# Problem Sheet 1

## Discrete Fourier Transforms

### Questions

1. Explain why it is common to work with the transform of an image instead of the image itself.
**Very often by observing the transform of the images provides more insight into the properties and characteristics of the image that observing directly the pixel intensities. Furthermore, we can do easily many operations on the transform of an image. Very often the large and therefore important transform coefficients are concentrated close to the origin of the transform domain. It is quite common to keep only a subset of the transform coefficients, i.e., the ones close to the origin and discard the rest. By doing so, we can easily achieve image compression. We can also achieve various filtering operations, i.e., low pass, band pass or high pass filtering by selecting and keeping certain subsets of the transform coefficients. Please look at the course slides on Discrete Fourier Transform.**

2. Explain why the Fourier transform amplitude of an image alone often **DOES NOT** capture the intelligibility of the image.
**In viewing a picture, some of the most important visual information is contained in the edges and regions of high contrast. Intuitively, regions of maximum and minimum intensity in a picture are places at which complex exponentials at different frequencies are in phase. Therefore, it seems plausible to expect the phase of the Fourier transform of a picture to contain much of the information in the picture, and in particular, the phase should capture the information about the edges.**

3. Explain why the Fourier transform phase of an image alone often **DOES** capture most of the intelligibility of the image.
**Same answer as in 2 above.**

4. In a specific experiment it is observed that the amplitude of the Fourier transform of an image exhibits high values only very close to the origin and takes very small values within the rest of the two-dimensional frequency plane. State the implications of this observation as far as the original image is concerned.
**The above observation implies that the original image contains mainly low frequency components, i.e., it is quite smooth.**

### Problems

1. Consider an $M \times N$-pixel image $f(x,y)$ which is zero outside $0 \le x \le M-1$ and $0 \le y \le N-1$. Let $F(u,v)$ denote the $M \times N$-point Discrete Fourier Transform (DFT) of $f(x,y)$. Let $G(u,v)$ denote $F(u,v)$ modified by

$$G(u,v) = \begin{cases} F(u,v), & \text{when } |F(u,v)| \text{ is large} \\ 0, & \text{otherwise} \end{cases}$$

Let

$$\frac{\sum_{u=0}^{M-1}\sum_{v=0}^{N-1} |G(u,v)|^2}{\sum_{u=0}^{M-1}\sum_{v=0}^{N-1} |F(u,v)|^2} = \frac{9}{10}$$

We reconstruct an image $g(x,y)$ by computing the $M \times N$-point inverse DFT of $G(u,v)$. Express $\sum_{x=0}^{M-1}\sum_{y=0}^{N-1}(f(x,y)-g(x,y))^2$ in terms of $\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)^2$.

**Solution**

**The signal $f(x,y)-g(x,y)$ is obtained by the Inverse DFT of the signal $F(u,v)-G(u,v)$. Therefore, according to Parseval's theorem the energy of the signal $f(x,y)-g(x,y)$ is equal to the energy of the signal $F(u,v)-G(u,v)$. The signal $F(u,v)-G(u,v)$ consists of the DFT samples of $F(u,v)$ which were excluded in forming $G(u,v)$. Since, $G(u,v)$ captures 0.9 of the energy of $F(u,v)$, the signal $F(u,v)-G(u,v)$ will capture 0.1 of the energy of $F(u,v)$. Therefore,**

$$\sum_{x=0}^{M-1}\sum_{y=0}^{N-1}(f(x,y)-g(x,y))^2 = 0.1\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)^2$$

2. Consider an $M \times N$-pixel gray level image $f(x,y)$ which is zero outside $0 \le x \le M-1$ and $0 \le y \le N-1$. The image intensity is given by the following relationship

$$f(x,y) = \begin{cases} c, & x = x_0, 0 \le y \le N-1 \\ 0, & \text{otherwise} \end{cases}$$

where $c$ is a constant value between 0 and 255 and $x_0$ is a constant value between 0 and $M-1$.

(i) Plot the image intensity.

(ii) Find the $M \times N$-point Discrete Fourier Transform (DFT) of $f(x,y)$. Plot its amplitude response.

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The following result holds: $\sum_{k=0}^{N-1} a^x = \frac{1-a^N}{1-a}, |a| \le 1$.

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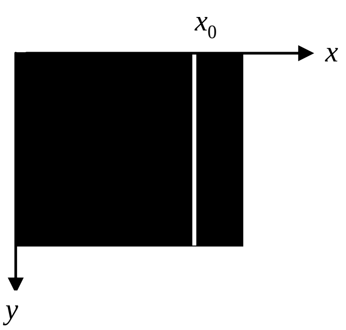
**Solution**

(i) The image is zero apart from a straight line $x = x_0$




(ii)

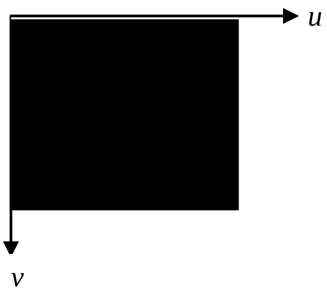
$$F(u,v) = \frac{1}{MN}\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)e^{-j2\pi(ux/M+vy/N)} = \frac{1}{MN}\sum_{y=0}^{N-1} f(x_0,y)e^{-j2\pi(ux_0/M+vy/N)}$$

$$= \frac{c}{MN}e^{-j2\pi ux_0/M}\sum_{y=0}^{N-1} e^{-j2\pi vy/N} = \frac{c}{MN}e^{-j2\pi ux_0/M}\frac{1-(e^{-j2\pi v/N})^N}{1-e^{-j2\pi v/N}} = \frac{c}{MN}e^{-j2\pi ux_0/M}\frac{1-e^{-j2\pi v}}{1-e^{-j2\pi v/N}}$$

$$F(u,v)=\begin{cases}\dfrac{c}{MN}Ne^{-j2\pi ux_0/M}=\dfrac{c}{M}e^{-j2\pi ux_0/M}, & v=0\\ 0, & \text{otherwise}\end{cases}$$

$$\left|F(u,v)\right|=\begin{cases}\dfrac{c}{M}, & v=0\\ 0, & \text{otherwise}\end{cases}$$

The resulting image in transform domain is zero apart from a straight non-zero line at $v=0$




**As seen a straight line in space implies a straight line perpendicular to the original one in frequency.**

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The following result holds: $\sum_{k=0}^{N-1} a^x = \dfrac{1-a^N}{1-a}, |a| \le 1$.

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