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Digital Image Processing

Image Restoration – Inverse Filtering

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What is Image Restoration?

- Image Restoration refers to a class of methods that aim at reducing or removing various types of distortions of the image of interest. These can be:

  - **Distortion due to sensor noise**.
  - **Out-of-focus camera**.
  - **Motion blur**.
  - Weather conditions.
  - Scratches, holes, cracks caused by aging of the image.
  - Others.

Classification of restoration methods

- Deterministic or stochastic methods.
  - In deterministic methods, we work directly with the image values in either space or frequency domain.
  - In stochastic methods, we work with the statistical properties of the image of interest (autocorrelation function, covariance function, variance, mean etc.)

- Non-blind or semi-blind or blind methods
  - In non-blind methods the degradation process is known.
  - In semi-blind methods the degradation process is partly-known.
  - In blind methods the degradation process is unknown.

Classification of implementation types of restoration methods

- Direct methods

  The signals we are looking for (original “undistorted” image and degradation model) are finally obtained through a signle closed-form expression.

- Iterative methods

  The signals we are looking for are obtained through a mathematical procedure that generates a sequence of improving approximate solutions to the problem.

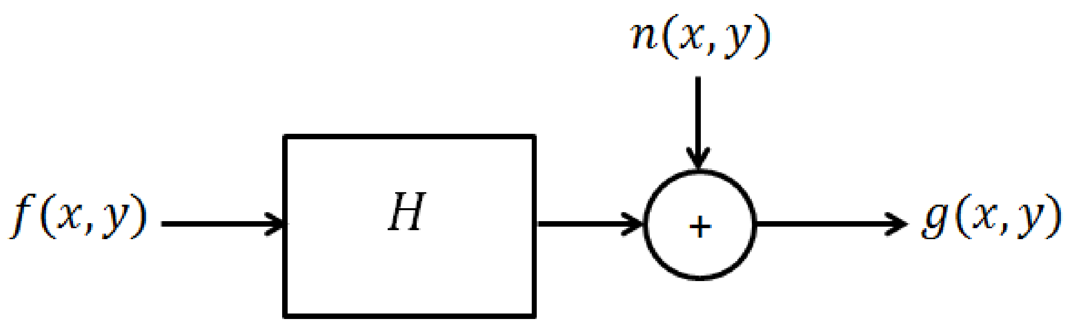
A generic model of an image restoration system

- A generic common model for an image restoration system is given by the following mathematical equation:

$$g(x,y) = H[f(x,y)] + n(x,y)$$

where

- (x,y) are the space coordinates
- $f(x,y)$ is the original (undistorted) image
- $H[\cdot]$ is a generic representation of the degradation function which is imposed onto the original image.
- $n(x,y)$ is a noise signal added to the distorted image.



Linear and space invariant (LSI) degradation model

- In the degradation model:

$$g(x,y) = H[f(x,y)] + n(x,y)$$

we are interested in the definitions of linearity and space-invariance.

- The degradation model is linear if

$$H[k_1 f_1(x,y) + k_2 f_2(x,y)] = k_1 H[f_1(x,y)] + k_2 H[f_2(x,y)]$$

- The degradation model is space or position invariant if

$$H[f(x - x_0, y - y_0)] = g(x - x_0, y - y_0)$$

- In the above definitions we ignore the presence of external noise.

- In real life scenarios, various types of degradations can be **approximated** by linear, space-invariant operators.

Advantage and drawback of LSI assumptions

- Advantage

  It is much easier to deal with linear and space-invariant models.

  - Mathematics are easier.
  - The distorted image is the convolution of the original image and the distortion model.
  - Software tools are available.

- Disadvantage

  For various realistic types of image degradations assumptions for linearity and space-invariance are too strict and significantly deviate from the true degradation model.

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Motion blur: A typical type of degradation

Types of motion blur

- 1-D uniform motion blur

$$h(x) = \begin{cases} \dfrac{1}{(2L+1)} & -L \leq x \leq L \\ 0 & \text{otherwise} \end{cases}$$

- 2-D uniform motion blur

$$h(x,y) = \begin{cases} \dfrac{1}{(2L+1)^2} & -L \leq x, y \leq L \\ 0 & \text{otherwise} \end{cases}$$

$$\frac{1}{9} \times \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \qquad \frac{1}{25} \times \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} \qquad \frac{1}{49} \times \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

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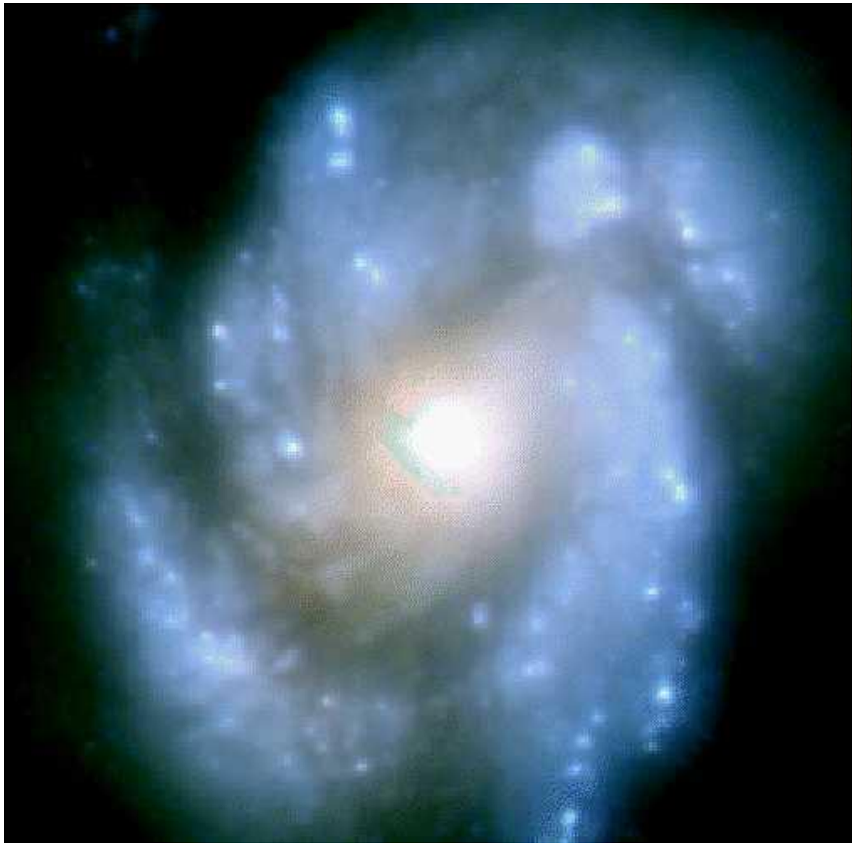
Atmospheric turbulence: A typical type of degradation



Typical model for atmospheric turbulence

- Atmospheric turbulence $h(x,y) = Kexp\left(-\frac{x^2+y^2}{2\sigma^2}\right)$

negligible distortion $\Rightarrow$

$\Leftarrow k = 0.0025$

$k = 0.001 \Rightarrow$

$\Leftarrow k = 0.00025$

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Uniform out-of-focus blur: A typical type of degradation

- Uniform out-of-focus blur: $h(x,y) = \begin{cases} \frac{1}{\pi R^2} & \sqrt{x^2+y^2} \le R \\ 0 & \text{otherwise} \end{cases}$
- Note that the model is defined within a circular disc.

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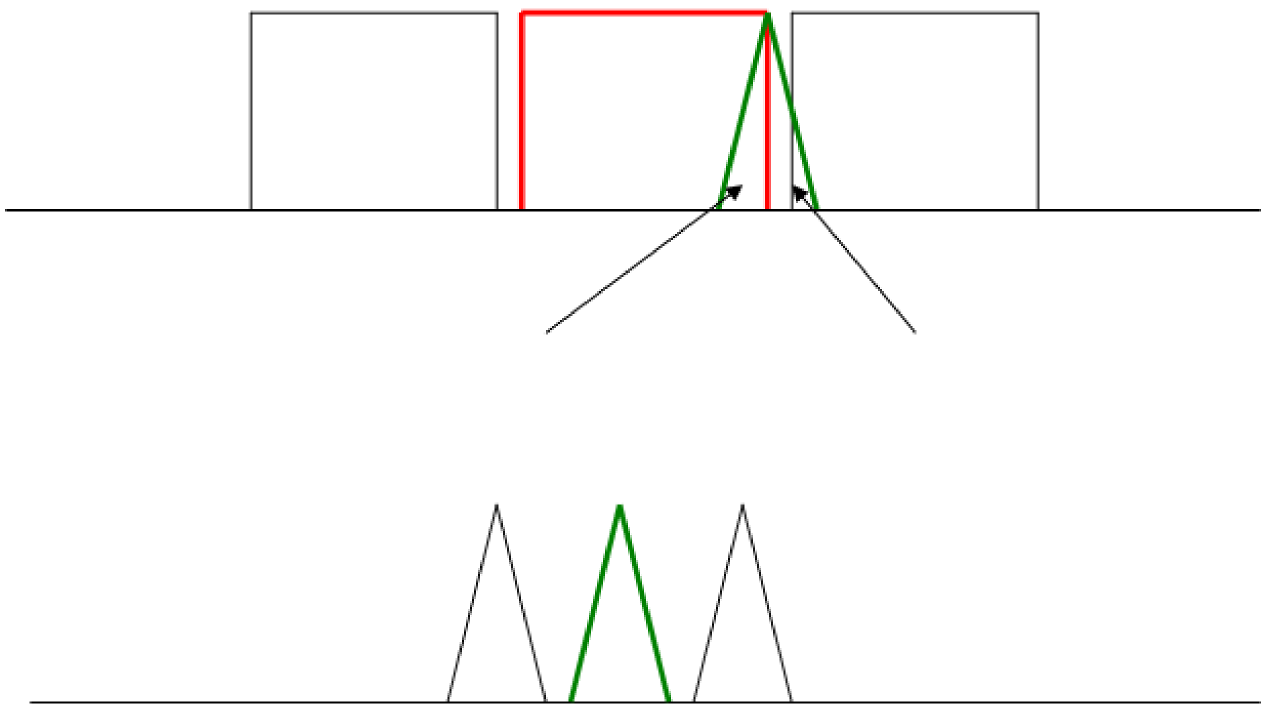
Periodic extension of images and degradation model

- In image restoration we often work with Discrete Fourier Transforms.
- DFT assumes periodicity of the signal in time or space.
- Therefore, periodic extension of signals is required.
- Distorted image is the convolution of the original image and the distortion model. We are able to assume this because of the linearity and space invariance assumptions!
- Convolution increases the size of signals.
- Periodic extension must take into consideration the presence of convolution: zero-padding is required!
- Every signal involved in an image restoration system must be extended by zero-padding and also treated as virtually periodic.

Wrong periodic extension of signals.

Red and green signal are convolved



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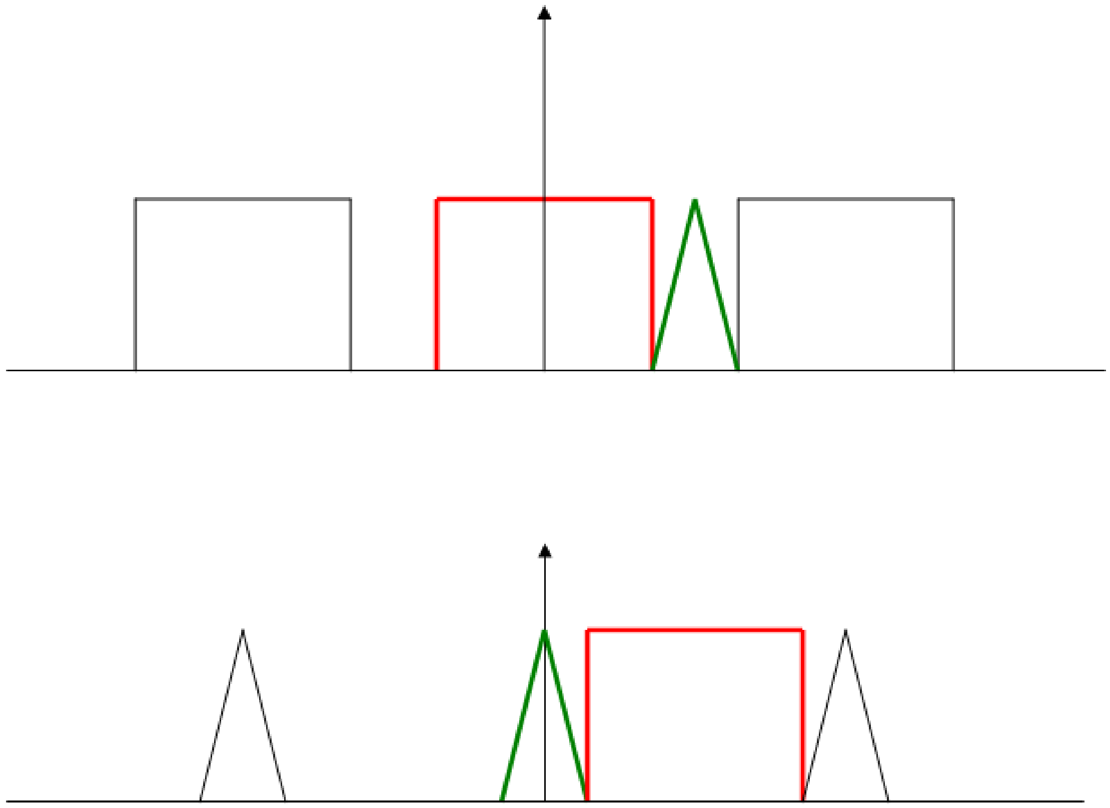
Correct periodic extension of signals

Red and green signal are convolved



Correct periodic extension of images and degradation model

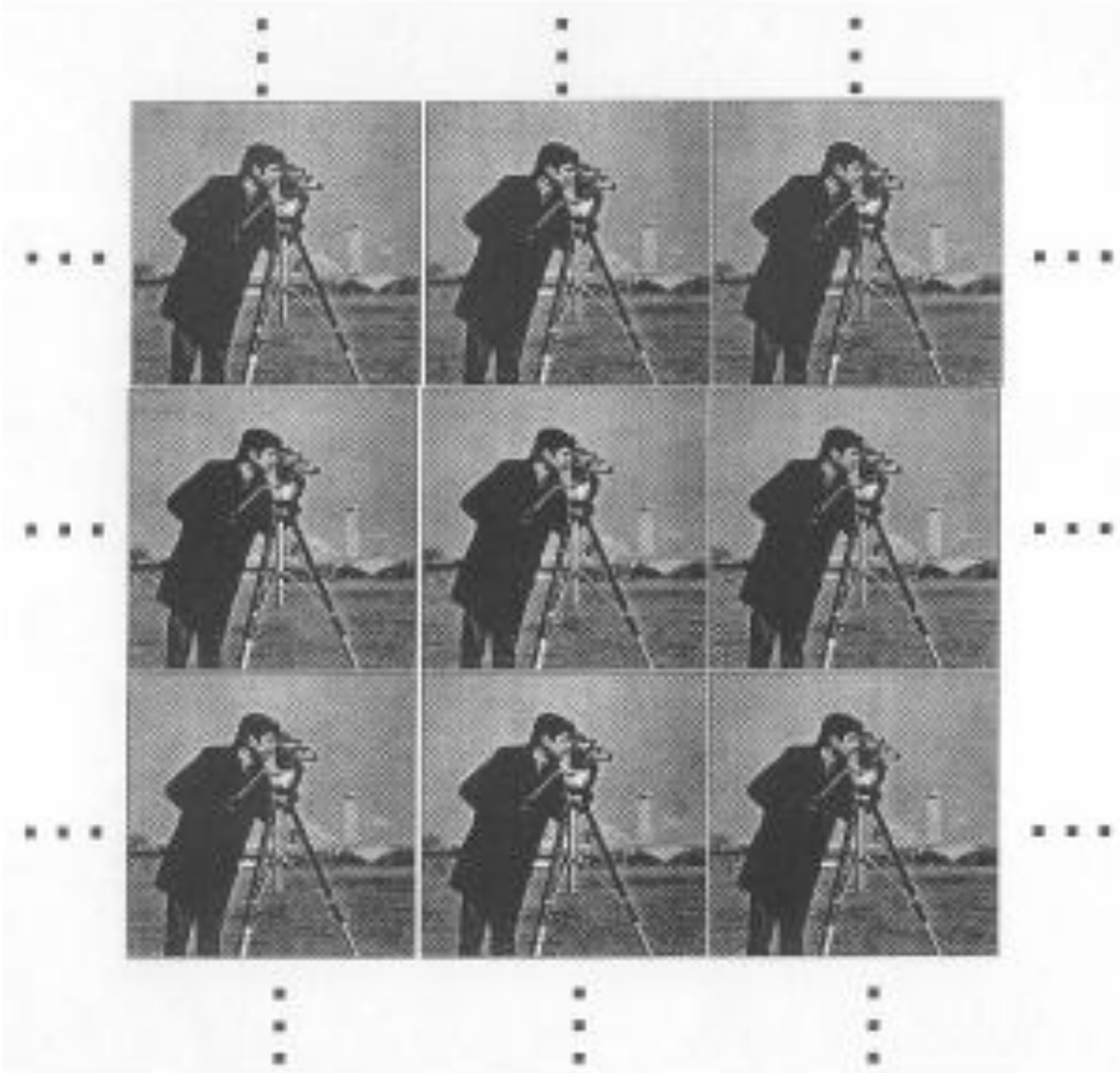
- The original image $f(x,y)$ is of size $A \times B$.
- The degradation model $h(x,y)$ is of size $C \times D$.
- We form the extended versions of $f(x,y)$ and $h(x,y)$ by zero padding, both of size $M \times N$ with

$$M \geq A + C - 1$$
$$N \geq B + D - 1$$

  both periodic with period $M \times N$.

- **Example**
  - Image 256x256
  - Degradation 3x3
  - With extension by zero padding both images have dimension at least (256+3-1)x(256+3-1)=258x258.
  - They are also assume to be periodic.

Correct periodic extension of images and degradation model



Inverse filtering for image restoration

- Inverse filtering is a deterministic and direct method for image restoration.
- The images involved must be lexicographically ordered. That means that an image is converted to a column vector by pasting the rows one by one after converting them to columns.
- An image of size 256×256 is converted to a column vector of size 65536×1.
- The degradation model is written in a matrix form, where the images are vectors and the degradation process is a **huge but sparse** matrix.

$$\mathbf{g} = \mathbf{H}\mathbf{f}$$

- The above relationship is ideal. What really happens is $\mathbf{g} = \mathbf{H}\mathbf{f} + \mathbf{n}$!

Inverse filtering for image restoration

- In this problem we know $\mathbf{H}$ and $\mathbf{g}$ and we are looking for a descent $\mathbf{f}$.

- The problem is formulated as follows:

  We are looking to minimize the Euclidian norm of the error, i.e.,

$$\|\mathbf{n}\|^2 = \|\mathbf{g} - \mathbf{Hf}\|^2$$

- The first derivative of the minimization function must be set to zero.

$$\|\mathbf{g} - \mathbf{Hf}\|^2 = (\mathbf{g} - \mathbf{Hf})^T(\mathbf{g} - \mathbf{Hf}) = \left(\mathbf{g}^T - \mathbf{f}^T\,\mathbf{H}^T\right)(\mathbf{g} - \mathbf{Hf}) =$$

$$\mathbf{g}^T\mathbf{g} - \mathbf{g}^T\,\mathbf{Hf} - \mathbf{f}^T\mathbf{H}^T\,\mathbf{g} + \mathbf{f}^T\mathbf{H}^T\,\mathbf{Hf}$$

$$\frac{\partial\|\mathbf{g} - \mathbf{Hf}\|^2}{\partial\mathbf{f}} = -2\mathbf{H}^T\,\mathbf{g} + 2\mathbf{H}^T\,\mathbf{Hf} = \mathbf{0} \Rightarrow \mathbf{H}^T\,\mathbf{Hf} = \mathbf{H}^T\,\mathbf{g}$$

$$\mathbf{H}^T\,\mathbf{Hf} = \mathbf{H}^T\,\mathbf{g}$$

$$\mathbf{f} = \left(\mathbf{H}^T\,\mathbf{H}\right)^{-1}\mathbf{H}^T\,\mathbf{g}$$

- If $\mathbf{H}$ is a square matrix and its inverse exists then $\mathbf{f} = \mathbf{H}^{-1}\mathbf{g}$

Inverse filtering for image restoration in frequency domain

- We have that

$$\mathbf{H}^T\,\mathbf{H}\mathbf{f} = \mathbf{H}^T\,\mathbf{g}$$

- If we take the DFT of the above relationship in both sides we have:

$$|H(u,v)|^2 F(u,v) = H(u,v)^* G(u,v)$$

$$F(u,v) = \frac{H(u,v)^*}{|H(u,v)|^2} G(u,v)$$

$$F(u,v) = \frac{G(u,v)}{H(u,v)}$$

- Note that the most popular types of degradations are low pass filters (out-of-focus blur, motion blur).

Inverse filtering for noise-free scenarios

- We have that

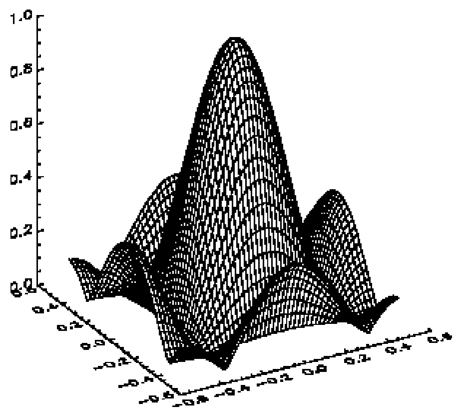
$$F(u,v) = \frac{G(u,v)}{H(u,v)}$$

- **Problem:** It is very likely that $H(u,v)$ is 0 or very small at certain frequency pairs.

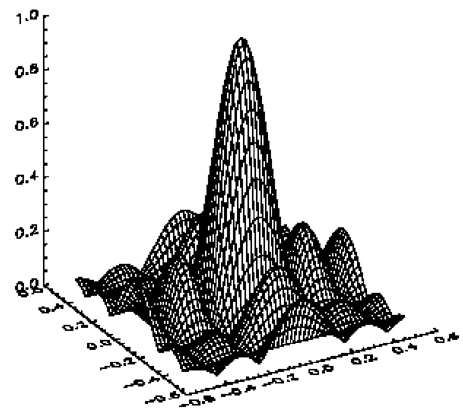
- For example, $H(u,v)$ could be a $sinc$ function.

- In general, since $H(u,v)$ is a low pass filter, it is very likely that its values drop off rapidly as the distance of (u,v) from the origin $(0,0)$ increases.

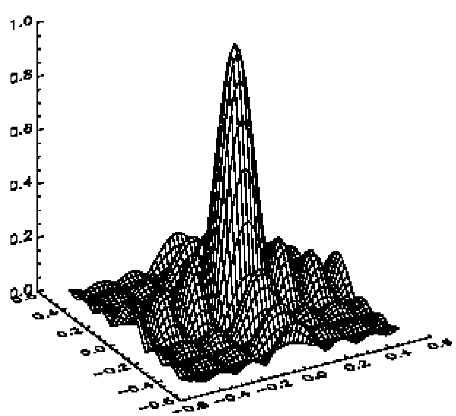
Frequency responses of various image degradation functions



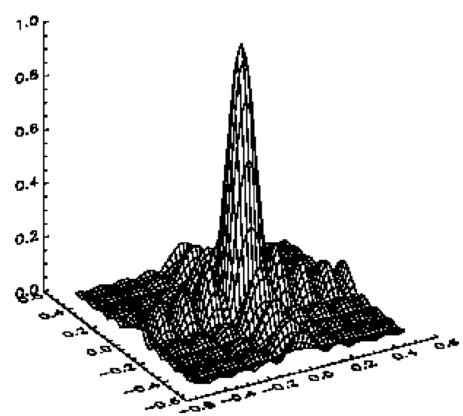
Frequency Response for M=3



Frequency Response for M=5



Frequency Response for M=7



Frequency Response for M=9

Inverse filtering for noisy scenarios

- We have that

$$F(u,v) = \frac{G(u,v) - N(u,v)}{H(u,v)} = \frac{G(u,v)}{H(u,v)} - \frac{N(u,v)}{H(u,v)}$$

- **Problem:** It is definite that while $H(u,v)$ is 0 or very small at certain frequency pairs, $N(u,v)$ is large.

- Note that $H(u,v)$ is a low pass filter, whereas $N(u,v)$ is an all pass function. Therefore, the term $\frac{N(u,v)}{H(u,v)}$ can be huge!

- Inverse filtering fails in that case ☹

Pseudo-inverse filtering

- Instead of the conventional inverse filter, we implement one of the following:

$$F(u,v) = \begin{cases} \dfrac{G(u,v)}{H(u,v)} & H(u,v) \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

$$F(u,v) = \begin{cases} \dfrac{G(u,v)}{H(u,v)} & |H(u,v)| \geq \epsilon \\ 0 & \text{otherwise} \end{cases}$$

- The parameter ϵ (called **threshold** in the figures in the next slides) is a small number chosen by the user.

- This filter is called **pseudo-inverse** or **generalized** inverse filter.

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Pseudo-inverse filtering with different thresholds





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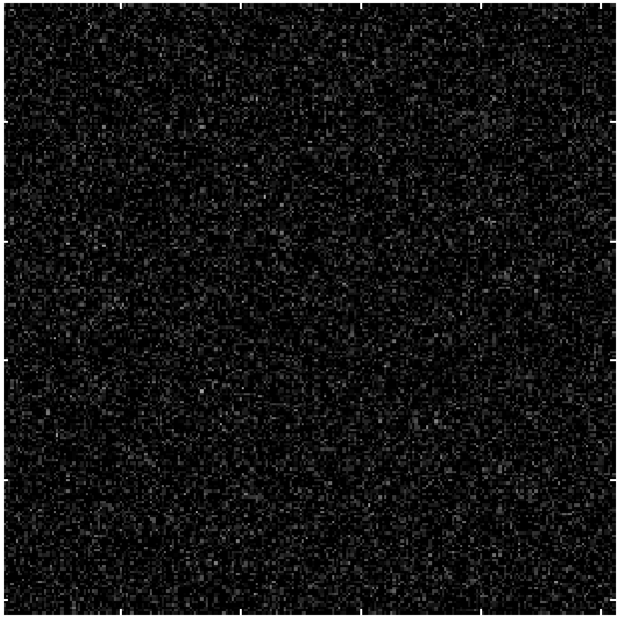
Pseudo-inverse filtering in the case of noise



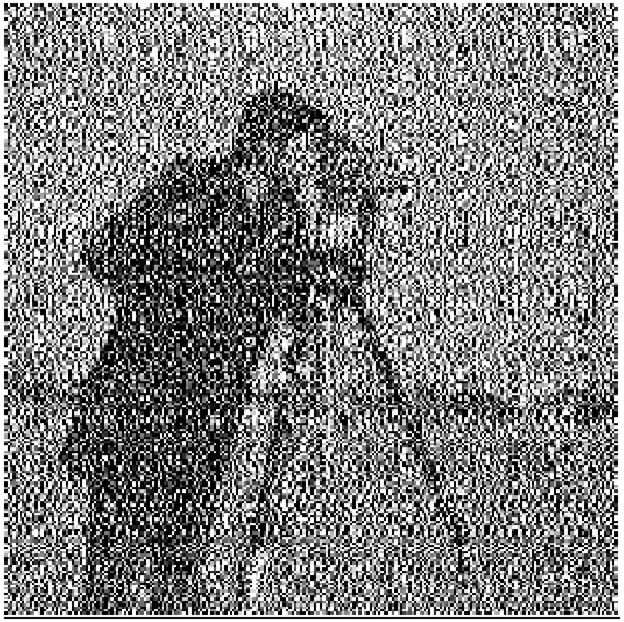


Figure 3: Degraded by a 7×7 pill-box blur, 20 dB BSNR

Figure 5: Degraded by a 5×5 Gaussian blur ($\sigma^2 = 1$), 20 dB BSNR

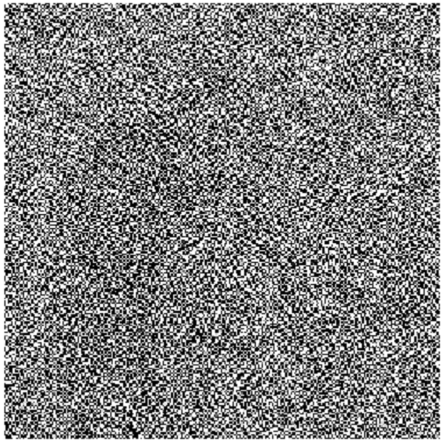


Figure 11: Result of Figure 3 restored by a generalized inverse filter with a threshold of 10^{-3}, ISNR $= -32.9$ dB

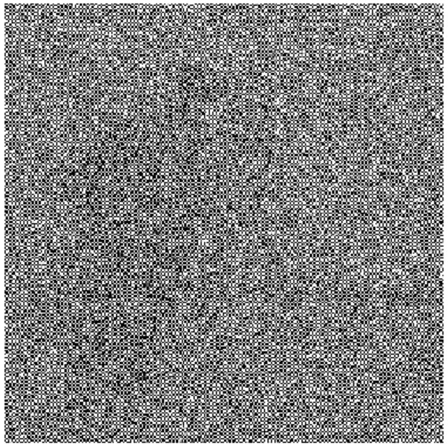


Figure 17: Result of Figure 5 restored by a generalized inverse filter with a threshold of 10^{-3}, ISNR $= -36.6$ dB

Figure 3: Degraded by a 7×7 pill-box blur, 20 dB BSNR

Figure 5: Degraded by a 5×5 Gaussian blur ($\sigma^2 = 1$), 20 dB BSNR

Figure 13: Result of Figure 3 restored by a generalized inverse filter with a threshold of 10^{-1}, ISNR $= 0.61$ dB

Figure 19: Result of Figure 5 restored by a generalized inverse filter with a threshold of 10^{-1}, ISNR $= -1.8$ dB