Problem Sheet 2

Discrete Walsh Transform

Problems

The following two problems aim at helping you practice calculating the Walsh-Hadamard transforms for a real life signal.

1. Let $f(x,y)$ denote an $N \times N$-point 2-D sequence, that has zero value outside $0 \le x, y \le N-1$, where N is an integer power of 2. In implementing the standard Walsh transform of $f(x,y)$, we relate $f(x,y)$ to a new $N \times N$-point 2-D sequence $W(u,v)$.

   (i) Define the sequence $W(u,v)$

   (ii) In the case of $N = 2$ and $f(x,y) = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$ calculate the Walsh transform coefficients.

<u>Solution</u>

(i) Book work.

(ii) We know that $N = 2^n$ and therefore, in case of $N = 2$ we have $n = 1$, x, y 0 or 1 and $b_0(0) = b_0(0) = 0$ and $b_0(1) = b_0(1) = 1$. For $f(x,y) = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$ we calculate the Walsh transform coefficients as follows.

$$W(u,v) = \frac{1}{N} \sum_{x=0}^{2-1} \sum_{y=0}^{2-1} f(x,y) \left[\prod_{i=0}^{n-1} (-1)^{(b_i(x) b_{n-1-i}(u) + b_i(y) b_{n-1-i}(v))} \right]$$

$$W(u,v) = \frac{1}{N} \sum_{x=0}^{1} \sum_{y=0}^{1} f(x,y) \left[\prod_{i=0}^{1-1} (-1)^{(b_i(x) b_{-i}(u) + b_i(y) b_{-i}(v))} \right]$$

$$W(u,v) = \frac{1}{N} \sum_{x=0}^{1} \sum_{y=0}^{1} f(x,y) (-1)^{(b_0(x) b_0(u) + b_0(y) b_0(v))}$$

$$= \frac{1}{2} f(0,0)(-1)^{(b_0(0)b_0(u) + b_0(0)b_0(v))} + \frac{1}{2} f(0,1)(-1)^{(b_0(0)b_0(u) + b_0(1)b_0(v))}$$

$$+ \frac{1}{2} f(1,0)(-1)^{(b_0(1)b_0(u) + b_0(0)b_0(v))} + \frac{1}{2} f(1,1)(-1)^{(b_0(1)b_0(u) + b_0(1)b_0(v))}$$

$$= \frac{1}{2} f(0,0)(-1)^{(0 \cdot b_0(u) + 0 \cdot b_0(v))} + \frac{1}{2} f(0,1)(-1)^{(0 \cdot b_0(u) + 1 \cdot b_0(v))}$$

$$+ \frac{1}{2} f(1,0)(-1)^{(1 \cdot b_0(u) + 0 \cdot b_0(v))} + \frac{1}{2} f(1,1)(-1)^{(1 \cdot b_0(u) + 1 \cdot b_0(v))}$$

$$= \frac{1}{2} f(0,0)(-1)^0 + \frac{1}{2} f(0,1)(-1)^{b_0(v)} + \frac{1}{2} f(1,0)(-1)^{b_0(u)} + \frac{1}{2} f(1,1)(-1)^{b_0(u)+b_0(v)}$$

$$= \frac{1}{2}(-1)^0 + \frac{1}{2} 2(-1)^{b_0(v)} + \frac{1}{2} 2(-1)^{b_0(u)} + \frac{1}{2} 3(-1)^{b_0(u)+b_0(v)}$$

$$= \frac{1}{2} + (-1)^{b_0(v)} + (-1)^{b_0(u)} + \frac{3}{2}(-1)^{b_0(u)+b_0(v)}$$

$$W(0,0)=\frac{1}{2}+(-1)^{b_0(0)}+(-1)^{b_0(0)}+\frac{3}{2}(-1)^{b_0(0)+b_0(0)}=\frac{1}{2}+(-1)^0+(-1)^0+\frac{3}{2}(-1)^0=\frac{1}{2}+1+1+\frac{3}{2}=4$$

$$W(0,1)=\frac{1}{2}+(-1)^{b_0(0)}+(-1)^{b_0(1)}+\frac{3}{2}(-1)^{b_0(0)+b_0(1)}=\frac{1}{2}+(-1)^0+(-1)^1+\frac{3}{2}(-1)^{0+1}$$

$$=\frac{1}{2}+1-1-\frac{3}{2}=-1$$

$$W(1,0)=\frac{1}{2}+(-1)^{b_0(1)}+(-1)^{b_0(0)}+\frac{3}{2}(-1)^{b_0(1)+b_0(0)}=\frac{1}{2}+(-1)^1+(-1)^0+\frac{3}{2}(-1)^{1+0}$$

$$=\frac{1}{2}-1+1-\frac{3}{2}=-1$$

$$W(1,1)=\frac{1}{2}+(-1)^{b_0(1)}+(-1)^{b_0(1)}+\frac{3}{2}(-1)^{b_0(1)+b_0(1)}=\frac{1}{2}+(-1)^1+(-1)^1+\frac{3}{2}(-1)^{1+1}$$

$$=\frac{1}{2}-1-1+\frac{3}{2}=0$$

Therefore, $W(u,v)=\begin{bmatrix}4 & -1\\ -1 & 0\end{bmatrix}$.

Discrete Hadamard Transform

1. Let $f(x,y)$ denote an $N\times N$-point 2-D sequence, that has zero value outside $0\le x,y\le N-1$, where N is an integer power of 2. In implementing the 2-D Discrete Hadamard Transform of $f(x,y)$, we relate $f(x,y)$ to a new $N\times N$-point 2-D sequence $H(u,v)$.

   (i) Define the sequence $H(u,v)$

   (ii) Find the Hadamard Transform of the following image

$$\begin{bmatrix}5 & 6 & 8 & 10\\ 6 & 6 & 5 & 7\\ 4 & 5 & 3 & 6\\ 8 & 7 & 5 & 5\end{bmatrix}$$

   Hint: Use the recursive relationship of the Hadamard transform;

   $H_{2N}=\begin{bmatrix}H_N & H_N\\ H_N & -H_N\end{bmatrix}$ with $H_2=\frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}$

   <u>Solution</u>

   (i) Book work.

   (ii) The 4x4 Hadamard matrix is $\frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1 & 1 & 1\\ 1 & -1 & 1 & -1\\ 1 & 1 & -1 & -1\\ 1 & -1 & -1 & 1\end{bmatrix}$

   We first calculate the Hadamard transform of the rows as follows.

$$\frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1 & 1 & 1\\ 1 & -1 & 1 & -1\\ 1 & 1 & -1 & -1\\ 1 & -1 & -1 & 1\end{bmatrix}\begin{bmatrix}5\\ 6\\ 8\\ 10\end{bmatrix}=\frac{1}{\sqrt{2}}\begin{bmatrix}29\\ -3\\ -7\\ 1\end{bmatrix}$$

$$\frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1 & 1 & 1\\ 1 & -1 & 1 & -1\\ 1 & 1 & -1 & -1\\ 1 & -1 & -1 & 1\end{bmatrix}\begin{bmatrix}6\\ 6\\ 5\\ 7\end{bmatrix}=\frac{1}{\sqrt{2}}\begin{bmatrix}24\\ -2\\ 0\\ 2\end{bmatrix}$$

$$\frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1 & 1 & 1\\ 1 & -1 & 1 & -1\\ 1 & 1 & -1 & -1\\ 1 & -1 & -1 & 1\end{bmatrix}\begin{bmatrix}4\\ 5\\ 3\\ 6\end{bmatrix}=\frac{1}{\sqrt{2}}\begin{bmatrix}18\\ -4\\ 0\\ 2\end{bmatrix}$$

$$\frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1 & 1 & 1\\ 1 & -1 & 1 & -1\\ 1 & 1 & -1 & -1\\ 1 & -1 & -1 & 1\end{bmatrix}\begin{bmatrix}8\\ 7\\ 5\\ 5\end{bmatrix}=\frac{1}{\sqrt{2}}\begin{bmatrix}25\\ 1\\ 5\\ 1\end{bmatrix}$$

After doing that we construct the intermediate matrix

$$\frac{1}{\sqrt{2}}\begin{bmatrix}29 & -3 & -7 & 1\\ 24 & -2 & 0 & 2\\ 18 & -4 & 0 & 2\\ 25 & 1 & 5 & 1\end{bmatrix}.$$

Then we apply the DHT on the columns of the intermediate image and we obtain:

$$\frac{1}{2}\begin{bmatrix}1 & 1 & 1 & 1\\ 1 & -1 & 1 & -1\\ 1 & 1 & -1 & -1\\ 1 & -1 & -1 & 1\end{bmatrix}\begin{bmatrix}29\\ 24\\ 18\\ 25\end{bmatrix}=\begin{bmatrix}96\\ -2\\ 10\\ 12\end{bmatrix}$$

$$\frac{1}{2}\begin{bmatrix}1 & 1 & 1 & 1\\ 1 & -1 & 1 & -1\\ 1 & 1 & -1 & -1\\ 1 & -1 & -1 & 1\end{bmatrix}\begin{bmatrix}-3\\ -2\\ -4\\ 1\end{bmatrix}=\begin{bmatrix}-8\\ -6\\ -2\\ 4\end{bmatrix}$$

$$\frac{1}{2}\begin{bmatrix}1 & 1 & 1 & 1\\ 1 & -1 & 1 & -1\\ 1 & 1 & -1 & -1\\ 1 & -1 & -1 & 1\end{bmatrix}\begin{bmatrix}-7\\ 0\\ 0\\ 5\end{bmatrix}=\begin{bmatrix}-2\\ -12\\ -12\\ -2\end{bmatrix}$$

$$\frac{1}{2}\begin{bmatrix}1 & 1 & 1 & 1\\ 1 & -1 & 1 & -1\\ 1 & 1 & -1 & -1\\ 1 & -1 & -1 & 1\end{bmatrix}\begin{bmatrix}1\\ 2\\ 2\\ 1\end{bmatrix}=\begin{bmatrix}6\\ 0\\ 0\\ -2\end{bmatrix}$$

The transformed image is obtained by combining the above columns and therefore,

$$\frac{1}{2}\begin{bmatrix} 96 & -8 & -2 & 6 \\ -2 & -6 & -12 & 0 \\ 10 & -2 & -12 & 0 \\ 12 & 4 & -2 & 2 \end{bmatrix}$$

Karhunen Loeve Transform

Questions

Consider the population of random vectors $\underline{f}$ of the form

$$\underline{f} = \begin{bmatrix} f_1(x,y) \\ f_2(x,y) \\ \vdots \\ f_n(x,y) \end{bmatrix}.$$

Each component $f_i(x,y)$ represents an image. The population arises from the formation of the images across the entire collection of pixels. Suppose that $n > 2$, i.e. you have at least three images.
Consider now a population of random vectors of the form

$$\underline{g} = \begin{bmatrix} g_1(x,y) \\ g_2(x,y) \\ \vdots \\ g_n(x,y) \end{bmatrix}$$

where the vectors $\underline{g}$ are the Karhunen-Loeve transforms of the vectors $\underline{f}$.

1. Write down the relationship between $\underline{g}$ and $\underline{f}$.

<u>**Answer**</u>

The mean vector of the population is defined as

$$\underline{m}_f = E\{\underline{f}\} \Rightarrow \begin{bmatrix} m_1 \\ m_2 \\ \vdots \\ m_n \end{bmatrix} = \begin{bmatrix} E\{f_1\} \\ E\{f_2\} \\ \vdots \\ E\{f_n\} \end{bmatrix}$$

The covariance matrix of the population is defined as

$$\underline{C}_f = E\{(\underline{f} - \underline{m}_f)(\underline{f} - \underline{m}_f)^T\}$$

For M vectors from a random population, where M is large enough, the mean vector and covariance matrix can be approximately calculated by summations

$$\underline{m}_f = \frac{1}{M}\sum_{k=1}^{M}\underline{f}_k \text{ , } \underline{C}_f = \frac{1}{M}\sum_{k=1}^{M}\underline{f}_k \underline{f}_k^T - \underline{m}_f \underline{m}_f^T$$

Very easily it can be seen that $\underline{C}_f$ is real and symmetric. We assume that a set of n orthonormal eigenvectors always exists.

Let $\underline{A}$ be a matrix whose rows are formed from the eigenvectors of $\underline{C}_f$, ordered so that the first row of $\underline{A}$ is the eigenvector corresponding to the largest eigenvalue, and the last row the eigenvector corresponding to the smallest eigenvalue.

The Karhunen-Loeve transform maps the vectors $\underline{f}'s$ into vectors $\underline{g}'s$ with the relationship

$$\underline{g} = \underline{A}(\underline{f} - \underline{m}_f)$$

2. It is known that the covariance matrix of $\underline{g}$ is diagonal. Explain the relationship between these diagonal elements and the covariance matrix of $\underline{f}$.

   Answer

   The mean of the $\underline{g}$ vectors resulting from the above transformation is zero ($\underline{m}_g = \underline{0}$) and the covariance matrix is $\underline{C}_g = \underline{A}\underline{C}_x\underline{A}^T$, where $\underline{C}_g$ is a diagonal matrix whose elements along the main diagonal are the eigenvalues of $\underline{C}_f$

$$\underline{C}_g = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix}$$

   The off-diagonal elements of the covariance matrix are 0, so the elements of the $\underline{g}$ vectors are uncorrelated.

3. Suppose some elements of the diagonal are very small. Comment on the significance of this observation in relation to processing the images.

   Answer

   The element λ_i represents the variance of the image $g_i(x,y)$. If λ_i has very small value then the variance of the image $g_i(x,y)$ is very small. Practically, $g_i(x,y)$ does not contain any useful information so it can be neglected in the computation of the inverse KL transform. This is very useful for compression purposes.

4. Suppose that a credible job could be done of reconstructing approximations to the n original images by using only the two principal component images associated with the largest eigenvalues. What would be the mean square error incurred in doing so? Express your answer as a percentage of the maximum possible error.

   Answer

   Mean square error $\sum_{j=3}^{n} \lambda_j$.

Maximum error occurs when we keep only the largest eigenvalue and is equal to $\sum_{j=2}^{n}\lambda_j = \lambda_2 + \lambda_3$. The error as a percentage of the maximum possible error is $100\frac{\sum_{j=3}^{n}\lambda_j}{\sum_{j=2}^{n}\lambda_j} = 100\frac{\lambda_2}{\lambda_2+\lambda_3}$.

5. Suppose that a credible job could be done of reconstructing approximations to the n original images, with n even number, by using only the half of the principal component images associated with the largest eigenvalues. What would be the mean square error incurred in doing so? Express your answer as a percentage of both the maximum and the minimum possible error.

   <u>**Answer**</u>

   Mean square error $\sum_{j=\frac{n}{2}+1}^{n}\lambda_j$.

   Maximum error occurs when we keep only the largest eigenvalue and is equal to $\sum_{j=2}^{n}\lambda_j$. The error as a percentage of the maximum possible error is $100\frac{\sum_{j=\frac{n}{2}+1}^{n}\lambda_j}{\sum_{j=2}^{n}\lambda_j}$.

6. Suppose that the covariance matrix of $\underline{g}$ turns out to be the identity matrix. Is the Karhunen-Loeve transform useful in that case? Justify your answer.

   <u>**Answer**</u>

   It is not useful since the original images are all equally important.

Problems

1. The covariance matrix of the population $\underline{f}$ calculated as part of the transform is

$$\underline{C}_{\underline{f}} = \begin{bmatrix} a & b & 0 \\ b & a & 0 \\ 0 & 0 & c \end{bmatrix}$$

   (i) Suppose that a credible job could be done of reconstructing approximations to the three original images by using one principal component image. What would be the mean square error incurred in doing so, if it is known that $b < a - c$?
   (ii) Suppose that a credible job could be done of reconstructing approximations to the three original images by using one principal component image. What would be the mean square error incurred in doing so, if it is known that $b < a - c$?

   <u>**Solution**</u>

The eigenvalues of the covariance matrix $\underline{C}_{\underline{f}} = \begin{bmatrix} a & b & 0 \\ b & a & 0 \\ 0 & 0 & c \end{bmatrix}$ **are found by the following relationship:**

$$\det\begin{bmatrix} a-\lambda & b & 0 \\ b & a-\lambda & 0 \\ 0 & 0 & c-\lambda \end{bmatrix} = (c-\lambda)[(a-\lambda)^2 - b^2] = (c-\lambda)[(a-\lambda)-b](c-\lambda)[(a-\lambda)+b]$$

$= 0 \Rightarrow \lambda_1 = c, \lambda_2 = a-b, \lambda = a+b$

(i) **If** $b < a-c$ **then because** $c \geq 0$ **since it represents variance of an image the eigenvalues will be sorted according to magnitude as** $a+c \geq a-c > b$ **and therefore by using only one principal component the error of reconstruction will be** $a-c+b$**.**

(ii) **If** $b > a+c$ **the eigenvalues will be sorted according to magnitude as** $b > a+c \geq a-c$ **and therefore by using only two principal components the error of reconstruction will be** $a-c$**.**

2. The covariance matrix of the population $\underline{f}$ calculated as part of the transform is

$$\underline{C}_{\underline{f}} = \begin{bmatrix} a & 0 & b^2 \\ 0 & a & b^2 \\ b^2 & b^2 & a \end{bmatrix}$$

Suppose that a credible job could be done of reconstructing approximations to the three original images by using one or two principal component images. What would be the mean square error incurred in doing so in each case?

<u>Solution</u>

The eigenvalues of the covariance matrix $\underline{C}_{\underline{f}} = \begin{bmatrix} a & 0 & b^2 \\ 0 & a & b^2 \\ b^2 & b^2 & a \end{bmatrix}$ **are found by the following relationship:**

$$\det\begin{bmatrix} a-\lambda & 0 & b^2 \\ 0 & a-\lambda & b^2 \\ b^2 & b^2 & a-\lambda \end{bmatrix} = (a-\lambda)\left[(a-\lambda)^2 - b^4\right] - b^4(a-\lambda) = (a-\lambda)\left[(a-\lambda)^2 - 2b^4\right] = 0 \Rightarrow$$

$\lambda_1 = a$

$\lambda - a = \pm\sqrt{2}b^2 \Rightarrow \lambda_{2,3} = a \pm \sqrt{2}b^2$

We rearrange the indices according to size

$\lambda_1 = \lambda_{\max} = a + \sqrt{2}b^2$

$\lambda_2 = a$

$\lambda_3 = \lambda_{\min} = a - \sqrt{2}b^2$

The maximum error is $\lambda_2 + \lambda_3 = 2a - \sqrt{2}b^2$

If we use one principal component the error is equal to the maximum possible error.

If we use two principal component the error is equal to $\lambda_3 = \lambda_{\min} = a - \sqrt{2}b^2$ **and as a percentage of the maximum possible error is** $\dfrac{a-\sqrt{2}b}{2a-\sqrt{2}b}$**.**