SHORT REVISION

Start with a Discrete Time signal

Linear Convolution ⇔ Discrete Time Fourier Transform (DTFT)

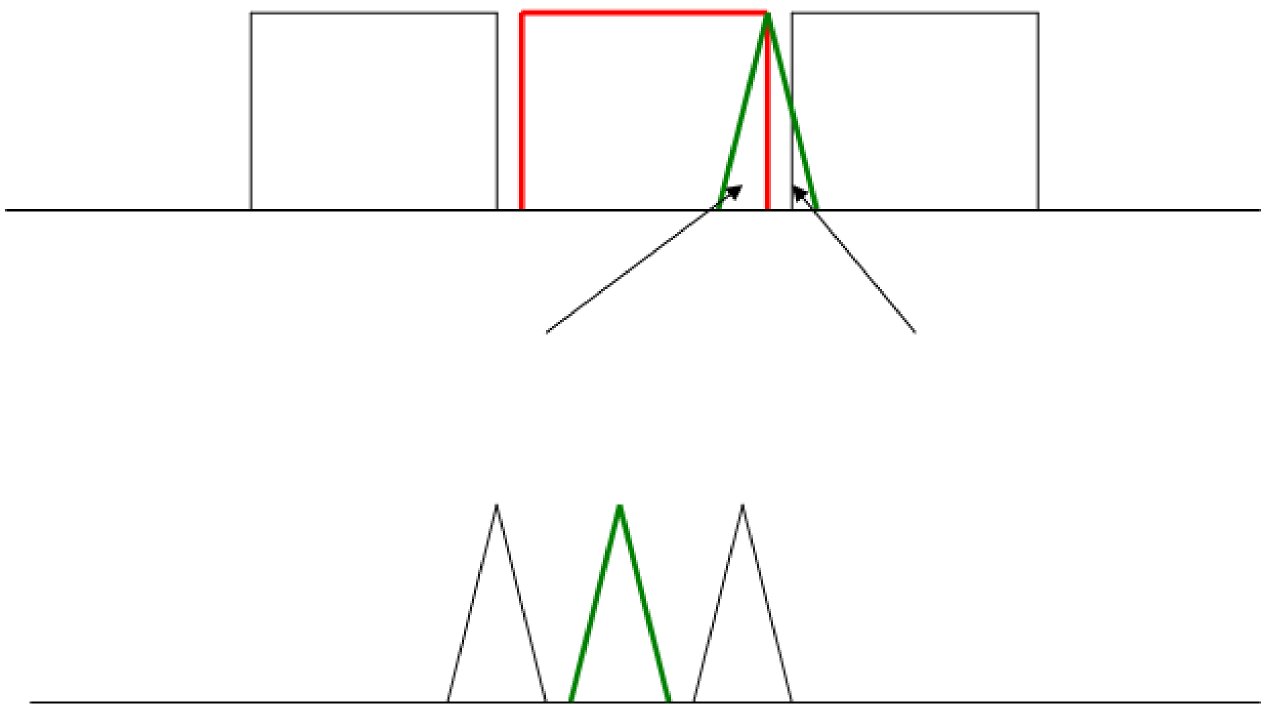
Circular Convolution ⇔ Discrete Fourier Transform (DFT)

periodic in frequency $\Longrightarrow$ periodic in time

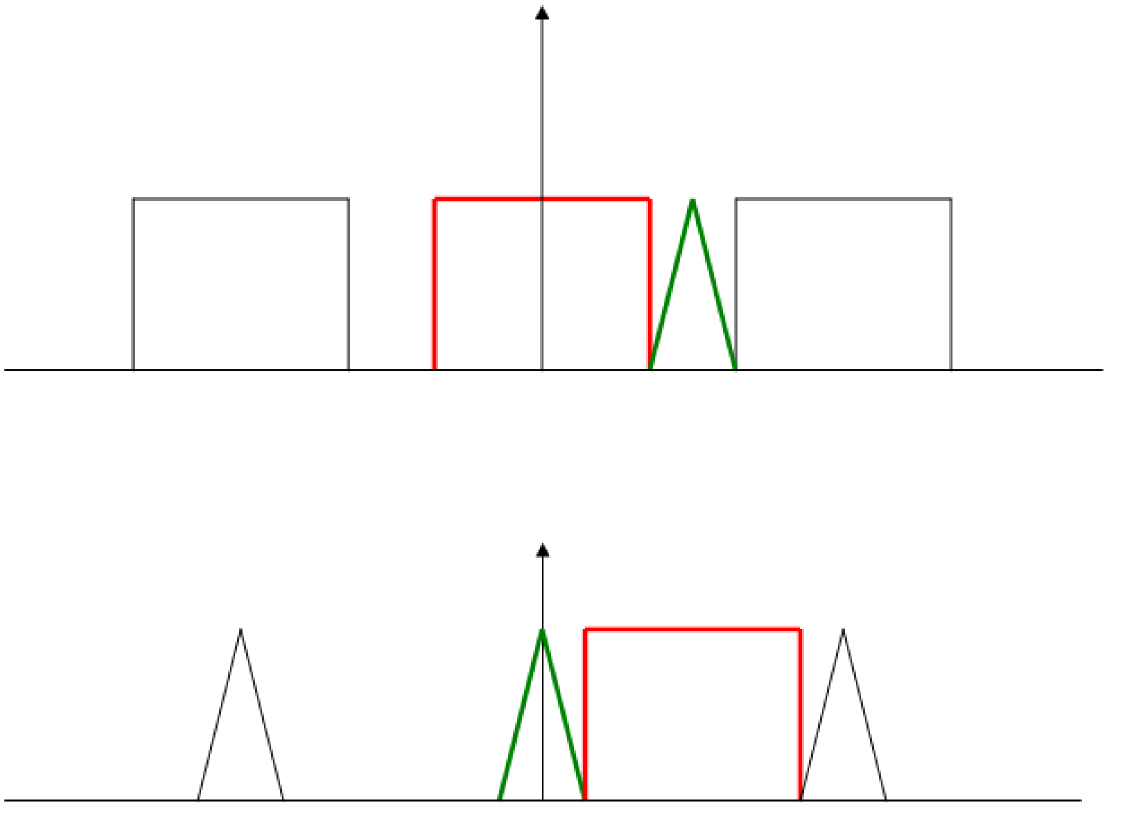
If $x(i)$ and $y(i)$ are both periodic with period M the circular convolution is defined as

$$z(i) = \sum_{m=0}^{M-1} x(m)\, y(i-m)$$

Periodic Extension of Signals: **Wrong!**



Periodic Extension of Signals: **Correct!**



One dimensional discrete degradation model

Suppose we have a one-dimensional discrete signal $f(i)$ of size A samples **which is due to a degradation process**.

The degradation can be modeled by a one-dimensional discrete impulse response $h(i)$ of size B samples.

AIM: **Work with the Discrete Fourier Transform instead of Time**

THIS MEANS: Each function in time should be

- Extended
- Periodic

We form the extended versions of $f(i)$ and $h(i)$, both of size $M \geq A + B - 1$ and periodic with period M. These can be denoted as $f_e(i)$ and $h_e(i)$.

Extension is done by zero-padding

$$y_e(i) = \sum_{m=0}^{M-1} f_e(m) h_e(i-m) + n_e(i)$$

Using matrix notation we can write the following form

$$\mathbf{y} = \mathbf{H}\mathbf{f} + \mathbf{n}$$

$$\underset{(M \times M)}{\mathbf{H}} = \begin{bmatrix} h_e(0) & h_e(M-1) & \dots & h_e(1) \\ h_e(1) & h_e(0) & \dots & h_e(2) \\ \vdots & \vdots & \ddots & \vdots \\ h_e(M-1) & h_e(M-2) & \dots & h_e(0) \end{bmatrix}$$

$$Y(k) = MH(k)F(k), k = 0,1,\dots,M-1$$

$$\mathbf{w}(k) = \begin{bmatrix} 1 \\ \exp(j\frac{2\pi}{M}k) \\ \vdots \\ \exp[j\frac{2\pi}{M}(M-1)k] \end{bmatrix}$$

$$\mathbf{W} = \begin{bmatrix} \mathbf{w}(0) & \mathbf{w}(1) & \dots & \mathbf{w}(M-1) \end{bmatrix}$$

$$\mathbf{H} = \mathbf{W}\mathbf{D}\mathbf{W}^{\mathbf{-1}} \Rightarrow \mathbf{D} = \mathbf{W}^{\mathbf{-1}}\mathbf{H}\mathbf{W}$$

$$\mathbf{D} = \begin{bmatrix} \lambda(0) & & & \mathbf{0} \\ & \lambda(1) & & \\ & & \ddots & \\ \mathbf{0} & & & \lambda(M-1) \end{bmatrix}$$

$$D(k,k) = \lambda(k) = MH(k)$$

The proof is in the notes based on the following points

- The eigenvalues of a circulant matrix are the DFT values of the signal that forms the matrix.

- The eigenvectors of a circulant matrix are the DFT basis functions!

- Diagonalisation of the degradation matrix yields the proof **circular convolution** $\Rightarrow$ **DFT**

2-D Case

For a space invariant degradation process we obtain

$$y_e(i,j) = \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} f_e(m,n) h_e(i-m, j-n) + n_e(i,j)$$

Using matrix notation

$$\mathbf{y} = \mathbf{H}\mathbf{f} + \mathbf{n}$$

where $\mathbf{f}$ and $\mathbf{y}$ are $MN-$dimensional column vectors that represent the **lexicographic ordering of images** $f_e(i,j)$ and $h_e(i,j)$ respectively.

In that case we end up with the following set of $M \times N$ scalar problems.

$$Y(u,v) = MNH(u,v)F(u,v)(+N(u,v))$$

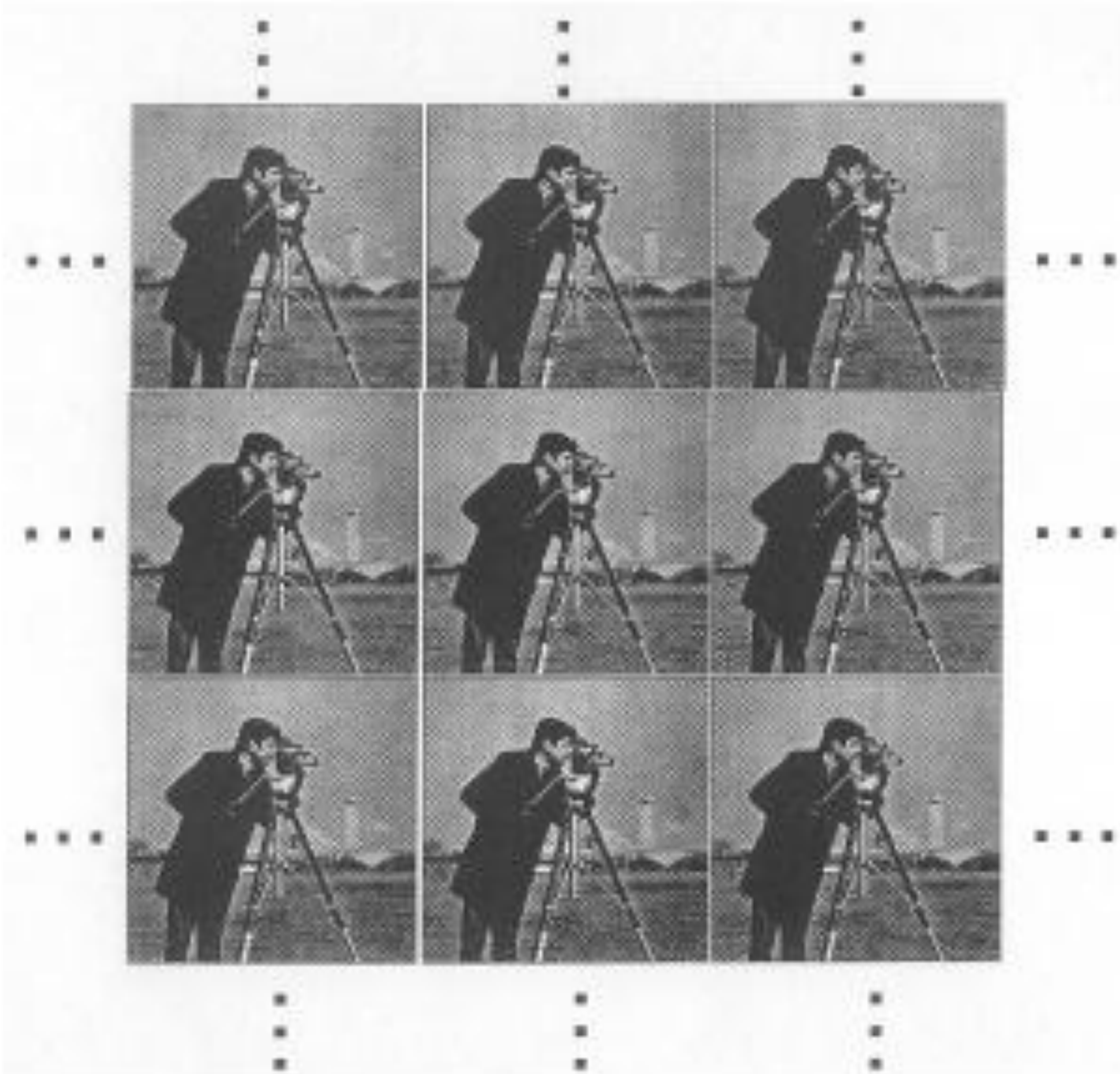
$$u = 0,1,\ldots,M-1, v = 0,1,\ldots,N-1$$

Example

- Image 256x256
- Degradation 3x3

- Finally both at least
  (256+3-1)x(256+3-1)=258x258
  and also periodic

Example



DETERMINISTIC APPROACHES TO RESTORATION

DIRECT METHODS

Inverse filtering

$$\mathbf{y} = \mathbf{Hf} + \mathbf{n}$$

Formulation of the problem: minimize $J(\mathbf{f}) = \|\mathbf{n}(\mathbf{f})\|^2 = \|\mathbf{y} - \mathbf{Hf}\|^2$

We set the first derivative of the cost function equal to zero

$$\frac{\partial J(\mathbf{f})}{\partial \mathbf{f}} = 0 \Rightarrow -2\mathbf{H}^{\mathbf{T}}(\mathbf{y} - \mathbf{Hf}) = \mathbf{0} \Rightarrow \mathbf{H}^{\mathbf{T}}\mathbf{Hf} = \mathbf{H}^{\mathbf{T}}\mathbf{y} \Rightarrow$$

$$\boxed{\mathbf{f} = (\mathbf{H}^{\mathbf{T}}\mathbf{H})^{-1}\mathbf{H}^{\mathbf{T}}\mathbf{y}}$$

If $M = N$ then if $\mathbf{H}^{-1}$ exists then $\mathbf{f} = (\mathbf{H}^{\mathbf{T}}\mathbf{H})^{-1}\mathbf{H}^{\mathbf{T}}\mathbf{y} = \mathbf{H}^{-1}(\mathbf{H}^{\mathbf{T}})^{-1}\mathbf{H}^{\mathbf{T}}\mathbf{y} \Rightarrow$

$$\boxed{\mathbf{f} = \mathbf{H}^{-1}\mathbf{y}}$$

Spatial Domain-Matrix Form-Lexicographic Ordering

$$\mathbf{f} = (\mathbf{H}^T\mathbf{H})^{-1}\mathbf{H}^T\mathbf{y}$$

$$\mathbf{f} = \mathbf{H}^{-1}\mathbf{y}$$

Frequency Domain-Scalar Form-Point Processing

$$F(u,v) = \frac{H^*(u,v)Y(u,v)}{|H(u,v)|^2} = \frac{Y(u,v)}{H(u,v)} \Rightarrow$$

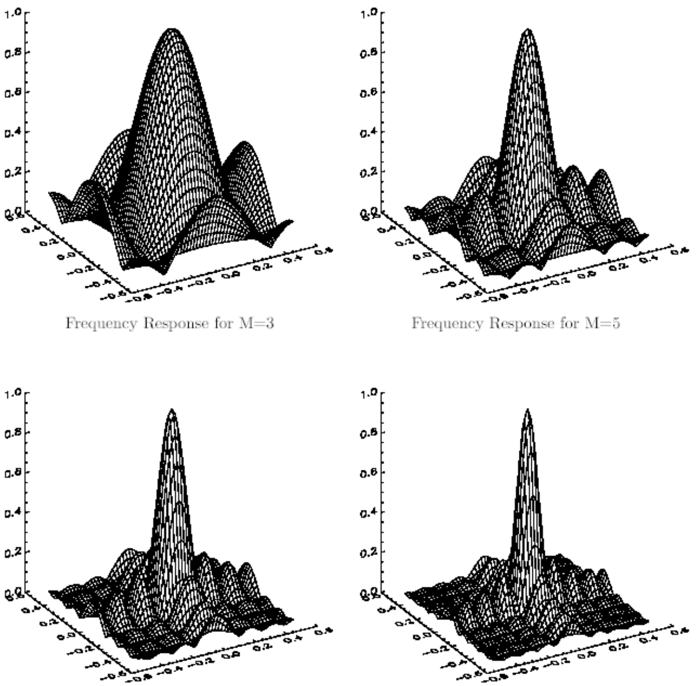
$$f(i,j) = \Im^{-1}\left[\frac{H^*(u,v)Y(u,v)}{|H(u,v)|^2}\right] = \Im^{-1}\left[\frac{Y(u,v)}{H(u,v)}\right]$$

Problems – Suppose there isn't any additive noise

$$F(u,v) = \frac{Y(u,v)}{H(u,v)}$$

A problem arises if $H(u,v)$ becomes very small or zero for some point (u,v) or for a whole region in the (u,v) plane. In that region inverse filtering cannot be applied.

Note that in most real applications $H(u,v)$ drops off rapidly as a function of distance from the origin!


Frequency Response for M=3
Frequency Response for M=5
Frequency Response for M=7
Frequency Response for M=9

Solution: carry out the restoration process in a limited neighborhood about the origin where $H(u,v)$ in not very small.

This procedure is called **pseudoinverse filtering**.

$$\hat{F}(u,v) = \begin{cases} \dfrac{H^*(u,v)Y(u,v)}{|H(u,v)|^2} & H(u,v) \neq 0 \\ 0 & H(u,v) = 0 \end{cases}$$

Problems – Suppose THERE IS ADDITIVE NOISE

$$\hat{F}(u,v) = \frac{H^*(u,v)\left(Y(u,v) + N(u,v)\right)}{\left|H(u,v)\right|^2} =$$

$$\frac{H^*(u,v)Y(u,v)}{\left|H(u,v)\right|^2} + \frac{H^*(u,v)N(u,v)}{\left|H(u,v)\right|^2} \Rightarrow$$

CAN BE HUGE !!

$$\hat{F}(u,v) = F(u,v) + \frac{N(u,v)}{H(u,v)}$$

If $H(u,v)$ becomes very small, the term $N(u,v)$ dominates the result.

Solution: carry out the restoration process in a limited neighborhood about the origin where $H(u,v)$ in not very small.

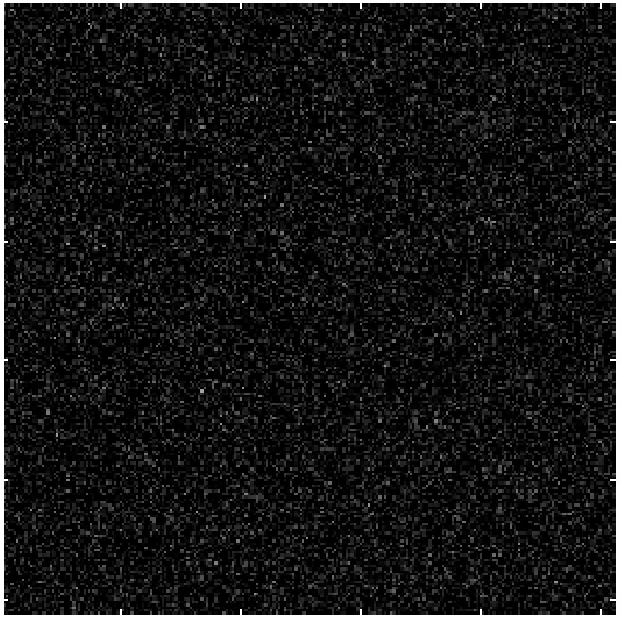
This procedure is called **pseudoinverse filtering**.

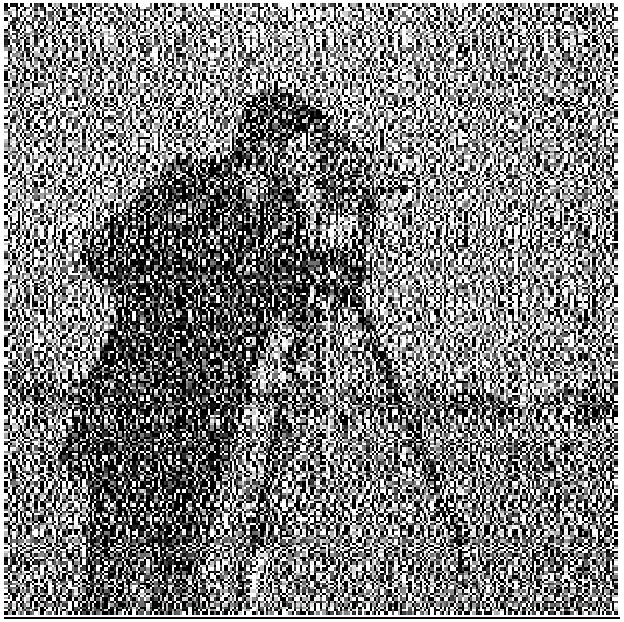
$$\hat{F}(u,v) = \begin{cases} \dfrac{H^*(u,v)Y(u,v)}{|H(u,v)|^2} & H(u,v) \neq 0 \\ 0 & H(u,v) = 0 \end{cases}$$

Pseudo-inverse Filtering with Different Thresholds

Observe the ghosts!!!

Pseudo-inverse Filtering in the Presence of Noise





2. Constrained least squares (CLS) restoration

It refers to a very large number of restoration algorithms.

The problem can be formulated as follows.

minimize

$$J(\mathbf{f}) = \|\mathbf{n}(\mathbf{f})\|^2 = \|\mathbf{y} - \mathbf{H}\mathbf{f}\|^2$$

subject to

$$\|\mathbf{C}\mathbf{f}\|^2 < \varepsilon$$

where

$\mathbf{C}\mathbf{f}$ is a high pass filtered version of the image.

The idea behind the above constraint is that the highpass version of the image contains a considerably large amount of noise!

Algorithms of the above type can be handled using optimization techniques.

Constrained least squares (CLS) restoration can be formulated by choosing an $\mathbf{f}$ to minimize the Lagrangian

$$\min\left(\|\mathbf{y}-\mathbf{Hf}\|^2+\alpha\|\mathbf{Cf}\|^2\right)$$

Typical choice for $\mathbf{C}$ is the 2-D Laplacian operator given by

$$\mathbf{C}=\begin{bmatrix} 0.00 & -0.25 & 0.00 \\ -0.25 & 1.00 & -0.25 \\ 0.00 & -0.25 & 0.00 \end{bmatrix}$$

α represents the Lagrange multiplier that is commonly known as the **regularisation parameter**.

α controls the relative contribution between the term $\|\mathbf{y}-\mathbf{Hf}\|^2$ and the term $\|\mathbf{Cf}\|^2$.

The minimization of the above leads to the following estimate for the original image

$$\mathbf{f} = \left(\mathbf{H}^{\mathbf{T}}\mathbf{H} + \alpha \mathbf{C}^{\mathbf{T}}\mathbf{C}\right)^{-1}\mathbf{H}^{\mathbf{T}}\mathbf{y}$$

$$\left(\mathbf{H}^{\mathbf{T}}\mathbf{H} + \alpha\mathbf{C}^{\mathbf{T}}\mathbf{C}\right)\mathbf{f} = \mathbf{H}^{\mathbf{T}}\mathbf{y}$$

$$\mathbf{H} = \mathbf{W}\mathbf{D}_{\mathbf{H}}\mathbf{W}^{-\mathbf{1}} = \mathbf{W}\mathbf{D}_{\mathbf{H}}\mathbf{W}^{*}$$

$$\mathbf{H}^{\mathbf{T}} = \mathbf{W}\mathbf{D}_{\mathbf{H}}^{*}\mathbf{W}^{-\mathbf{1}} = \mathbf{W}\mathbf{D}_{\mathbf{H}}^{*}\mathbf{W}^{*}$$

$$\mathbf{H}^{\mathbf{T}}\mathbf{H} = \mathbf{W}\mathbf{D}_{\mathbf{H}}^{*}\mathbf{W}^{-\mathbf{1}}\mathbf{W}\mathbf{D}_{\mathbf{H}}\mathbf{W}^{-\mathbf{1}} = \mathbf{W}\mathbf{D}_{\mathbf{H}}^{*}\mathbf{D}_{\mathbf{H}}\mathbf{W}^{*} = \mathbf{W}\left|\mathbf{D}_{\mathbf{H}}\right|^{\mathbf{2}}\mathbf{W}^{*}$$

$$\mathbf{C}^{\mathbf{T}}\mathbf{C} = \mathbf{W}\left|\mathbf{D}_{\mathbf{C}}\right|^{\mathbf{2}}\mathbf{W}^{*}$$

$$\left(\mathbf{H}^{\mathbf{T}}\mathbf{H} + \alpha\mathbf{C}^{\mathbf{T}}\mathbf{C}\right)\mathbf{f} = \mathbf{H}^{\mathbf{T}}\mathbf{y} \Rightarrow \mathbf{W}\left(\left|\mathbf{D}_{\mathbf{H}}\right|^{\mathbf{2}} + \alpha\left|\mathbf{D}_{\mathbf{C}}\right|^{\mathbf{2}}\right)\mathbf{W}^{*}\mathbf{f} = \mathbf{W}\mathbf{D}_{\mathbf{H}}^{*}\mathbf{W}^{*}\mathbf{y} \Rightarrow$$
$$\left(\left|\mathbf{D}_{\mathbf{H}}\right|^{\mathbf{2}} + \alpha\left|\mathbf{D}_{\mathbf{C}}\right|^{\mathbf{2}}\right)\mathbf{W}^{*}\mathbf{f} = \mathbf{D}_{\mathbf{H}}^{*}\mathbf{W}^{*}\mathbf{y}$$

$$\left(|\mathbf{D_H}|^2 + \alpha|\mathbf{D_C}|^2\right)\mathbf{W}^*\mathbf{f} = \mathbf{D_H^*}\mathbf{W}^*\mathbf{y}$$

$|\mathbf{D_H}|^2$: A diagonal matrix with the $(\mathrm{DFT})^2$ values of $\mathbf{H}$

$|\mathbf{D_C}|^2$: A diagonal matrix with the $(\mathrm{DFT})^2$ values of $\mathbf{C}$

$\mathbf{W}^*\mathbf{f}$: A vector with the DFT values of $\mathbf{f}$

$\mathbf{W}^*\mathbf{y}$: A vector with the DFT values of $\mathbf{y}$

$$(|H(u,v)|^2 + \alpha|C(u,v)|^2)F(u,v) = H^*(u,v)Y(u,v) \Rightarrow$$

$$F(u,v) = \frac{H^*(u,v)}{(|H(u,v)|^2 + \alpha|C(u,v)|^2)}Y(u,v)$$

Computational issues concerning the CLS method

(I) Choice of α

A restored image is approximated by an image which lies in the intersection of the two ellipsoids defined by **(set theoretic approach)**.

$$Q_{\mathbf{f}|\mathbf{y}} = \{\mathbf{f} \mid \|\mathbf{y} - \mathbf{Hf}\|^2 \le E^2\} \text{ and } Q_{\mathbf{f}} = \{\mathbf{f} \mid \|\mathbf{Cf}\|^2 \le \varepsilon^2\}$$

The center of one of the ellipsoids which bounds the intersection of $Q_{\mathbf{f}|\mathbf{y}}$ and $Q_{\mathbf{f}}$, is given by the equation

$$\mathbf{f} = \left(\mathbf{H}^{\mathbf{T}}\mathbf{H} + \alpha \mathbf{C}^{\mathbf{T}}\mathbf{C}\right)^{-1}\mathbf{H}^{\mathbf{T}}\mathbf{y}$$

with $\alpha = (E/\varepsilon)^2$.

Another problem: choice of E^2 and ε^2.

Computational issues concerning the CLS method

(I) Choice of α (cont.)

A popular choice is

$$\alpha = \frac{1}{\text{BSNR}}$$

- **Blurred Signal-to-Noise Ratio (BSNR)**: a metric that describes the degradation model.

$$\mathrm{BSNR} = 10\log_{10}\left\{\frac{\left(\frac{1}{MN}\sum_i\sum_j\left[g(i,j)-\overline{g}(i,j)\right]^2\right)}{\sigma_n^2}\right\}$$

variance of degraded noiseless signal

$g(i,j) = y(i,j) - n(i,j)$

$\overline{g}(i,j) = E\{g(i,j)\}$

σ_n^2: variance of additive noise

- **Improvement in SNR (ISNR)**: validates the performance of the image restoration algorithm.

$$\mathrm{ISNR} = 10\log_{10}\left\{\frac{\sum_i \sum_j \left[f(i,j) - y(i,j)\right]^2}{\sum_i \sum_j \left[f(i,j) - \hat{f}(i,j)\right]^2}\right\}$$

ISNR can only be used for simulation with artificial data!!! since the original image is NOT AVAILABLE in a real life situation

Comments

With larger values of α, and thus more regularisation, the restored image tends to have more ***ringing***.

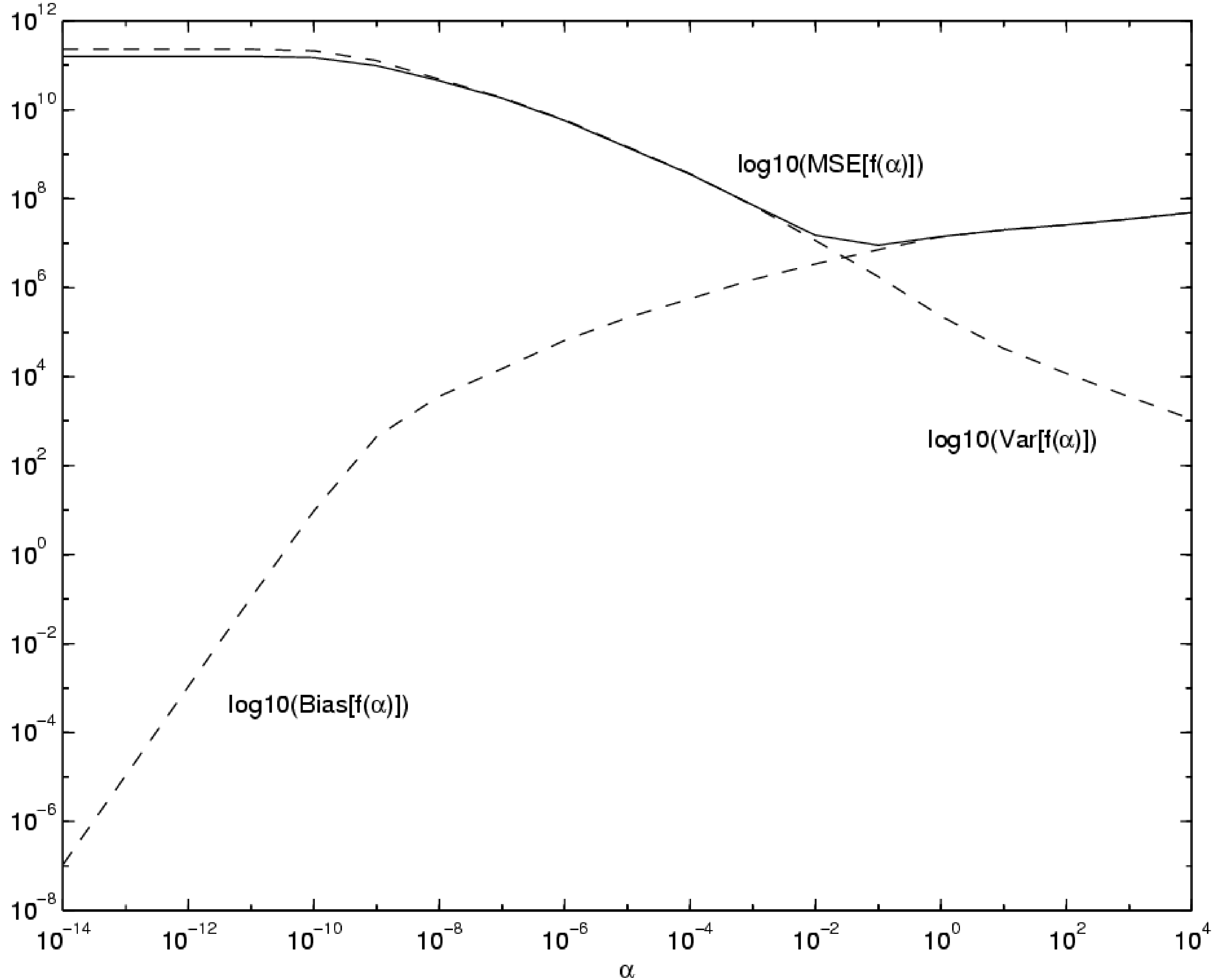
With smaller values of α, the restored image tends to have more ***amplified noise effects***.

The **variance** and **bias** of the error image in frequency domain are

$$\mathrm{Var}(\alpha) = \sigma_n^2 \sum_{u=0}^{M} \sum_{v=0}^{N} \frac{|H(u,v)|^2}{\left(|H(u,v)|^2 + \alpha|C(u,v)|^2\right)^2}$$

$$\mathrm{Bias}(\alpha) = \sigma_n^2 \sum_{u=0}^{M-1} \sum_{v=0}^{N-1} \frac{|F(u,v)|^2 \alpha^2 |C(u,v)|^4}{\left(|H(u,v)|^2 + \alpha|C(u,v)|^2\right)^2}$$

The minimum $\mathrm{MSE}(\alpha)$ is encountered close to the intersection of the above functions.


log10(MSE[f(α)])
log10(Var[f(α)])
log10(Bias[f(α)])
α
10^{12}
10^{10}
10^{8}
10^{6}
10^{4}
10^{2}
10^{0}
10^{-2}
10^{-4}
10^{-6}
10^{-8}
10^{-14}
10^{-12}
10^{-10}
10^{-8}
10^{-6}
10^{-4}
10^{-2}
10^{0}
10^{2}
10^{4}

A good choice of α would be one that gives the best compromise between the variance and bias of the error image.

ITERATIVE METHODS

new estimate=old estimate+function(old estimate)

- There is no need to explicitly implement the inverse of an operator. The restoration process is monitored as it progresses. Termination of the algorithm may take place before convergence.
- The effects of noise can be controlled in each iteration.
- The algorithms used can be spatially adaptive.
- The problem specifications are very flexible with respect to the type of degradation. Iterative techniques can be applied in cases of spatially varying or nonlinear degradations or in cases where the type of degradation is completely unknown (blind restoration).

A general formulation

In general, iterative restoration refers to any technique that attempts to minimize a function of the form

$$\Phi(\mathbf{f})$$

using an updating rule for the partially restored image.

Method of **successive approximations**:

$$\mathbf{f_0} = \mathbf{0}$$

$$\mathbf{f_{k+1}} = \mathbf{f_k} - \beta\Phi(\mathbf{f_k}) = \Psi(\mathbf{f_k})$$

Least squares iteration

In that case we seek for a solution that minimizes the function

$$M(\mathbf{f}) = \|\mathbf{y} - \mathbf{H}\mathbf{f}\|^2$$

A necessary condition for $M(\mathbf{f})$ to have a minimum is that its gradient with respect to $\mathbf{f}$ is equal to zero, which results in the normal equations

$$\frac{\partial M(\mathbf{f})}{\partial \mathbf{f}} = \mathbf{0} \Rightarrow \mathbf{H}^{\mathbf{T}}\mathbf{H}\mathbf{f} - \mathbf{H}^{\mathbf{T}}\mathbf{y} = \mathbf{0} \Rightarrow -\mathbf{H}^{\mathbf{T}}(\mathbf{y} - \mathbf{H}\mathbf{f}) = \mathbf{0}$$

and

$$\Phi(\mathbf{f}) = -\mathbf{H}^{\mathbf{T}}(\mathbf{y} - \mathbf{H}\mathbf{f})$$

$$\mathbf{f}_{\mathbf{0}} = \beta \mathrm{H}^{\mathbf{T}}\mathbf{y}$$

$$\mathbf{f}_{\mathbf{k+1}} = \mathbf{f}_{\mathbf{k}} + \beta \mathrm{H}^{\mathbf{T}}(\mathbf{y} - \mathrm{H}\mathbf{f}_{\mathbf{k}}) = \beta \mathrm{H}^{\mathbf{T}}\mathbf{y} + (\mathrm{I} - \beta \mathrm{H}^{\mathbf{T}}\mathrm{H})\mathbf{f}_{\mathbf{k}}$$

Constrained least squares iteration

$$M(\mathbf{f},\alpha) = \|\mathbf{y} - \mathbf{Hf}\|^2 + \alpha\|\mathbf{Cf}\|^2$$

$$\Phi(\mathbf{f}) = \nabla_{\mathbf{f}} M(\mathbf{f},\alpha) = (\mathbf{H}^{\mathbf{T}}\mathbf{H} + \alpha\mathbf{C}^{\mathbf{T}}\mathbf{C})\mathbf{f} - \mathbf{H}^{\mathbf{T}}\mathbf{y} = \mathbf{0}$$

$$\mathbf{f}_{\mathbf{0}} = \beta\mathbf{H}^{\mathbf{T}}\mathbf{y}$$

$$\mathbf{f}_{\mathbf{k+1}} = \mathbf{f}_{\mathbf{k}} + \beta[\mathbf{H}^{\mathbf{T}}\mathbf{y} - (\mathbf{H}^{\mathbf{T}}\mathbf{H} + \alpha\mathbf{C}^{\mathbf{T}}\mathbf{C})\mathbf{f}_{\mathbf{k}}]$$

It can be proved that the above iteration (known as **Iterative CLS** or **Tikhonov-Miller Method**) converges if

$$0 < \beta < \frac{2}{|\lambda_{\max}|}$$

where $\lambda_{\max}$ is the maximum eigenvalue of the matrix

$$(\mathbf{H}^{\mathbf{T}}\mathbf{H} + \alpha\mathbf{C}^{\mathbf{T}}\mathbf{C})$$

If the matrices $\mathbf{H}$ and $\mathbf{C}$ are block-circulant the iteration can be implemented in the frequency domain.

Figure 3: Degraded by a 7×7 pill-box blur, 20 dB BSNR

Figure 5: Degraded by a 5×5 Gaussian blur ($\sigma^2 = 1$), 20 dB BSNR

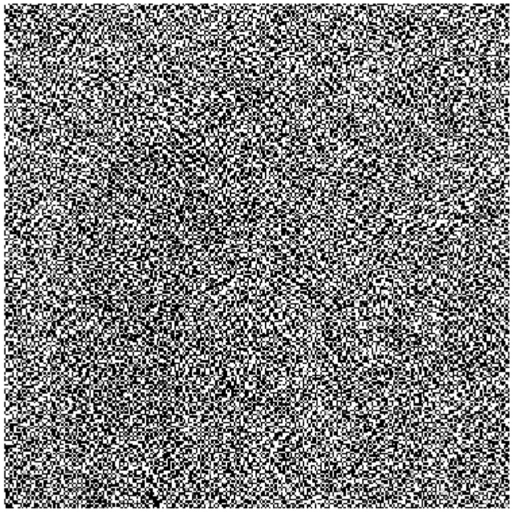


Figure 11: Result of Figure 3 restored by a generalized inverse filter with a threshold of 10^{-3}, ISNR $= -32.9$ dB

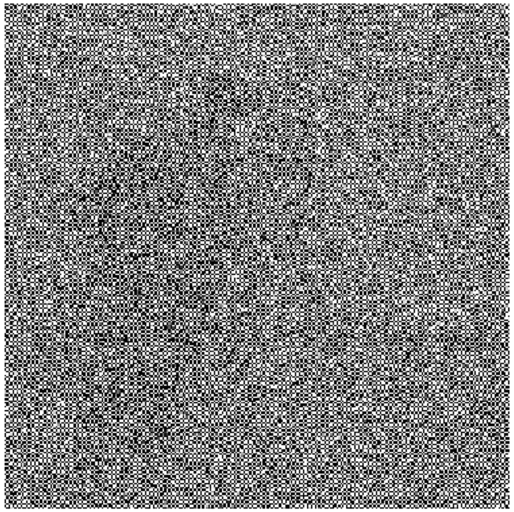


Figure 17: Result of Figure 5 restored by a generalized inverse filter with a threshold of 10^{-3}, ISNR $= -36.6$ dB

Figure 3: Degraded by a 7×7 pill-box blur, 20 dB BSNR

Figure 5: Degraded by a 5×5 Gaussian blur ($\sigma^2 = 1$), 20 dB BSNR

Figure 13: Result of Figure 3 restored by a generalized inverse filter with a threshold of 10^{-1}, ISNR $= 0.61$ dB

Figure 19: Result of Figure 5 restored by a generalized inverse filter with a threshold of 10^{-1}, ISNR $= -1.8$ dB

Figure 3: Degraded by a 7×7 pill-box blur, 20 dB BSNR

Figure 5: Degraded by a 5×5 Gaussian blur ($\sigma^2 = 1$), 20 dB BSNR

Figure 26: CLS restoration of Figure 3 with $\alpha = 1$, ISNR = 2.5 dB

Figure 40: CLS restoration of Figure 5 with $\alpha = 1$, ISNR = 1.3 dB

Figure 3: Degraded by a 7×7 pill-box blur, 20 dB BSNR

Figure 5: Degraded by a 5×5 Gaussian blur ($\sigma^2 = 1$), 20 dB BSNR

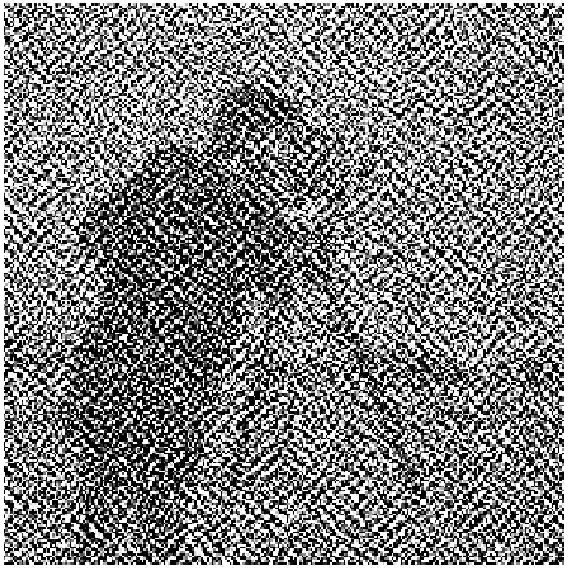


Figure 30: CLS restoration of Figure 3 with $\alpha = 0.0001$, ISNR $= -21.9$ dB

e 44: CLS restoration of Figure 5 with $\alpha = 0.0001$, ISNR $= -22.1$ dB

Figure 3: Degraded by a 7 × 7 pill-box blur, 20 dB BSNR

Figure 27: Corresponding error image for Figure 26 (|original − restored|, scaled for display)

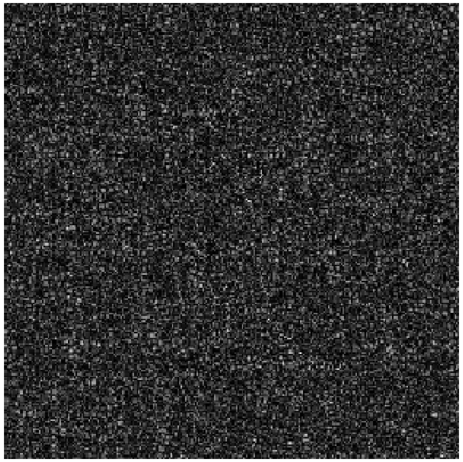


Figure 29: Corresponding error image for Figure 28 (|original − restored|, scaled for display)

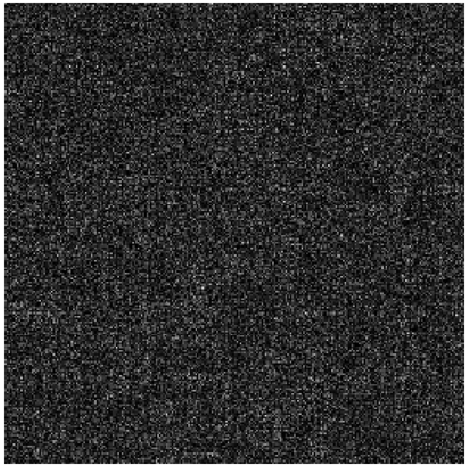


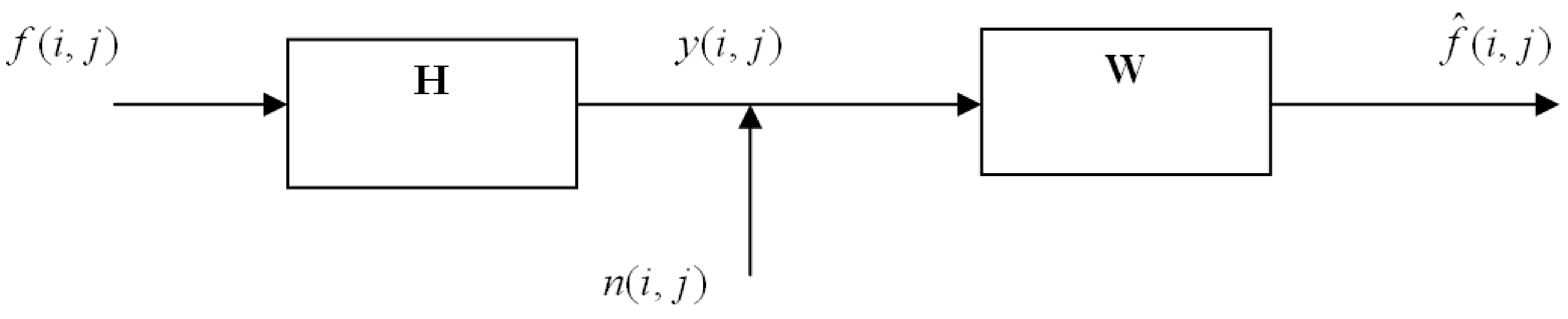
Figure 31: Corresponding error image for Figure 30 (|original − restored|, scaled for display)

Stochastic approaches to restoration

DIRECT METHODS

Wiener estimator (stochastic regularisation)

The image restoration problem can be viewed as a system identification problem of the following form



The objective is to minimize the following function

$$E\{(\mathbf{f}-\hat{\mathbf{f}})^{\mathbf{T}}(\mathbf{f}-\hat{\mathbf{f}})\}$$

To do so the following conditions should hold:

(i) $E\{\hat{\mathbf{f}}\}=E\{\mathbf{f}\}\Rightarrow E\{\mathbf{f}\}=\mathbf{W}E\{\mathbf{y}\}$

(ii) the error must be orthogonal to the observation about the mean

$$E\{(\hat{\mathbf{f}}-\mathbf{f})(\mathbf{y}-E\{\mathbf{y}\})^{\mathbf{T}}\}=0$$

$$\mathbf{w} = \mathrm{R}_{\mathbf{ff}}\mathrm{H}^{\mathbf{T}}(\mathrm{H}\mathrm{R}_{\mathbf{ff}}\mathrm{H}^{\mathbf{T}} + \mathrm{R}_{\mathbf{nn}})^{-\mathbf{1}}$$

$$\hat{\mathbf{f}} = \mathrm{R}_{\mathbf{ff}}\mathrm{H}^{\mathbf{T}}(\mathrm{H}\mathrm{R}_{\mathbf{ff}}\mathrm{H}^{\mathbf{T}} + \mathrm{R}_{\mathbf{nn}})^{-\mathbf{1}}\mathbf{y}$$

Note that knowledge of $\mathrm{R}_{\mathbf{ff}}$ and $\mathrm{R}_{\mathbf{nn}}$ is assumed.

$$\mathrm{R}_{\mathbf{ff}} = E\{\mathbf{ff}^{\mathbf{T}}\} \text{ and } \mathrm{R}_{\mathbf{nn}} = E\{\mathbf{nn}^{\mathbf{T}}\}$$

In frequency domain

$$W(u,v) = \frac{S_{ff}(u,v)H^*(u,v)}{S_{ff}(u,v)\left|H(u,v)\right|^2 + S_{nn}(u,v)}$$

$$\hat{F}(u,v) = \frac{S_{ff}(u,v)H^*(u,v)}{S_{ff}(u,v)\left|H(u,v)\right|^2 + S_{nn}(u,v)}Y(u,v)$$

Computational issues

The noise variance has to be known, otherwise it is estimated from a flat region of the observed image.

In practical cases where a single copy of the degraded image is available, it is quite common to use $S_{yy}(u,v)$ as an estimate of $S_{ff}(u,v)$. **This is very often a poor estimate !**

Wiener smoothing filter in the absence of blur (only noise present)

In the absence of any blur, $H(u,v)=1$ and

$$W(u,v)=\frac{S_{ff}(u,v)}{S_{ff}(u,v)+S_{nn}(u,v)}=\frac{(SNR)}{(SNR)+1}$$

(a) Low spatial frequencies $(SNR) >> 1 \Rightarrow W(u,v) \cong 1$

(b) High spatial frequencies $(SNR) << 1 \Rightarrow W(u,v) \cong (SNR)$ SMALL

$W(u,v)$ can be implemented with a lowpass (smoothing) filter.

Relation with inverse filtering

$$W(u,v) = \frac{S_{ff}(u,v)H^*(u,v)}{S_{ff}(u,v)\left|H(u,v)\right|^2 + S_{nn}(u,v)}$$

If $S_{nn}(u,v) = 0 \Rightarrow W(u,v) = \dfrac{1}{H(u,v)}$ which is the inverse filter

If $S_{nn}(u,v) \to 0$

$$\lim_{S_{nn}\to 0} W(u,v) = \begin{cases} \dfrac{1}{H(u,v)} & H(u,v) \neq 0 \\ 0 & H(u,v) = 0 \end{cases}$$

which is the **pseudoinverse** filter.

Tikhonov-Miller Method

Iterative restoration of Figure 2

Figure 48: Result of Figure 2 restored by the iterative Tikhonov-Miller algorithm after 20 iterations, $\alpha = 10^{-3}$, ISNR = 0.88 dB

Figure 49: Result of Figure 2 restored by the iterative Tikhonov-Miller algorithm after 60 iterations, $\alpha = 10^{-3}$, ISNR = 4.17 dB

Figure 50: Final result of Figure 2 restored by the iterative Tikhonov-Miller algorithm (74 iterations), $\alpha = 10^{-3}$, ISNR = 4.15 dB

Iterative restoration of Figure 3

Figure 51: Result of Figure 3 restored by the iterative Tikhonov-Miller algorithm after 20 iterations, $\alpha = 10^{-2}$, ISNR = 0.23 dB

Figure 52: Result of Figure 3 restored by the iterative Tikhonov-Miller algorithm after 30 iterations, $\alpha = 10^{-2}$, ISNR = 1.42 dB

Figure 53: Result of Figure 3 restored by the iterative Tikhonov-Miller algorithm after 80 iterations, $\alpha = 10^{-2}$, ISNR $= -0.63$ dB

Figure 54: Final result of Figure 3 restored by the iterative Tikhonov-Miller algorithm (92 iterations), $\alpha = 10^{-2}$, ISNR = −1.01 dB

Figure 55: Result of Figure 4 restored by the iterative Tikhonov-Miller algorithm after 40 iterations, $\alpha = 10^{-3}$, ISNR = 2.73 dB

Figure 56: Final result of Figure 4 restored by the iterative Tikhonov-Miller algorithm (73 iterations), $\alpha = 10^{-3}$, ISNR = 2.38 dB

Figure 57: Result of Figure 5 restored by the iterative Tikhonov-Miller algorithm after 20 iterations, $\alpha = 10^{-2}$, ISNR = -2.70 dB

Figure 58: Result of Figure 5 restored by the iterative Tikhonov-Miller algorithm after 40 iterations, $\alpha = 10^{-2}$, ISNR = -1.73 dB

Figure 59: Final result of Figure 5 restored by the iterative Tikhonov-Miller algorithm (81 iterations), $\alpha = 10^{-2}$, ISNR $= -3.67$ dB

Wiener Filtering

Figure 60: Result of Figure 2 restored by a direct Wiener filter, ISNR = 2.6 dB

Figure 61: Result of Figure 3 restored by a direct Wiener filter, ISNR = 1.7 dB

Figure 62: Result of Figure 4 restored by a direct Wiener filter, ISNR = 1.8 dB

Figure 63: Result of Figure 5 restored by a direct Wiener filter, ISNR = 1.0 dB

ITERATIVE METHODS

Step 0: Initial estimate of $\mathbf{R_{ff}}$

$$\mathbf{R_{ff}}(0) = \mathbf{R_{yy}} = E\{\mathbf{yy}^{\mathbf{T}}\}$$

Step 1: Construct the i^{th} restoration filter

$$\mathbf{W}(i+1) = \mathbf{R_{ff}}(i)\mathbf{H}^{\mathbf{T}}(\mathbf{H}\mathbf{R_{ff}}(i)\mathbf{H}^{\mathbf{T}} + \mathbf{R_{nn}})^{-1}$$

Step 2: Obtain the $(i+1)^{\text{th}}$ estimate of the restored image

$$\hat{\mathbf{f}}(i+1) = \mathbf{W}(i+1)\mathbf{y}$$

Step 3: Use $\hat{\mathbf{f}}(i+1)$ to compute an improved estimate of $\mathbf{R_{ff}}$ given by

$$\mathbf{R_{ff}}(i+1) = E\{\hat{\mathbf{f}}(i+1)\hat{\mathbf{f}}^{\mathbf{T}}(i+1)\}$$

Step 4: Increase i and repeat steps 1,2,3,4.