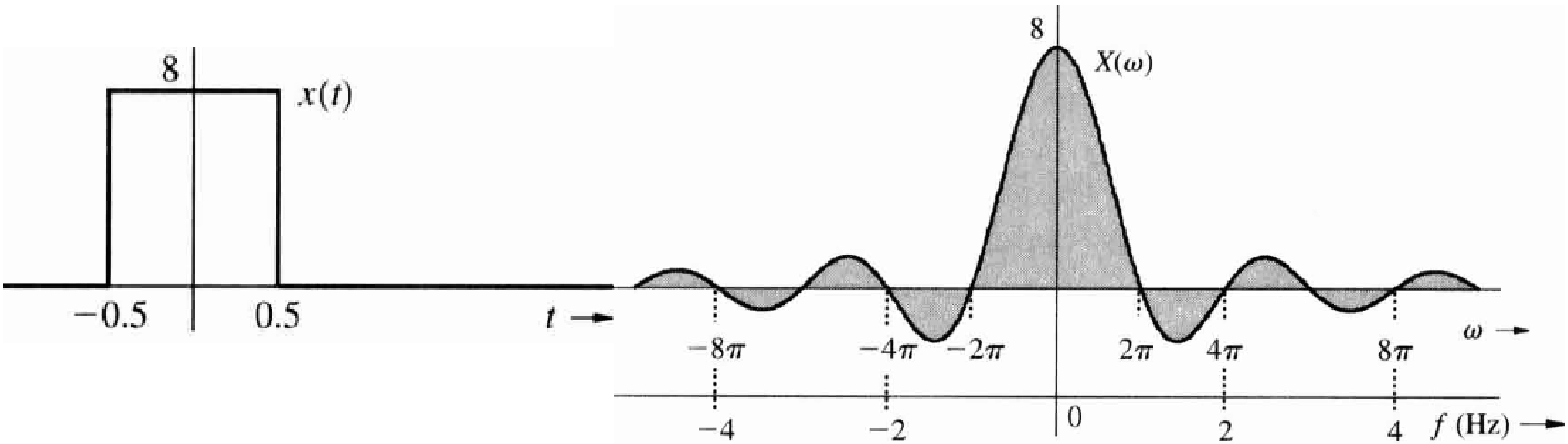
$$X[k] = \sum_{n=0}^{N_0-1} x[n]e^{-jnk\Omega_0} \quad (1)$$

$$x[n] = \frac{1}{N_0}\sum_{k=0}^{N_0-1} X[k]e^{jkn\Omega_0},\ \Omega_0 = \omega_0 T = \frac{2\pi}{N_0} \quad (2)$$

- The equations (1) and (2) above are the direct and inverse Discrete Fourier Transforms respectively, known as DFT and IDFT.
- In the above equations, the summation is performed from 0 to $N_0 - 1$. It can be shown that the summation can be performed over any successive N_0 values of n or k.

Example

- Use DFT to compute the Fourier transform of $8\text{rect}(t)$ (Lathi page 808.)

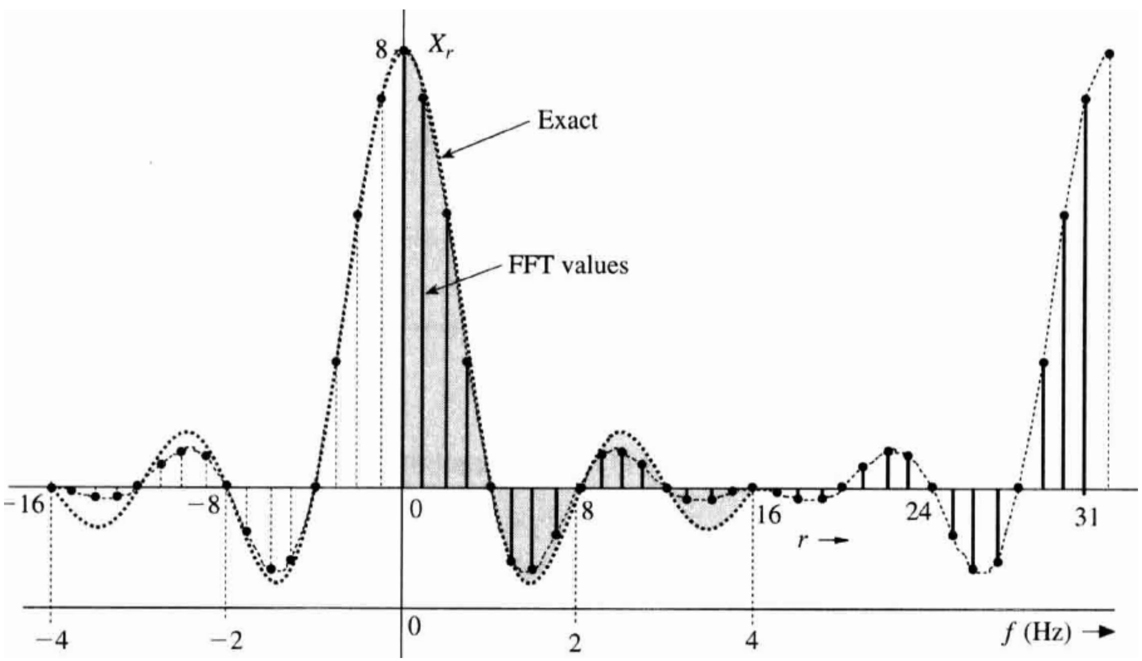


- The essential bandwidth B (calculated by finding where the amplitude response drops to 1% of its peak value) is well above $16Hz$. However, we select $B = 4Hz$:
  - To observe the effects of aliasing.
  - In order not to end up with a huge number of samples in time.

Imperial College London

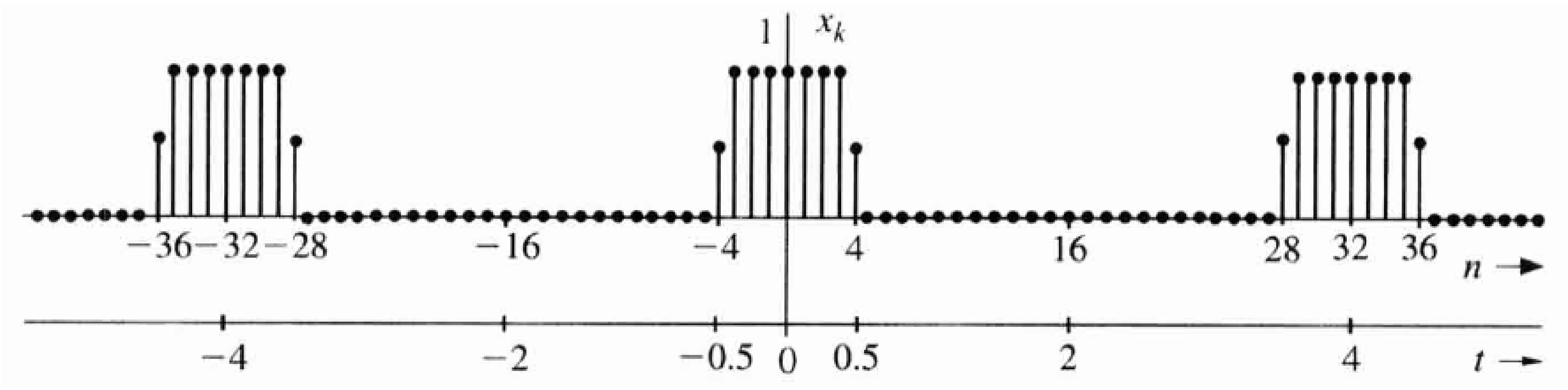
Example cont.

- $B = 4Hz,\ f_s = 8Hz,\ T = \frac{1}{f_s} = \frac{1}{8}.$
- For the frequency resolution we choose $f_0 = \frac{1}{4}Hz.$ This choice gives us 4 samples in each lobe of $X(\omega)$ and $T_0 = \frac{1}{f_0} = 4s$.



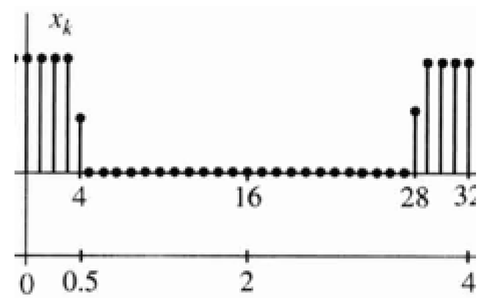
Example cont.

- $N_0 = \frac{T_0}{T} = \frac{4}{1/8} = 32$. Therefore, we must repeat $x(t)$ every $4s$ and take samples every $\frac{1}{8}s$. This yields 32 samples in a period.
- $x[n] = Tx(nT) = \frac{1}{8}x(\frac{n}{8})$ with $x(t) = 8\text{rect}(t)$.
- The DFT of the signal $x[n]$ is obtained by taking any full period of $x[n]$ (i.e., N_0 samples) and not necessarily N_0 over the interval $(0, T_0)$ as we assumed in the theoretical analysis of DFT.

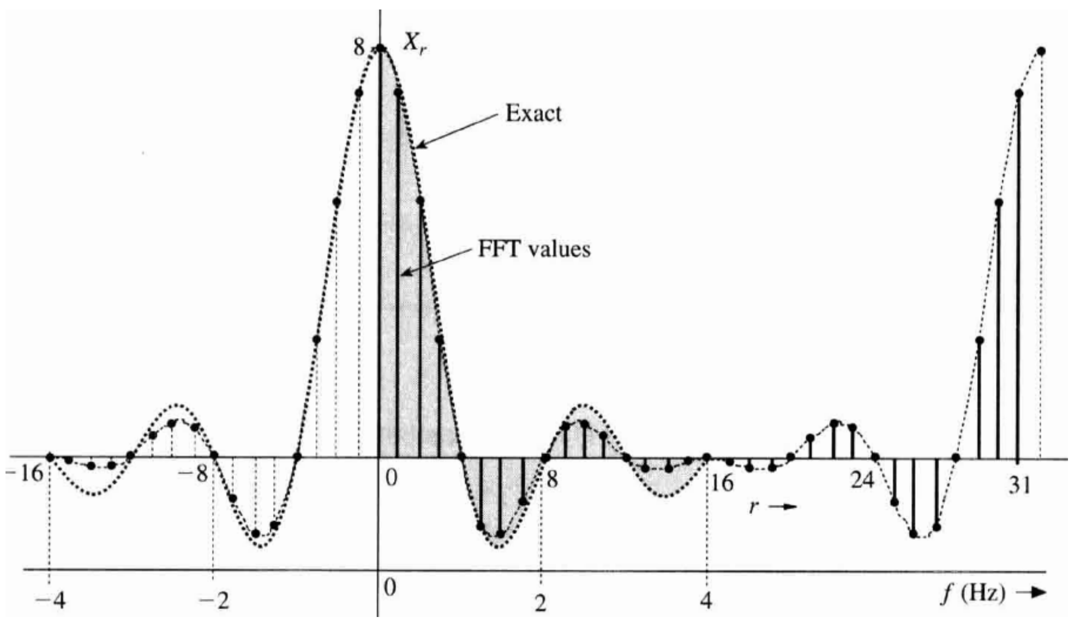


Example cont.

- $x[n] = \begin{cases} 1 & 0 \le n \le 3 \quad \text{and} \quad 29 \le n \le 31 \\ 0 & 5 \le n \le 27 \\ 0.5 & n = 4{,}28 \end{cases}$

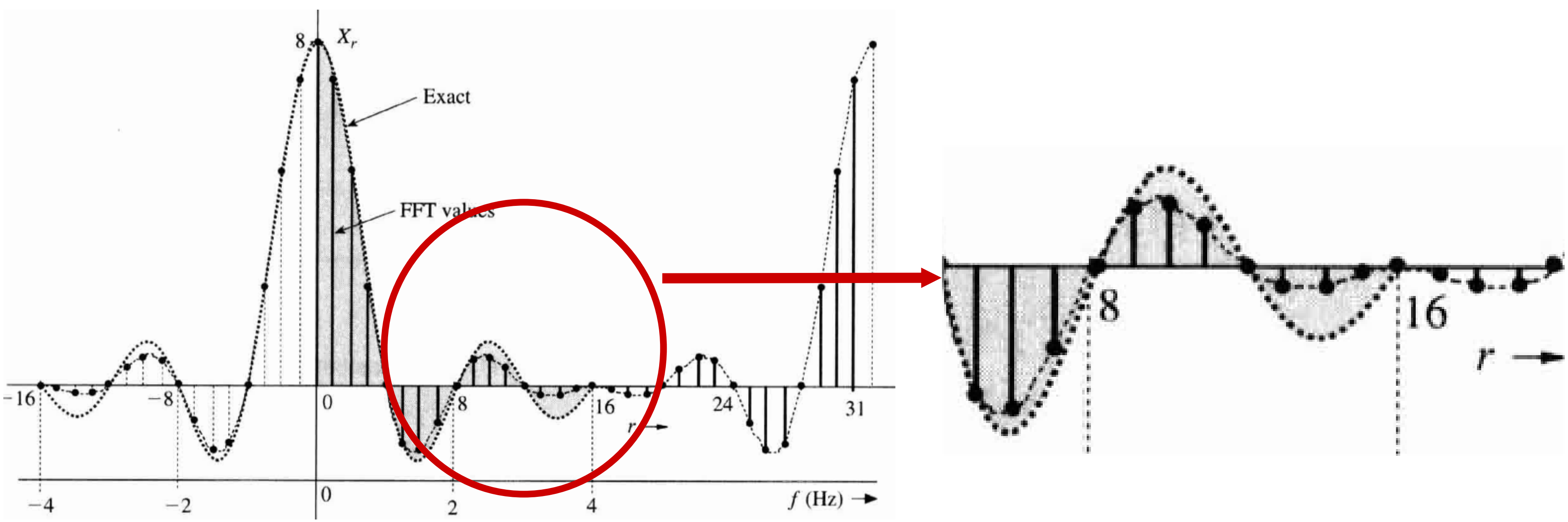


- $\Omega_0 = \frac{2\pi}{32} = \frac{\pi}{16}$
- $X[k] = \sum_{n=0}^{N_0-1} x[n]e^{-jk\Omega_0 n} = \sum_{n=0}^{31} x[n]e^{-jk(\pi/16)n}$. See figure below.



Example cont.

- Observe that $X[k]$ is periodic.
- The dotted curve depicts the Fourier transform of $x(t) = 8\text{rect}(t)$.
- The aliasing error is quite visible when we use a single graph to compare the superimposed plots. The error increases rapidly with k.



Appendix: Proof of DFT relationships

- For the sampled signal we have:

$$\overline{x(t)} = \sum_{n=0}^{N_0-1} x(nT)\delta(t-nT).$$

- Since $\delta(t-nT) \Leftrightarrow e^{-jn\omega T}$

$$\overline{X(\omega)} = \sum_{n=0}^{N_0-1} x(nT)e^{-jn\omega T}$$

- For $|\omega| \le \frac{\omega_s}{2}$, $\overline{X(\omega)}$ the Fourier transform of $\overline{x(t)}$ is $\frac{X(\omega)}{T}$, i.e.,

$$X(\omega) = T\overline{X(\omega)} = T\sum_{n=0}^{N_0-1} x(nT)e^{-jn\omega T},\ |\omega| \le \frac{\omega_s}{2}$$

$$X[k] = X(k\omega_0) = T\sum_{n=0}^{N_0-1} x(nT)e^{-jnk\omega_0 T}$$

- If we let $\omega_0 T = \Omega_0$ then $\Omega_0 = \omega_0 T = 2\pi f_0 T = \frac{2\pi}{N_0}$ and also $Tx(nT) = x[n]$.
- Therefore, $X[k] = \sum_{n=0}^{N_0-1} x[n]e^{-jnk\Omega_0}$

Appendix: Proof of DFT relationships

- To prove the inverse relationship write:

$$\sum_{k=0}^{N_0-1} X[k]e^{jkm\Omega_0} = \sum_{k=0}^{N_0-1}\left[\sum_{n=0}^{N_0-1} x[n]e^{-jnk\Omega_0}\right]e^{jkm\Omega_0} \Rightarrow$$

$$\sum_{k=0}^{N_0-1} X[k]e^{jkm\Omega_0} = \sum_{n=0}^{N_0-1} x[n]\left[\sum_{k=0}^{N_0-1} e^{-jk(n-m)\Omega_0}\right]$$

- $\sum_{k=0}^{N_0-1} e^{-jk(n-m)\Omega_0} = \sum_{k=0}^{N_0-1} e^{-jk(n-m)\frac{2\pi}{N_0}} = \begin{cases} N_0 & n-m = rN_0, r \in \mathbb{Z} \\ 0 & \text{otherwise} \end{cases}$

- Since $0 \leq m, n \leq N_0 - 1$ the only multiple of N_0 that the term $(n-m)$ can be is 0. Therefore:

$$\sum_{k=0}^{N_0-1} e^{-jk(n-m)\frac{2\pi}{N_0}} = \begin{cases} N_0 & n-m = 0 \Rightarrow n = m \\ 0 & \text{otherwise} \end{cases}$$

- Therefore,

$$x_m = \frac{1}{N_0}\sum_{k=0}^{N_0-1} X[k]e^{jkm\Omega_0},\ \Omega_0 = \frac{2\pi}{N_0}$$

Continue with Dr Mike Brookes's notes

- For the rest of the material related to DFT refer to Dr Mike Brookes's notes Three Different Fourier Transforms, from section Symmetries to the end.