Digital Image Procesing

The Karhunen-Loeve Transform (KLT) in Image Processing

DR TANIA STATHAKI

READER (ASSOCIATE PROFFESOR) IN SIGNAL PROCESSING
IMPERIAL COLLEGE LONDON

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Eigenvalues and Eigenvectors

The concepts of eigenvalues and eigenvectors are important for understanding the KL transform.

If C is a matrix of dimension $n \times n$, then a scalar λ is called an eigenvalue of C if there is a nonzero vector $\underline{e}$ in R^n such that:

$$C\underline{e} = \lambda \underline{e}$$

The vector $\underline{e}$ is called an eigenvector of the matrix C corresponding to the eigenvalue λ.

Vector population

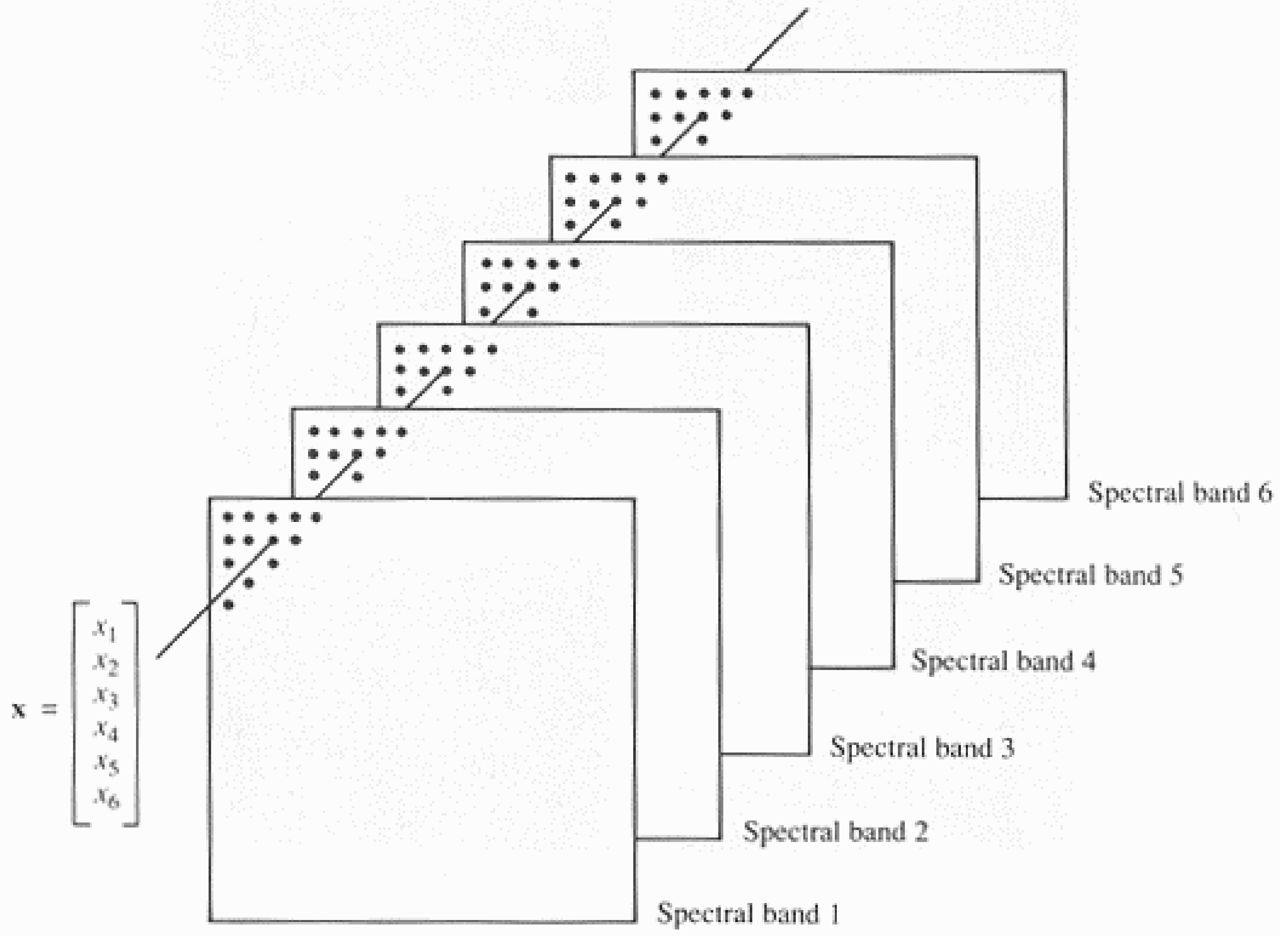
- Consider a population of random vectors of the following form:

$$\underline{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

- The quantity x_i may represent the value (grey level) of the image i.

- The population may arise from the formation of the above vectors for different image pixels.

Example: x vectors could be pixel values in several spectral bands (channels)



Formation of a vector from corresponding pixels in six images.

Mean and Covariance Matrix

- The mean vector of the population is defined as:

$$\underline{m}_x = E\{\underline{x}\} = [m_1 \quad m_2 \quad \ldots \quad m_n]^T = [E\{x_1\} \quad E\{x_2\} \quad \ldots \quad E\{x_n\}]^T$$

- The covariance matrix of the population is defined as:

$$C = E\left\{(\underline{x} - \underline{m}_x)(\underline{x} - \underline{m}_x)^T\right\}$$

- For M vectors of a random population, where M is large enough

$$\underline{m}_x = \frac{1}{M}\sum_{k=1}^{M} \underline{x}_k$$

Karhunen-Loeve Transform

- Let A be a matrix whose rows are formed from the eigenvectors of the covariance matrix C of the population.
- They are ordered so that the first row of A is the eigenvector corresponding to the largest eigenvalue, and the last row the eigenvector corresponding to the smallest eigenvalue.
- We define the following transform:

$$\underline{y} = A(\underline{x} - \underline{m}_x)$$

- It is called the Karhunen-Loeve transform.

Karhunen-Loeve Transform

- You can demonstrate very easily that:

$$E\{\underline{y}\} = 0$$

$$C_y = A C_x A^T$$

$$C_y = \begin{bmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \lambda_n \end{bmatrix}$$

Inverse Karhunen-Loeve Transform

- To reconstruct the original vectors $\underline{x}$ from its corresponding $\underline{y}$

$$A^{-1} = A^T$$

$$\underline{x} = A^T \underline{y} + \underline{m}_x$$

- We form a matrix A_K from the K eigenvectors which correspond to the K largest eigenvalues, yielding a transformation matrix of size $K \times n$.
- The $\underline{y}$ vectors would then be K dimensional.
- The reconstruction of the original vector $\hat{\underline{x}}$ is

$$\hat{\underline{x}} = A_K^T \underline{y} + \underline{m}_x$$

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Mean squared error of approximate reconstruction

- It can be proven that the mean square error between the perfect reconstruction $\underline{x}$ and the approximate reconstruction $\underline{\hat{x}}$ is given by the expression

$$e_{\text{ms}} = \left\| \underline{x} - \underline{\hat{x}} \right\|^2 = \sum_{j=1}^{n} \lambda_j - \sum_{j=1}^{K} \lambda_j = \sum_{j=K+1}^{n} \lambda_j$$

- By using A_K instead of A for the KL transform we can achieve compresssion of the available data.

Drawbacks of the KL Transform

Despite its favourable theoretical properties, the KLT is not used in practice for the following reasons.

- Its basis functions depend on the covariance matrix of the image, and hence they have to recomputed and transmitted for every image.

- Perfect decorrelation is not possible, since images can rarely be modelled as realisations of ergodic fields.

- There are no fast computational algorithms for its implementation.

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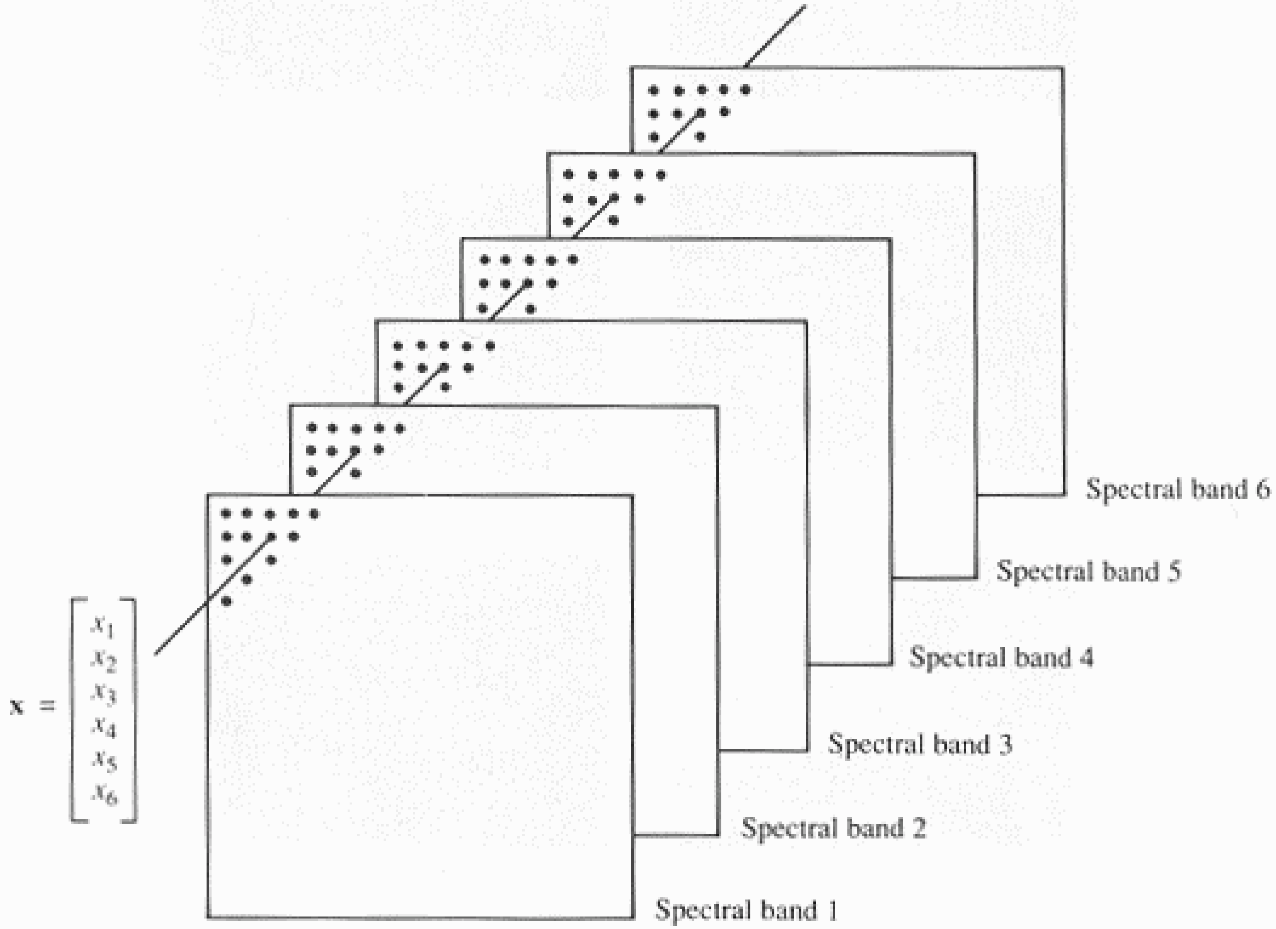


Formation of a vector from corresponding pixels in six images.

Example of the KLT: Original images

6 spectral images from an airborne Scanner.

Channel	Wavelength band (microns)
1	0.40–0.44
2	0.62–0.66
3	0.66–0.72
4	0.80–1.00
5	1.00–1.40
6	2.00–2.60

Channel 1

Channel 2

Channel 3

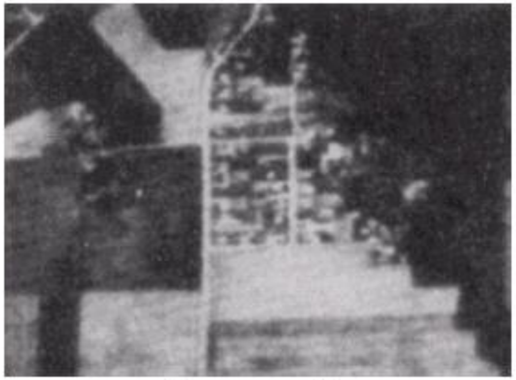
Channel 4

Channel 5

Channel 6

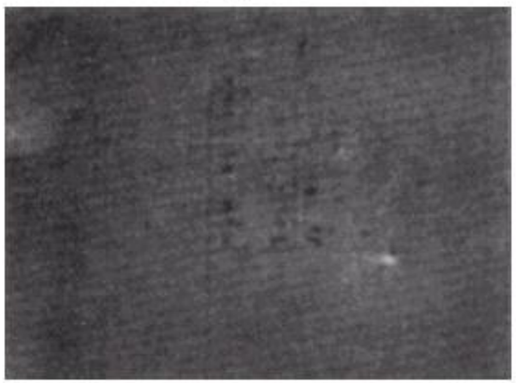
(Images from Rafael C. Gonzalez and Richard E. Wood, Digital Image Processing, 2nd Edition.

Example: Principal Components



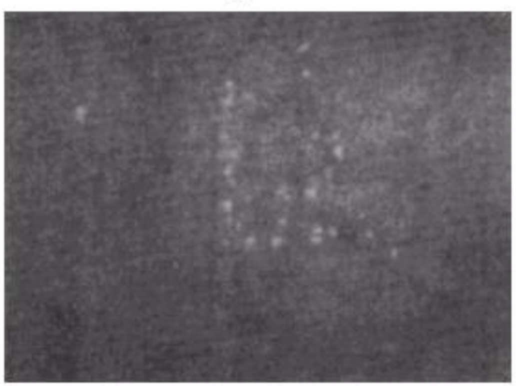
Component 1

Component 2

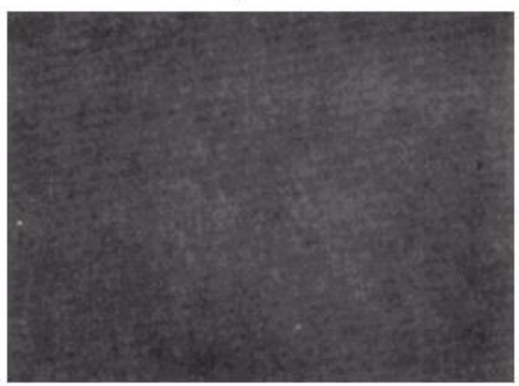


Component 3

Component 4



Component 5

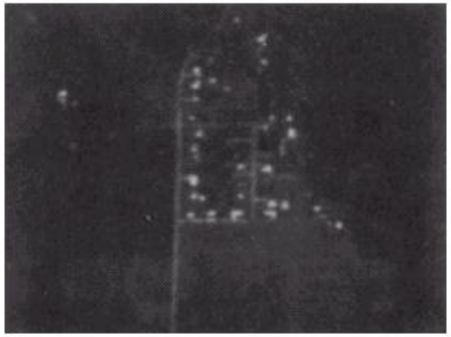


Component 6

Component	λ
1	3210
2	931.4
3	118.5
4	83.88
5	64.00
6	13.40

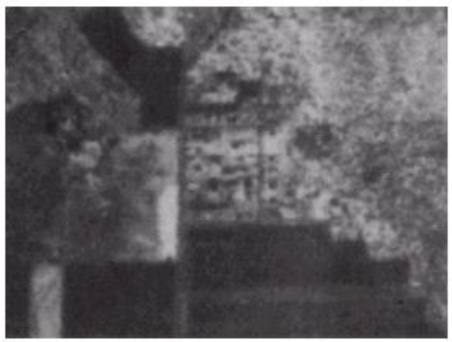
(Images from Rafael C. Gonzalez and Richard E. Wood, Digital Image Processing, 2nd Edition.

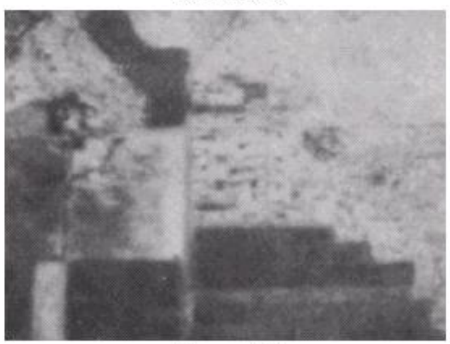
Example: Principal Components (cont.)

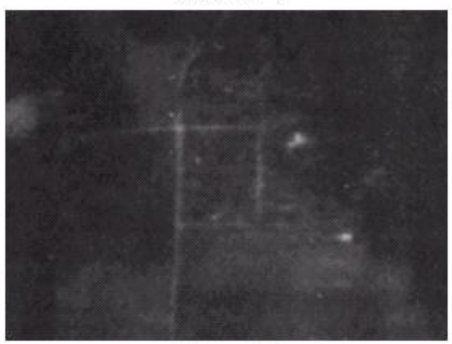

Channel 1


Channel 2

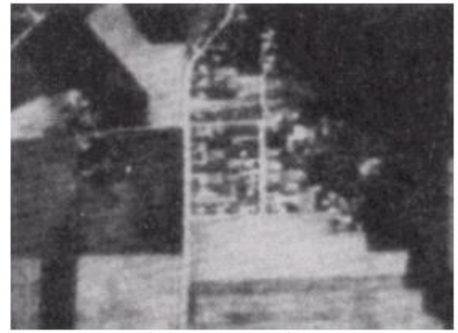

Channel 3


Channel 4

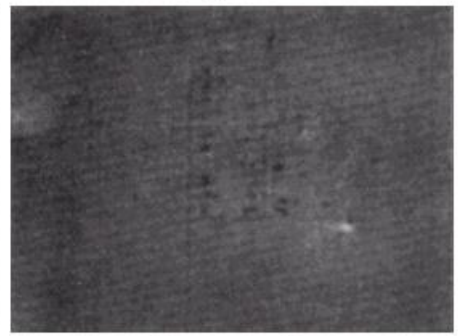

Channel 5

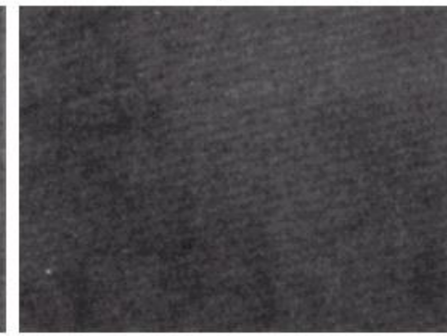

Channel 6

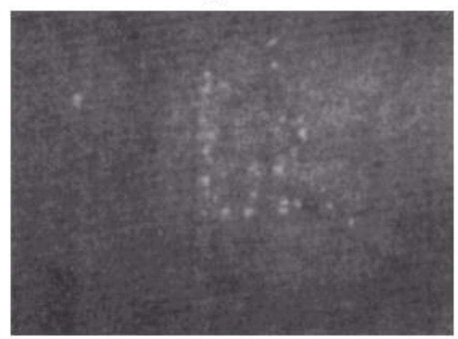
Original images (channels)


Component 1


Component 2


Component 3


Component 4


Component 5


Component 6

Six principal components after KL transform

Example: Original Images (left) and Principal Components (right)