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Digital Image Processing

Image Restoration – Inverse Filtering

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What is Image Restoration?

- Image Restoration refers to a class of methods that aim at reducing or removing various types of distortions of the image of interest. These can be:

  - **Distortion due to sensor noise**.
  - **Out-of-focus camera**.
  - **Motion blur**.
  - Weather conditions.
  - Scratches, holes, cracks caused by aging of the image.
  - Others.

Classification of restoration methods

- Deterministic or stochastic methods.
  - In deterministic methods, we work directly with the image values in either space or frequency domain.
  - In stochastic methods, we work with the statistical properties of the image of interest (autocorrelation function, covariance function, variance, mean etc.)

- Non-blind or semi-blind or blind methods
  - In non-blind methods the degradation process is known.
  - In semi-blind methods the degradation process is partly-known.
  - In blind methods the degradation process is unknown.

Classification of implementation types of restoration methods

- Direct methods

  The signals we are looking for (original “undistorted” image and degradation model) are finally obtained through a signle closed-form expression.

- Iterative methods

  The signals we are looking for are obtained through a mathematical procedure that generates a sequence of improving approximate solutions to the problem.

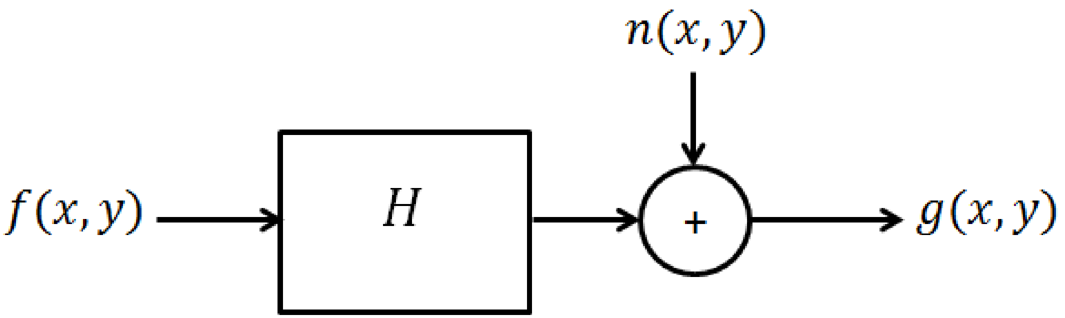
A generic model of an image restoration system

- A generic common model for an image restoration system is given by the following mathematical equation:

$$g(x,y) = H[f(x,y)] + n(x,y)$$

where

- (x,y) are the space coordinates
- $f(x,y)$ is the original (undistorted) image
- $H[\cdot]$ is a generic representation of the degradation function which is imposed onto the original image.
- $n(x,y)$ is a noise signal added to the distorted image.



Linear and space invariant (LSI) degradation model

- In the degradation model:

$$g(x,y) = H[f(x,y)] + n(x,y)$$

we are interested in the definitions of linearity and space-invariance.

- The degradation model is linear if

$$H[k_1 f_1(x,y) + k_2 f_2(x,y)] = k_1 H[f_1(x,y)] + k_2 H[f_2(x,y)]$$

- The degradation model is space or position invariant if

$$H[f(x-x_0, y-y_0)] = g(x-x_0, y-y_0)$$

- In the above definitions we ignore the presence of external noise.

- In real life scenarios, various types of degradations can be **approximated** by linear, space-invariant operators.

Advantage and drawback of LSI assumptions

- Advantage

  It is much easier to deal with linear and space-invariant models.

  - Mathematics are easier.
  - The distorted image is the convolution of the original image and the distortion model.
  - Software tools are available.

- Disadvantage

  For various realistic types of image degradations assumptions for linearity and space-invariance are too strict and significantly deviate from the true degradation model.

Motion blur: A typical type of degradation

Types of motion blur

- 1-D uniform motion blur

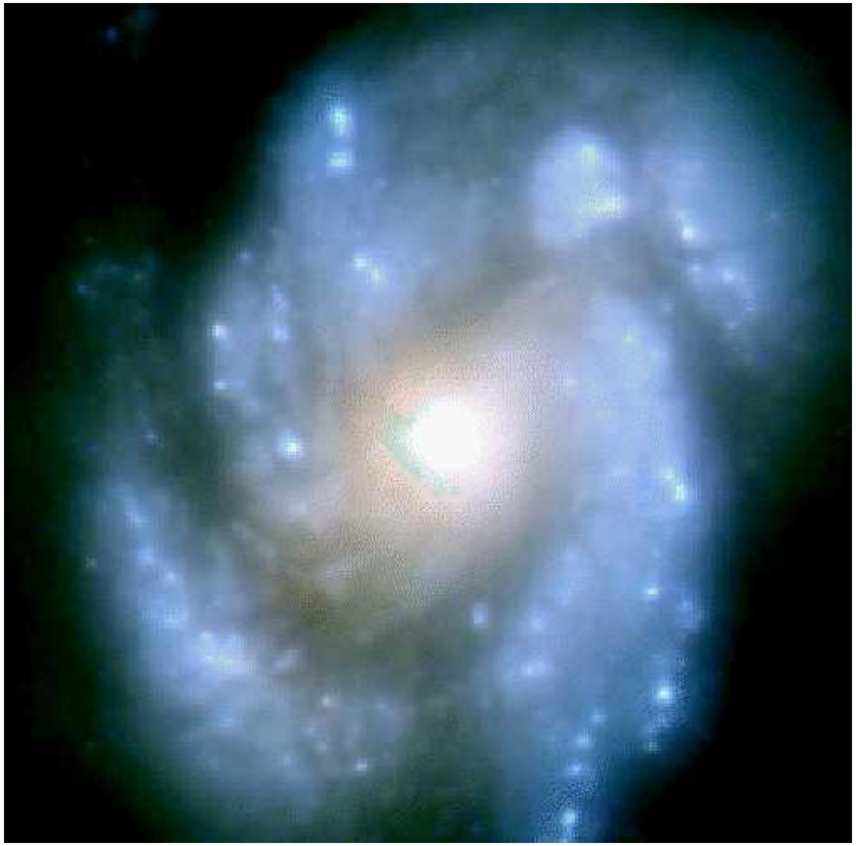
$$h(x) = \begin{cases} \dfrac{1}{(2L+1)} & -L \le x \le L \\ 0 & \text{otherwise} \end{cases}$$

- 2-D uniform motion blur

$$h(x,y) = \begin{cases} \dfrac{1}{(2L+1)^2} & -L \le x, y \le L \\ 0 & \text{otherwise} \end{cases}$$

$$\frac{1}{9} \times \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \qquad \frac{1}{25} \times \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} \qquad \frac{1}{49} \times \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

Atmospheric turbulence: A typical type of degradation



Typical model for atmospheric turbulence

- Atmospheric turbulence $h(x,y) = Kexp\left(-\frac{x^2+y^2}{2\sigma^2}\right)$

negligible distortion $\Rightarrow$

$\Leftarrow k = 0.0025$

$k = 0.001 \Rightarrow$

$\Leftarrow k = 0.00025$

Uniform out-of-focus blur: A typical type of degradation

- Uniform out-of-focus blur: $h(x,y) = \begin{cases} \frac{1}{\pi R^2} & \sqrt{x^2+y^2} \le R \\ 0 & \text{otherwise} \end{cases}$
- Note that the model is defined within a circular disc.

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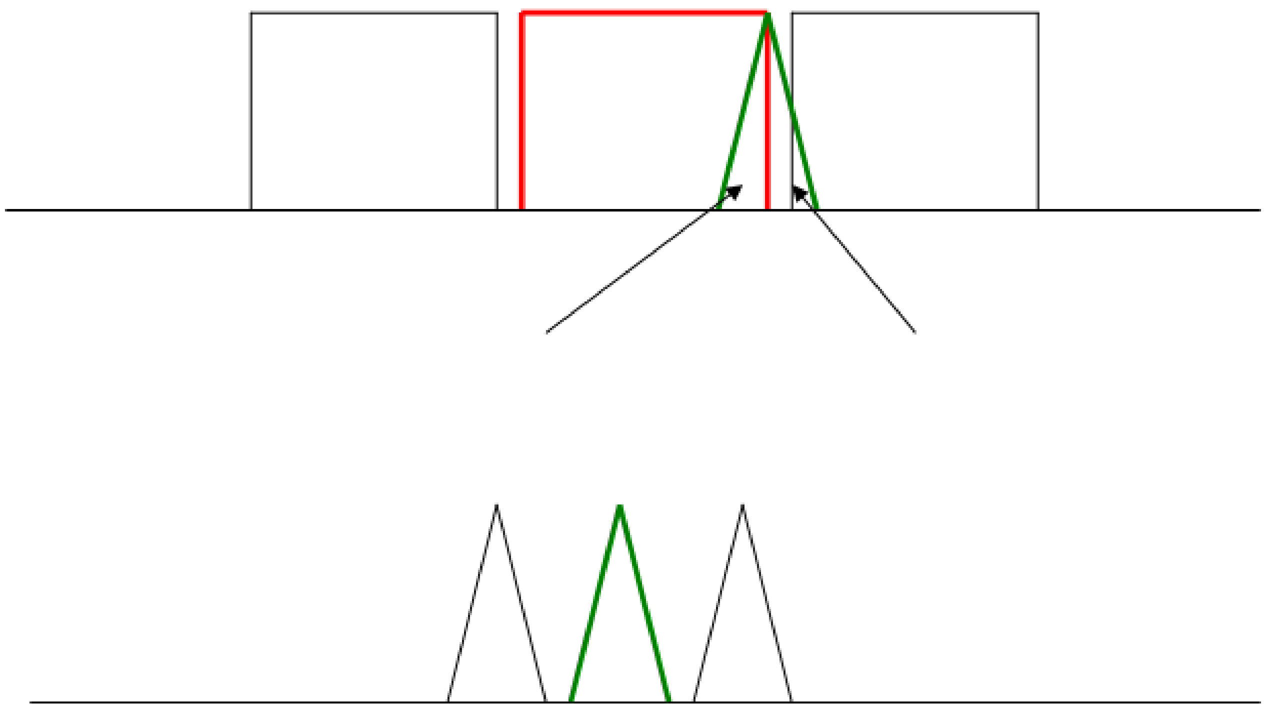
Periodic extension of images and degradation model

- In image restoration we often work with Discrete Fourier Transforms.
- DFT assumes periodicity of the signal in time or space.
- Therefore, periodic extension of signals is required.
- Distorted image is the convolution of the original image and the distortion model. We are able to assume this because of the linearity and space invariance assumptions!
- Convolution increases the size of signals.
- Periodic extension must take into consideration the presence of convolution: zero-padding is required!
- Every signal involved in an image restoration system must be extended by zero-padding and also treated as virtually periodic.

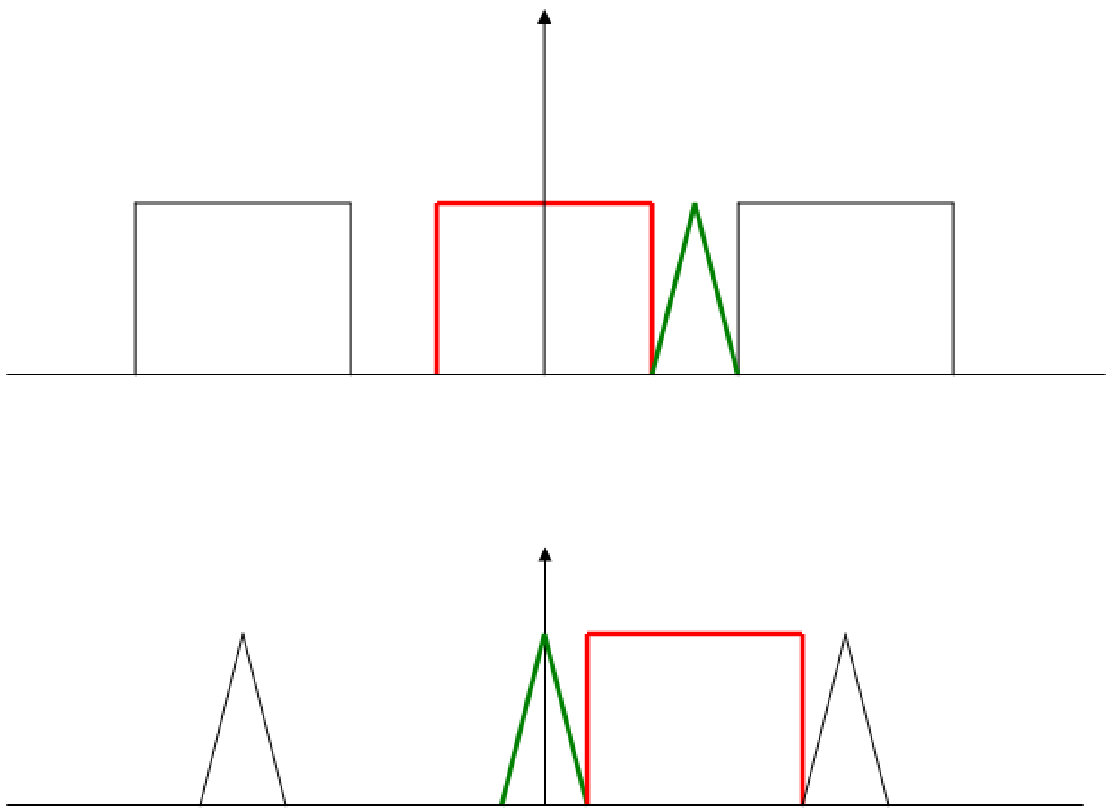
Wrong periodic extension of signals.

Red and green signal are convolved



Correct periodic extension of signals

Red and green signal are convolved



Correct periodic extension of images and degradation model

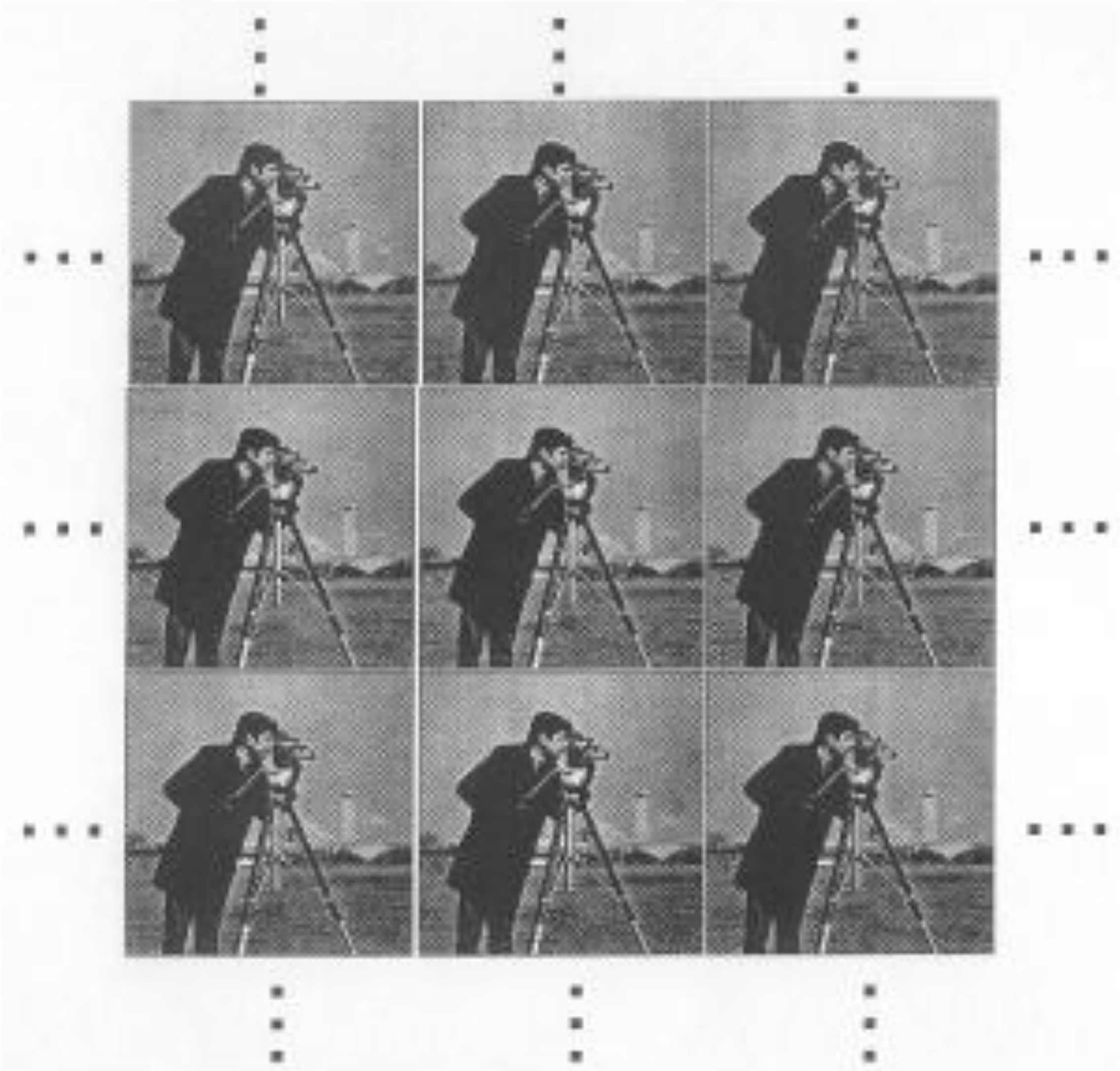
- The original image $f(x,y)$ is of size $A \times B$.
- The degradation model $h(x,y)$ is of size $C \times D$.
- We form the extended versions of $f(x,y)$ and $h(x,y)$ by zero padding, both of size $M \times N$ with

$$M \geq A + C - 1$$
$$N \geq B + D - 1$$

both periodic with period $M \times N$.

- **Example**
  - Image 256x256
  - Degradation 3x3
  - With extension by zero padding both images have dimension at least (256+3-1)x(256+3-1)=258x258.
  - They are also assume to be periodic.

Correct periodic extension of images and degradation model



Inverse filtering for image restoration

- Inverse filtering is a deterministic and direct method for image restoration.

- The images involved must be lexicographically ordered. That means that an image is converted to a column vector by pasting the rows one by one after converting them to columns.

- An image of size 256×256 is converted to a column vector of size 65536×1.

- The degradation model is written in a matrix form, where the images are vectors and the degradation process is a **huge but sparse** matrix.

$$\mathbf{g} = \mathbf{H}\mathbf{f}$$

- The above relationship is ideal. What really happens is $\mathbf{g} = \mathbf{H}\mathbf{f} + \mathbf{n}$!

Inverse filtering for image restoration

- In this problem we know $\mathbf{H}$ and $\mathbf{g}$ and we are looking for a descent $\mathbf{f}$.

- The problem is formulated as follows:

  We are looking to minimize the Euclidian norm of the error, i.e.,

$$\|\mathbf{n}\|^2 = \|\mathbf{g} - \mathbf{Hf}\|^2$$

- The first derivative of the minimization function must be set to zero.

$$\|\mathbf{g} - \mathbf{Hf}\|^2 = (\mathbf{g} - \mathbf{Hf})^T(\mathbf{g} - \mathbf{Hf}) = \left(\mathbf{g}^T - \mathbf{f}^T\,\mathbf{H}^T\right)(\mathbf{g} - \mathbf{Hf}) =$$

$$\mathbf{g}^T\mathbf{g} - \mathbf{g}^T\,\mathbf{Hf} - \mathbf{f}^T\mathbf{H}^T\,\mathbf{g} + \mathbf{f}^T\mathbf{H}^T\,\mathbf{Hf}$$

$$\frac{\partial\|\mathbf{g}-\mathbf{Hf}\|^2}{\partial\mathbf{f}} = -2\mathbf{H}^T\,\mathbf{g} + 2\mathbf{H}^T\,\mathbf{Hf} = \mathbf{0} \Rightarrow \mathbf{H}^T\,\mathbf{Hf} = \mathbf{H}^T\,\mathbf{g}$$

$$\mathbf{H}^T\,\mathbf{Hf} = \mathbf{H}^T\,\mathbf{g}$$

$$\mathbf{f} = \left(\mathbf{H}^T\,\mathbf{H}\right)^{-1}\mathbf{H}^T\,\mathbf{g}$$

- If $\mathbf{H}$ is a square matrix and its inverse exists then $\mathbf{f} = \mathbf{H}^{-1}\mathbf{g}$

Inverse filtering for image restoration in frequency domain

- We have that

$$\mathbf{H}^{T}\,\mathbf{H}\mathbf{f} = \mathbf{H}^{T}\,\mathbf{g}$$

- If we take the DFT of the above relationship in both sides we have:

$$|H(u,v)|^2 F(u,v) = H(u,v)^* G(u,v)$$

$$F(u,v) = \frac{H(u,v)^*}{|H(u,v)|^2} G(u,v)$$

$$F(u,v) = \frac{G(u,v)}{H(u,v)}$$

- Note that the most popular types of degradations are low pass filters (out-of-focus blur, motion blur).

Inverse filtering for noise-free scenarios

- We have that

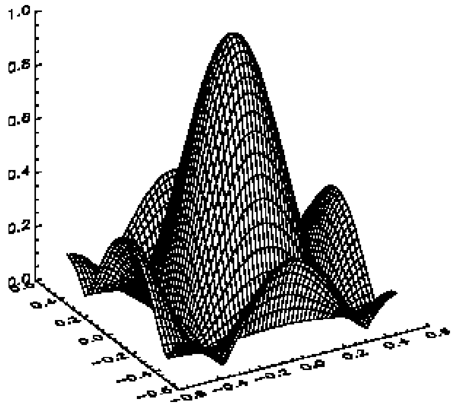
$$F(u,v) = \frac{G(u,v)}{H(u,v)}$$

- **Problem:** It is very likely that $H(u,v)$ is 0 or very small at certain frequency pairs.

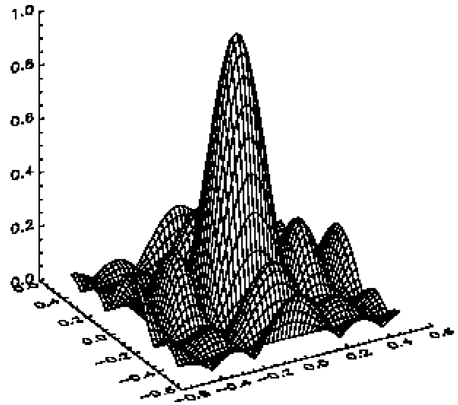
- For example, $H(u,v)$ could be a $sinc$ function.

- In general, since $H(u,v)$ is a low pass filter, it is very likely that its values drop off rapidly as the distance of (u,v) from the origin $(0,0)$ increases.

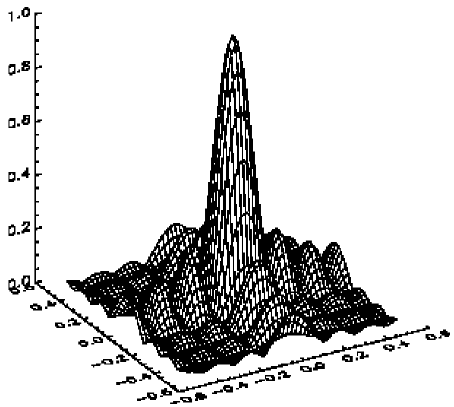
Frequency responses of various image degradation functions



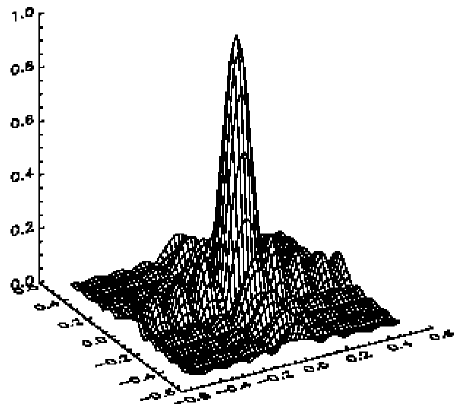
Frequency Response for M=3



Frequency Response for M=5



Frequency Response for M=7



Frequency Response for M=9

Inverse filtering for noisy scenarios

- We have that

$$F(u,v) = \frac{G(u,v) - N(u,v)}{H(u,v)} = \frac{G(u,v)}{H(u,v)} - \frac{N(u,v)}{H(u,v)}$$

- **Problem:** It is definite that while $H(u,v)$ is 0 or very small at certain frequency pairs, $N(u,v)$ is large.

- Note that $H(u,v)$ is a low pass filter, whereas $N(u,v)$ is an all pass function. Therefore, the term $\frac{N(u,v)}{H(u,v)}$ can be huge!

- Inverse filtering fails in that case ☹

Pseudo-inverse filtering

- Instead of the conventional inverse filter, we implement one of the following:

$$F(u,v) = \begin{cases} \dfrac{G(u,v)}{H(u,v)} & H(u,v) \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

$$F(u,v) = \begin{cases} \dfrac{G(u,v)}{H(u,v)} & |H(u,v)| \geq \epsilon \\ 0 & \text{otherwise} \end{cases}$$

- The parameter ϵ (called **threshold** in the figures in the next slides) is a small number chosen by the user.

- This filter is called **pseudo-inverse** or **generalized** inverse filter.

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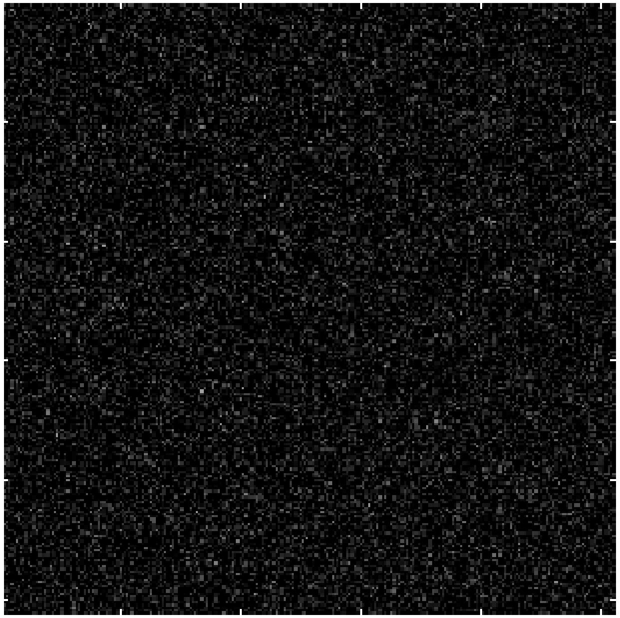
Pseudo-inverse filtering with different thresholds

Pseudo-inverse filtering in the case of noise



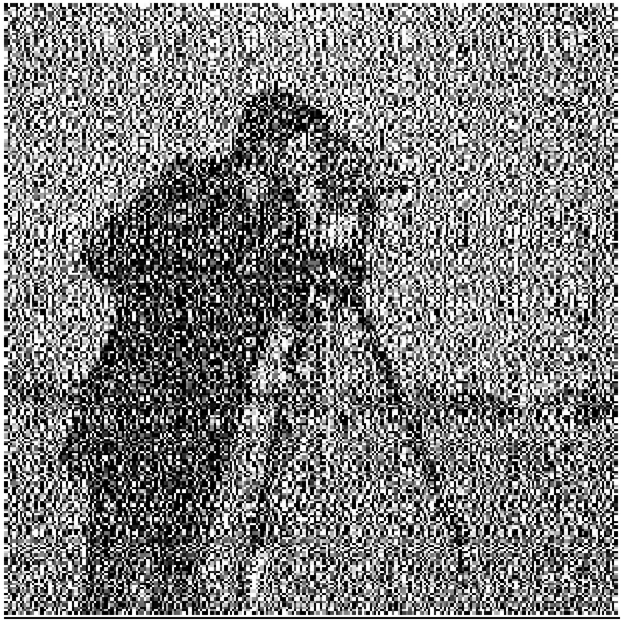


Figure 3: Degraded by a 7×7 pill-box blur, 20 dB BSNR

Figure 5: Degraded by a 5×5 Gaussian blur ($\sigma^2 = 1$), 20 dB BSNR

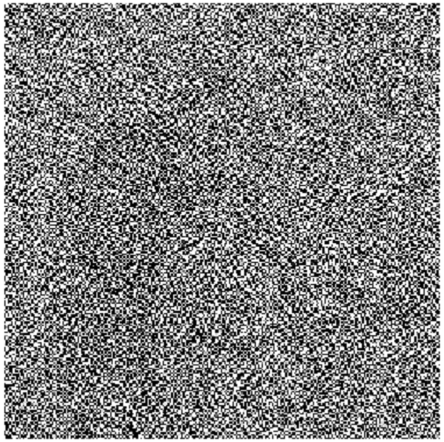


Figure 11: Result of Figure 3 restored by a generalized inverse filter with a threshold of 10^{-3}, ISNR = -32.9 dB

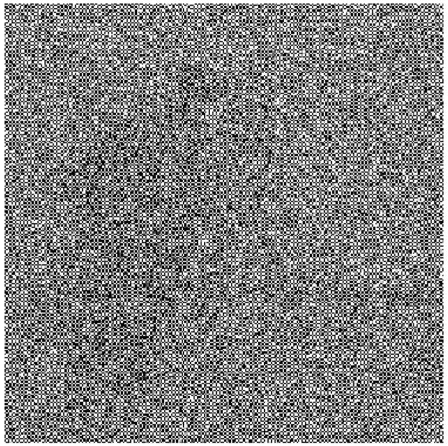


Figure 17: Result of Figure 5 restored by a generalized inverse filter with a threshold of 10^{-3}, ISNR = -36.6 dB

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Figure 3: Degraded by a 7×7 pill-box blur, 20 dB BSNR

Figure 5: Degraded by a 5×5 Gaussian blur ($\sigma^2 = 1$), 20 dB BSNR

Figure 13: Result of Figure 3 restored by a generalized inverse filter with a threshold of 10^{-1}, ISNR = 0.61 dB

Figure 19: Result of Figure 5 restored by a generalized inverse filter with a threshold of 10^{-1}, ISNR = -1.8 dB