4: Linear Time Invariant Systems

LTI Systems

$$x[n] \longrightarrow \boxed{\mathscr{H}} \longrightarrow y[n] = \mathscr{H}(x[n])$$

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Proof: $\sum_r \delta[r]x[n-r] \overset{\text{(i)}}{=} x[n]$ (i) all terms zero except $r = 0$.

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Define $x[n] = \begin{cases} 1 & h[-n] \geq 0 \\ -1 & h[-n] < 0 \end{cases}$

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&\leq \sum_{r=-\infty}^{\infty} |x[n-r]|\,|h[r]| \\
&\leq B \sum_{r=-\infty}^{\infty} |h[r]| \leq BS < \infty
\end{aligned}$$

Frequency Response

For a BIBO stable system $Y(e^{j\omega}) = X(e^{j\omega})H(e^{j\omega})$
where $H(e^{j\omega})$is the DTFT of $h[n]$ i.e. $H(z)$ evaluated at $z = e^{j\omega}$.

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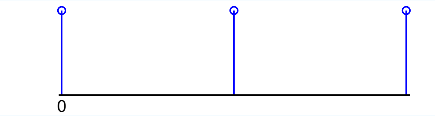

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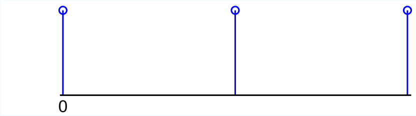

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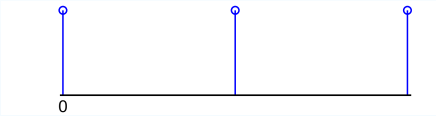

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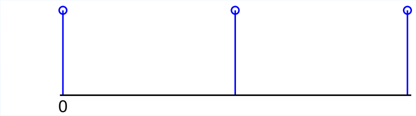

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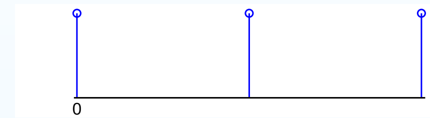

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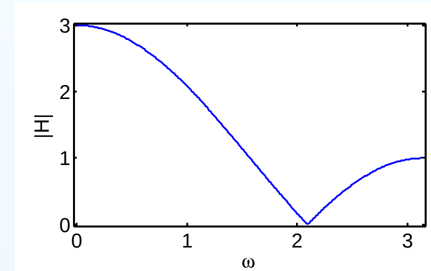

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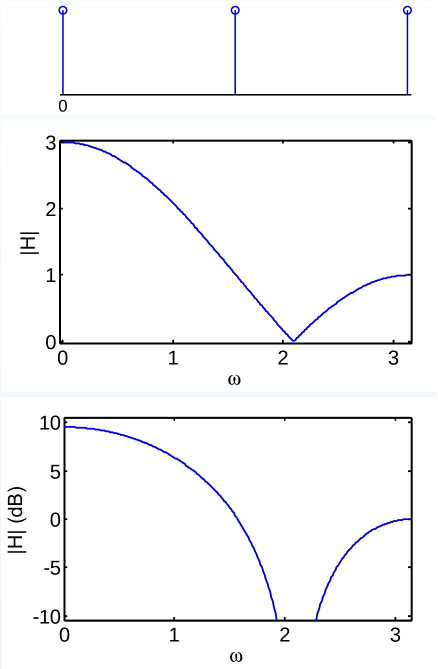

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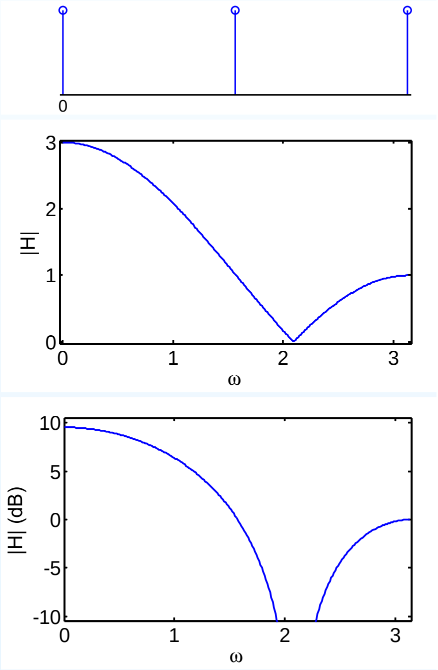

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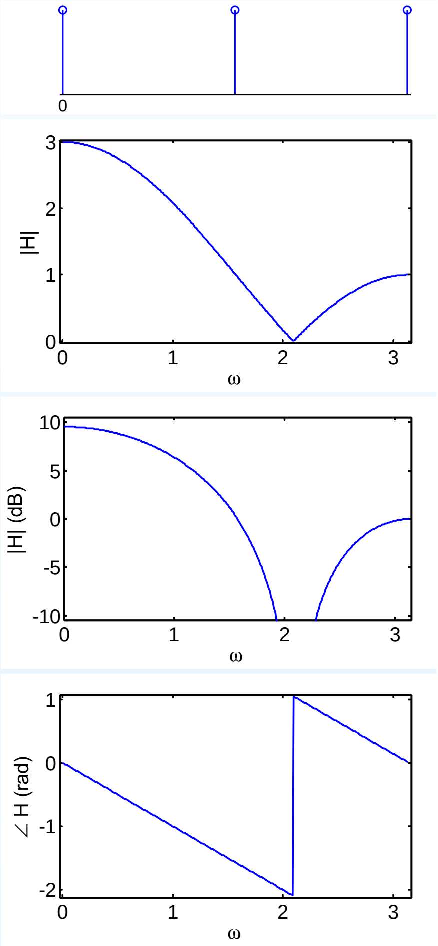

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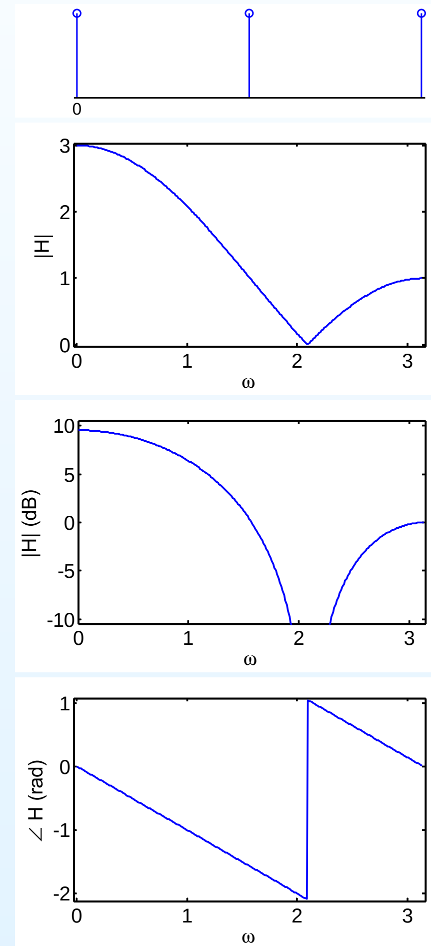

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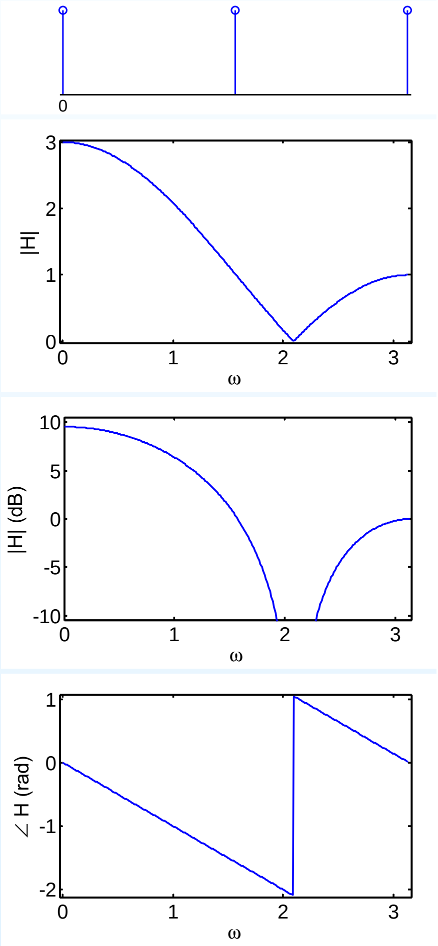

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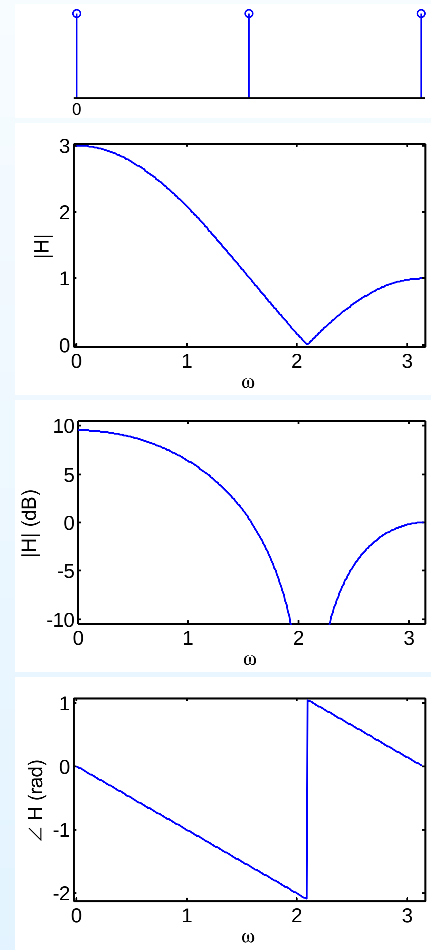

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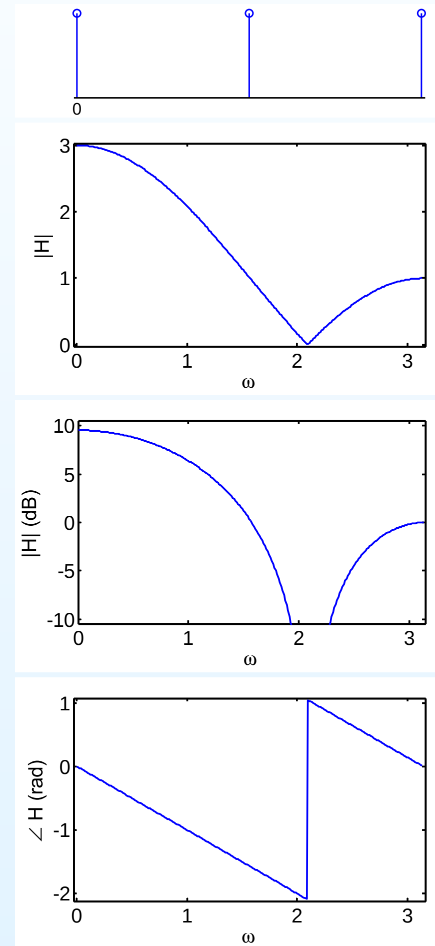

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Normally varies with ω but for a symmetric filter it is constant: in this case $+1$ samples.
Discontinuities of $\pm k\pi$ do not affect group delay.

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i.e. output at time n depends only on inputs up to time n.

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All the systems we will deal with are causal.

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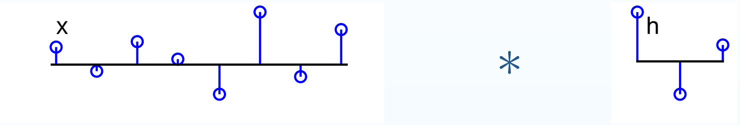
$y[n] = x[n] * h[n]$: convolve $x[0:N-1]$ with $h[0:M-1]$



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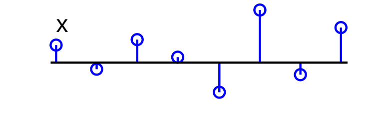

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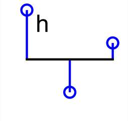

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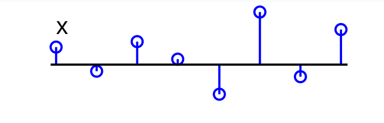

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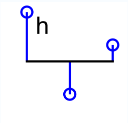

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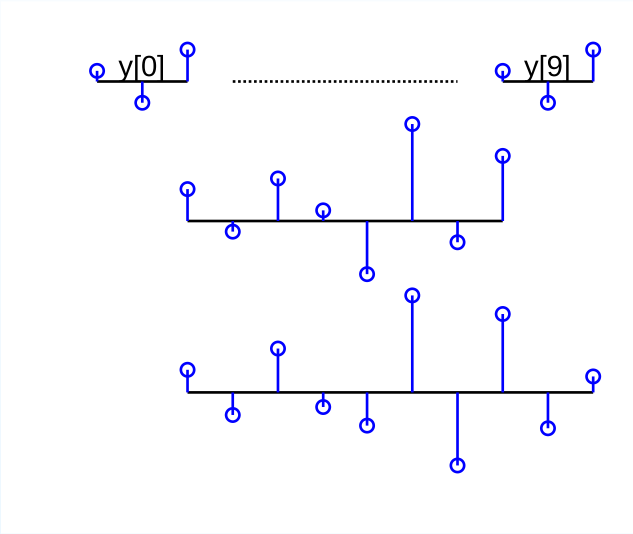

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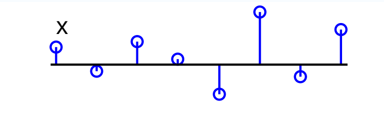

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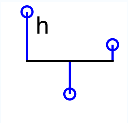

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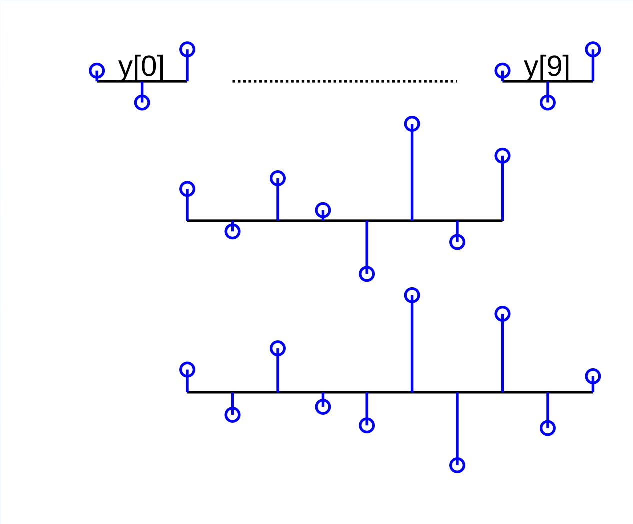

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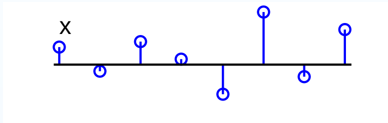

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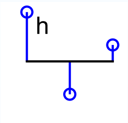

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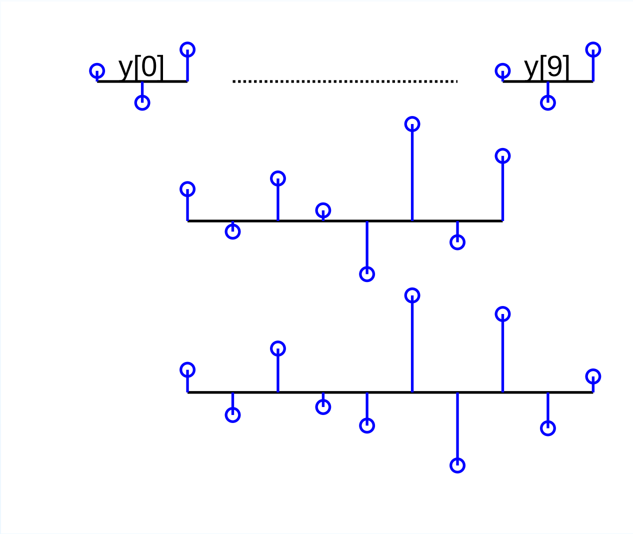

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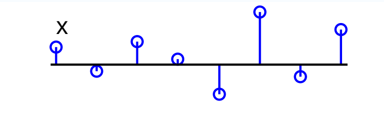

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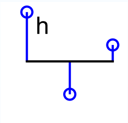

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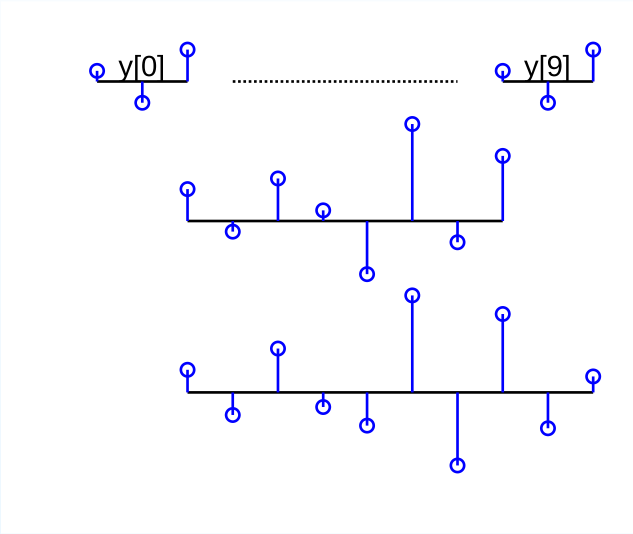

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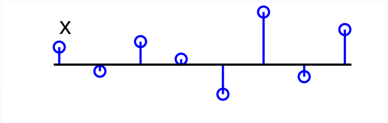

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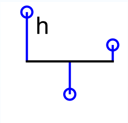

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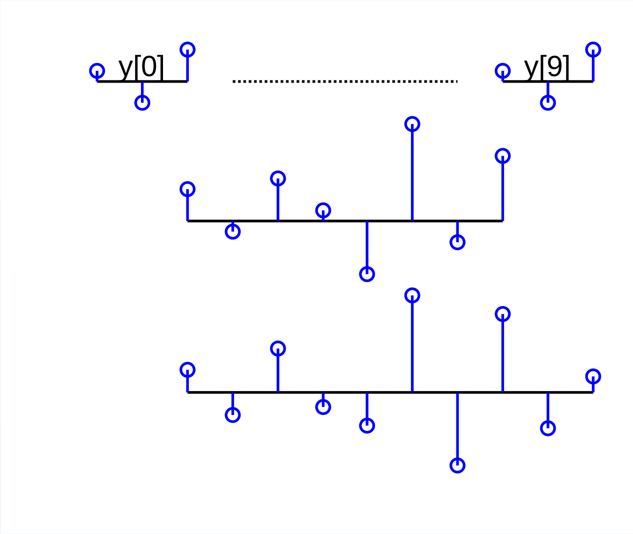

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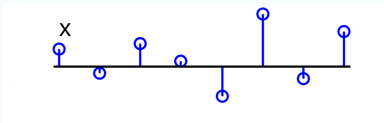

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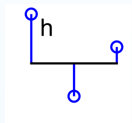

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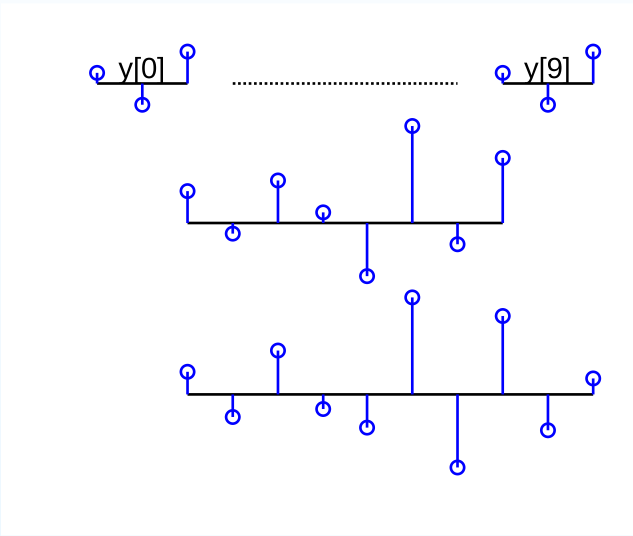

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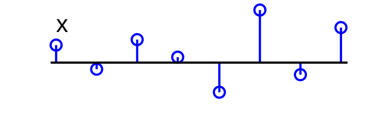

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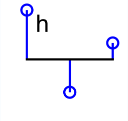

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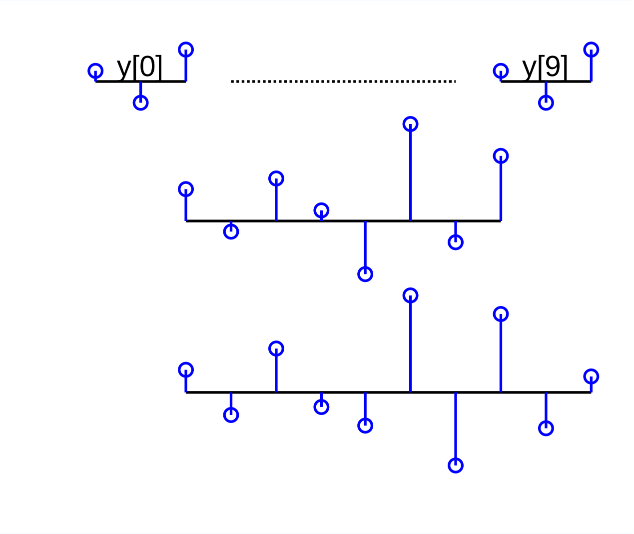

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We must multiply each $h[n]$ by each $x[n]$ and add them to a total

$\Rightarrow$ total arithmetic complexity ($\times$ or $+$ operations) $\approx 2MN$

Circular Convolution

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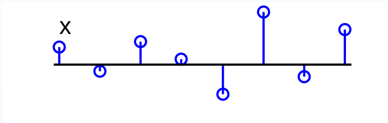

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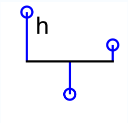

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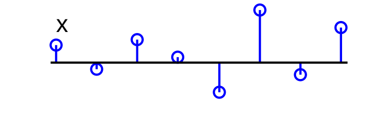

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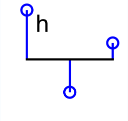

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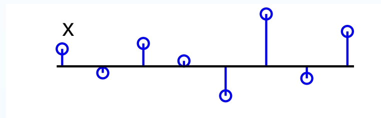

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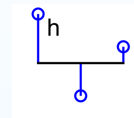

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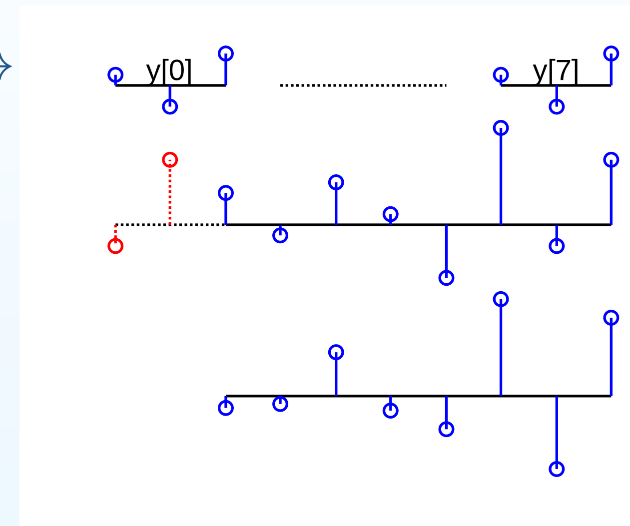

Convolution sum:

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$N = 8, M = 3$

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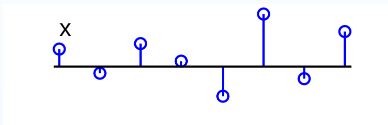
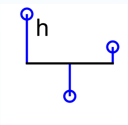


$\circledast_N$



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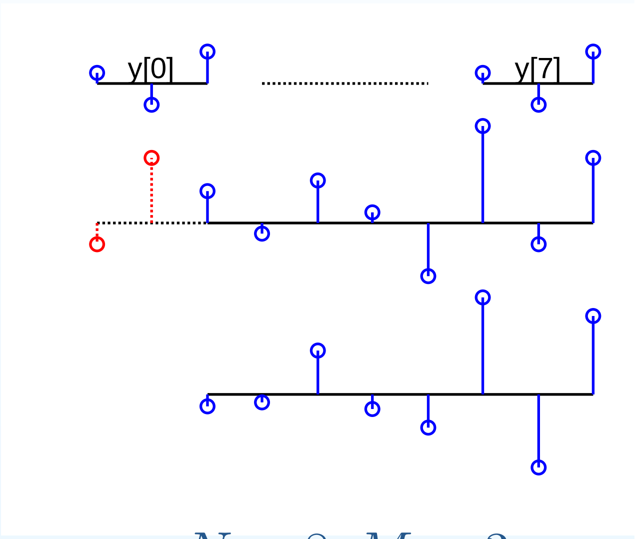

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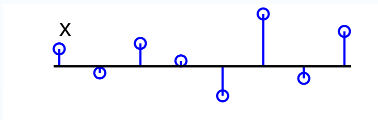

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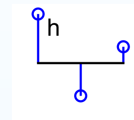

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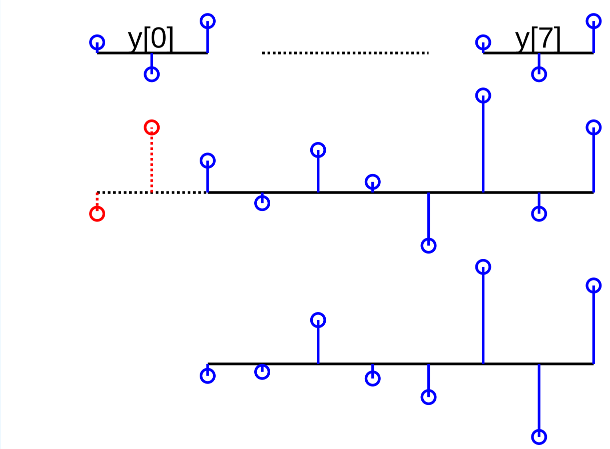

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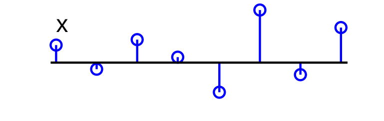

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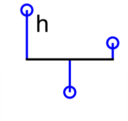

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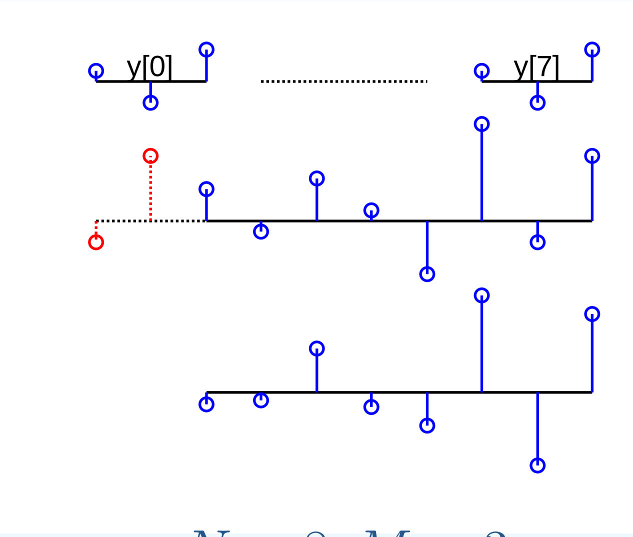

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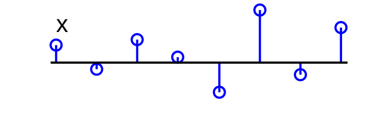

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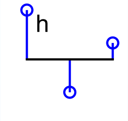

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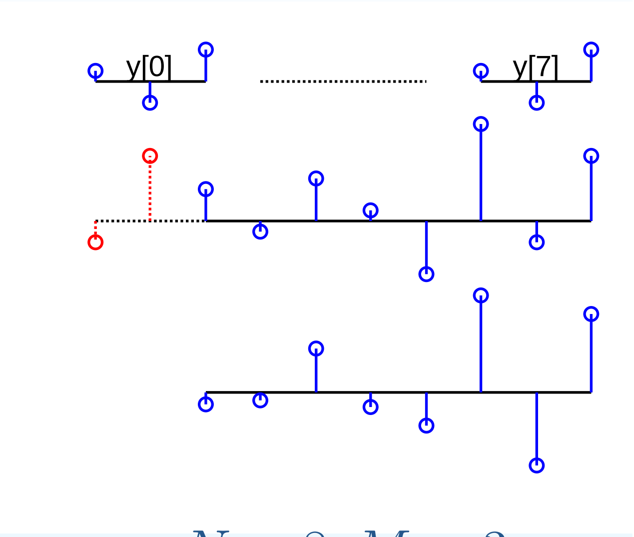

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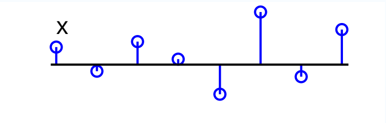

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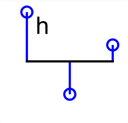

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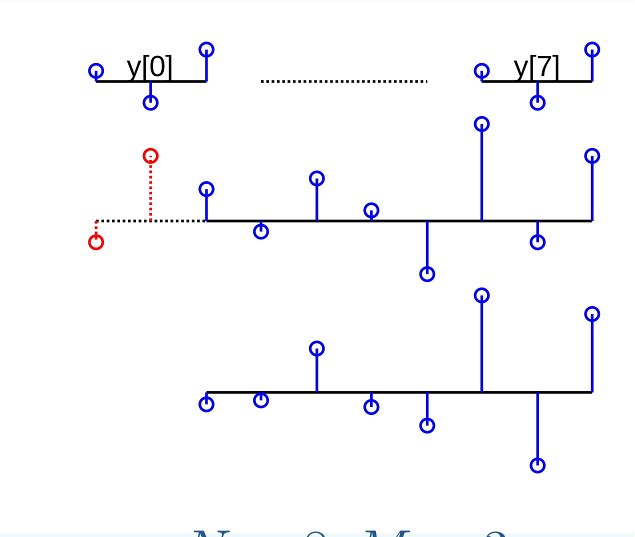

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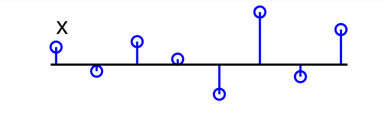

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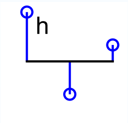

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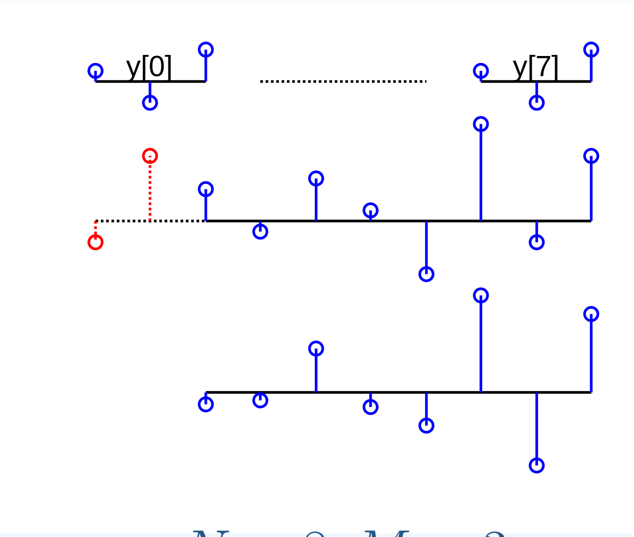

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Circular convolution is a necessary evil in exchange for using the DFT

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Idea: Use DFT to perform circular convolution - less computation

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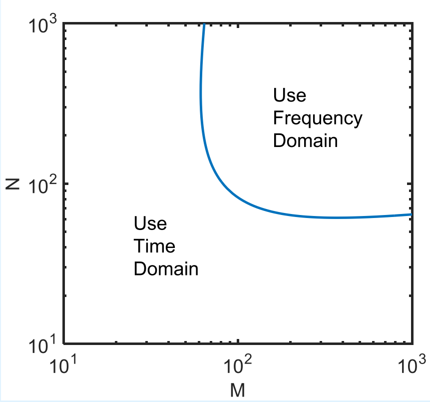
Beneficial if both M and N are $>\sim 70$.



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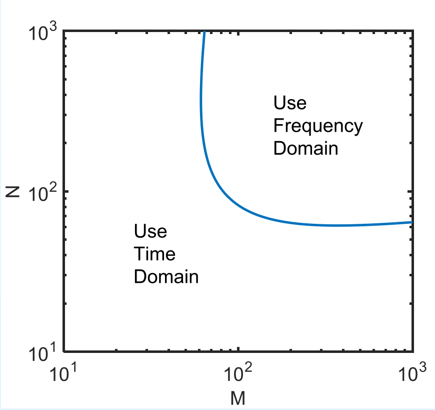

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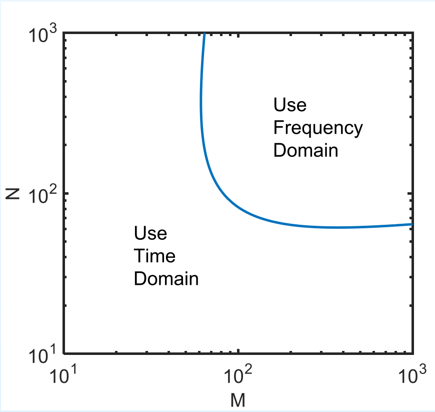

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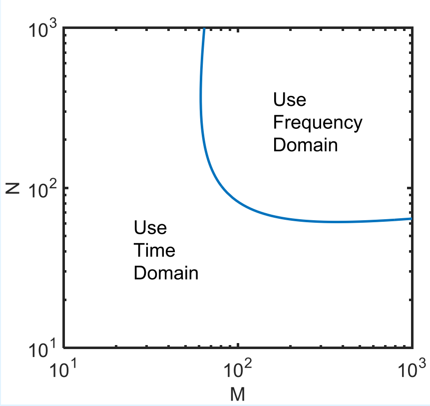
with DFT: $= 1.8 \times 10^6$ ☺



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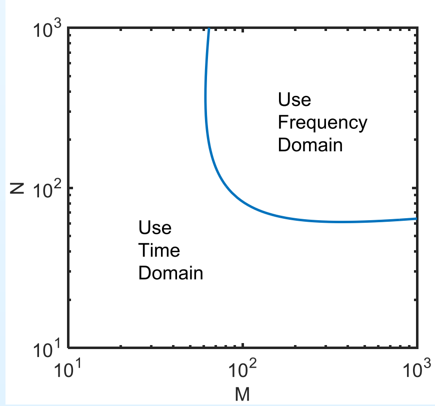

But: (a) DFT may be very long if N is large
(b) No outputs until all $x[n]$ has been input.

Overlap Add

If N is very large:

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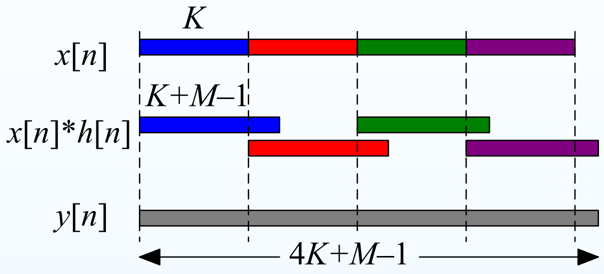
(1) chop $x[n]$ into $\frac{N}{K}$ chunks of length K



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If N is very large:
(1) chop $x[n]$ into $\frac{N}{K}$ chunks of length K
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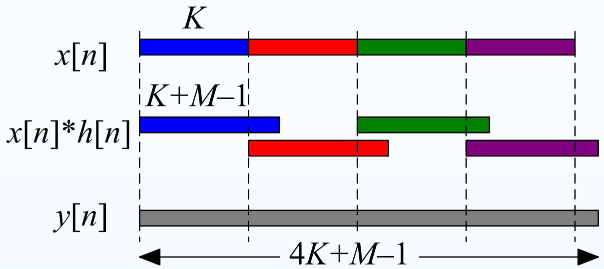

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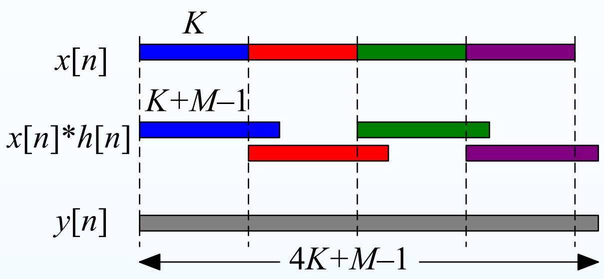

Each output chunk is of length $K + M - 1$ and overlaps the next chunk

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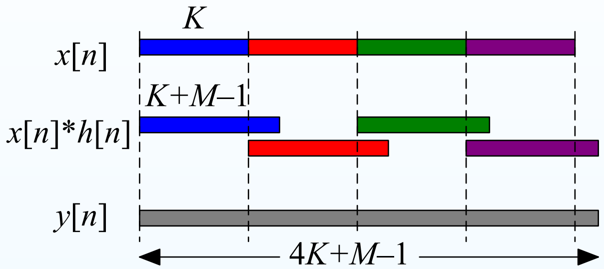

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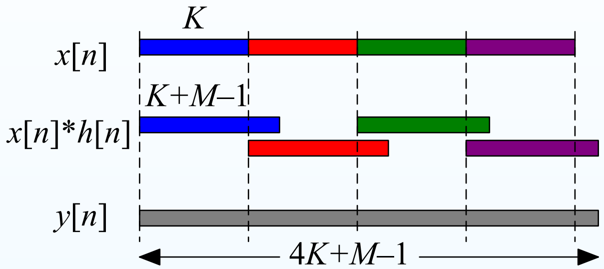

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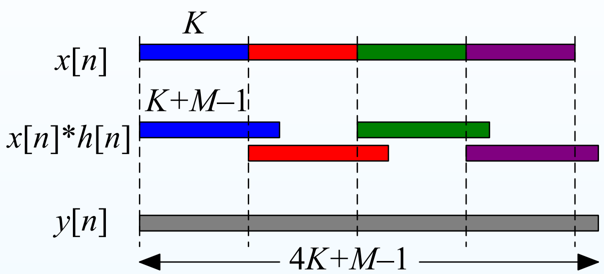

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Computational saving if $\approx 100 < M \ll K \ll N$

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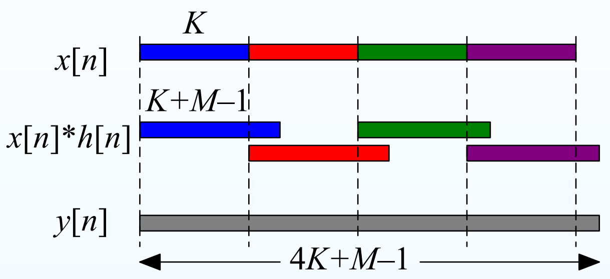

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Example: $M = 500$, $K = 10^4$, $N = 10^7$

Direct: $2MN = 10^{10}$

Overlap Add

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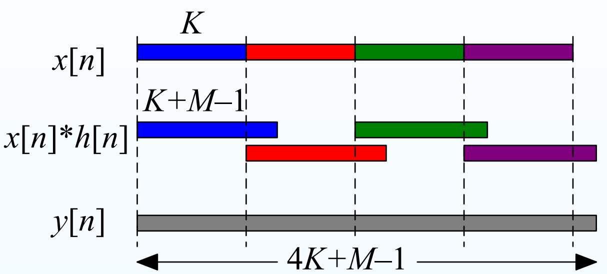

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Direct: $2MN = 10^{10}$

single DFT: $12\,(M+N)\log_2(M+N) = 2.8 \times 10^9$

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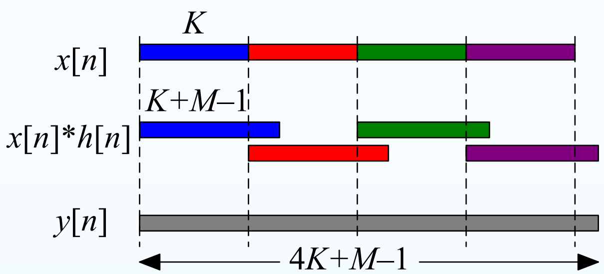
(2) convolve each chunk with $h[n]$
(3) add up the results



Each output chunk is of length $K+M-1$ and overlaps the next chunk

Operations: $\approx \frac{N}{K} \times 8\,(M+K)\log_2(M+K)$

Computational saving if $\approx 100 < M \ll K \ll N$

Example: $M = 500$, $K = 10^4$, $N = 10^7$

Direct: $2MN = 10^{10}$

single DFT: $12\,(M+N)\log_2(M+N) = 2.8 \times 10^9$

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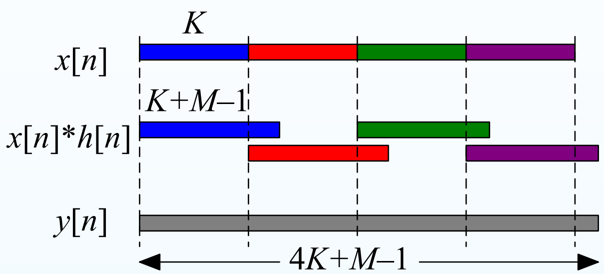

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(a) Shorter DFT

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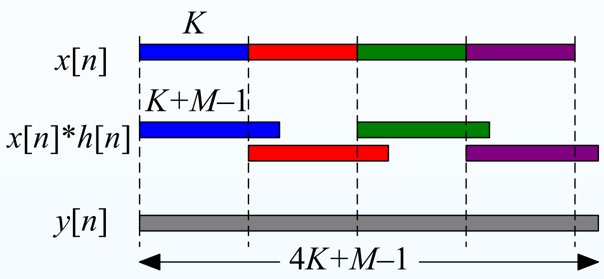

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Other advantages:
(a) Shorter DFT
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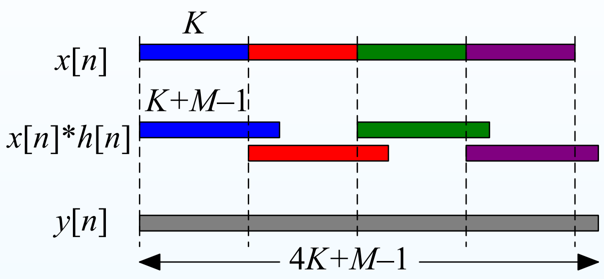

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Other advantages:

(a) Shorter DFT

(b) Can cope with $N = \infty$

(c) Can calculate $y[0]$ as soon as $x[K-1]$ has been read:

algorithmic delay = $K - 1$ samples

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Alternative method:

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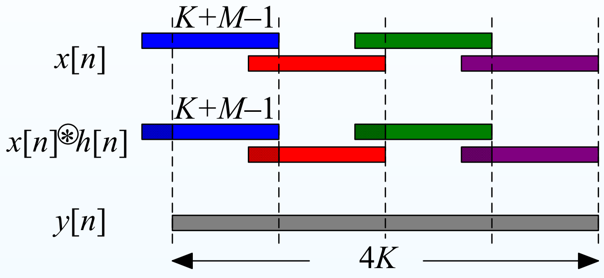

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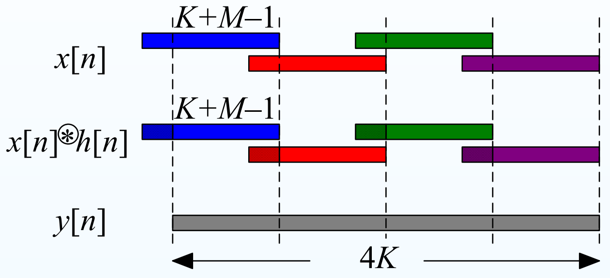

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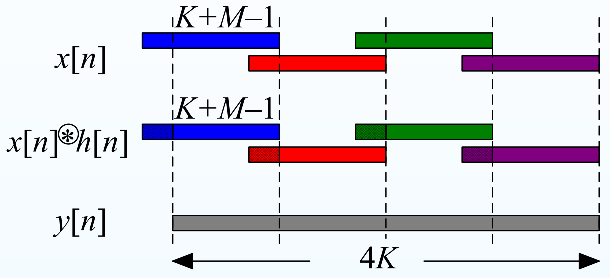

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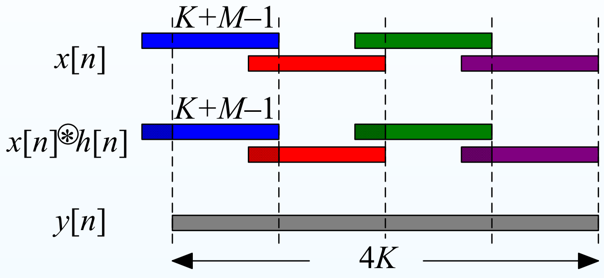

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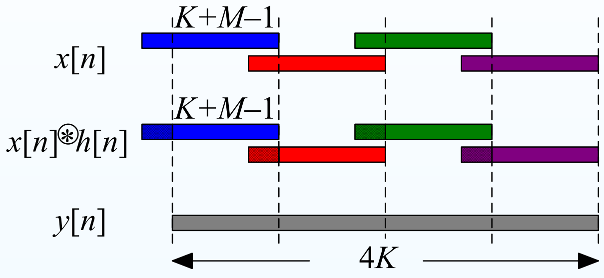

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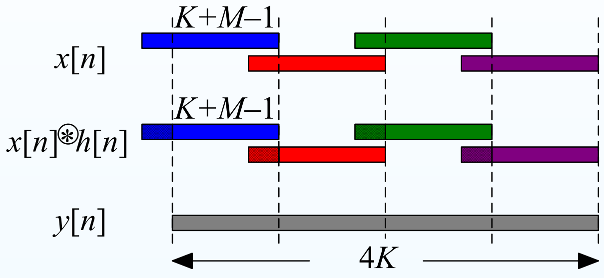

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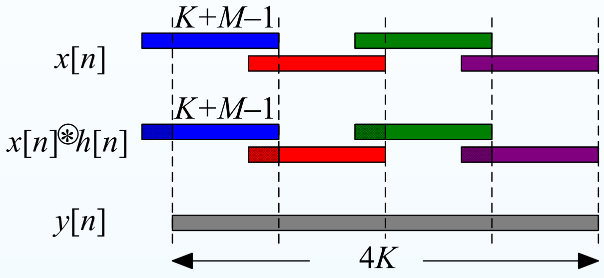

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Advantages: same as overlap add

Strangely, rather less popular than overlap-add

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For further details see Mitra: 4 & 5.

MATLAB routines

fftfilt	Convolution using overlap add
$x[n] \circledast y[n]$	real(ifft(fft(x).*fft(y)))