1. (a)

(i) $$F(u,v) = \frac{1}{(2M+1)^2} \sum_{-M}^{M} \sum_{-M}^{M} f(x,y)\, e^{-j\frac{2\pi}{M}(ux+vy)} \qquad (1)$$

$$F(-u,-v) = \frac{1}{(2M+1)^2} \sum_{-M}^{M} \sum_{-M}^{M} f(x,y)\, e^{j\frac{2\pi}{M}(ux+vy)} = F^*(u,v)$$

if $f(x,y) =$ real.

(ii) $$F(u,v) = \frac{1}{(2M+1)^2} \sum_{M}^{-M} \sum_{M}^{-M} f(-x,-y)\, e^{j\frac{2\pi}{M}(ux+vy)}$$

$$= \frac{1}{(2M+1)^2} \sum_{-M}^{M} \sum_{-M}^{M} f(-x,-y)\, e^{j\frac{2\pi}{M}(ux+vy)} \qquad (2)$$

If $f(x,y)$ is symmetric, (2) becomes

$$F(u,v) = \frac{1}{(2M+1)^2} \sum_{-M}^{M} \sum_{-M}^{M} f(x,y)\, e^{j\frac{2\pi}{M}(ux+vy)} \qquad (3)$$

From (1) we write

$$F(u,v) = \frac{1}{(2M+1)^2} \sum_{-M}^{M} \sum_{-M}^{M} f(x,y) \cos\left[\frac{2\pi}{M}(ux+vy)\right]$$

$$-j \frac{1}{(2M+1)^2} \sum_{-M}^{M} \sum_{-M}^{M} f(x,y) \sin\left[\frac{2\pi}{M}(ux+vy)\right] = A - jB \quad (4)$$

From (3) we write

$F(u,v) = A + jB$ (5)

From (4), (5) we see that $B = \operatorname{Im}\{F(u,v)\} = 0$

(iii) If $f(x,y)$ is antisymmetric, (2) becomes

$$F(u,v) = \frac{-1}{(2M+1)^2} \sum_{-M}^{M} \sum_{-M}^{M} f(x,y)\, e^{+j\frac{2\pi}{M}(ux+vy)} \qquad (6)$$

$$= -A - jB$$

From (4), (6) we see that $A = \operatorname{Re}\{F(u,v)\} = 0$

(b)

(i) Having imaginary part 0 requires the image to be symmetric.

Therefore: A, B, C, F, H

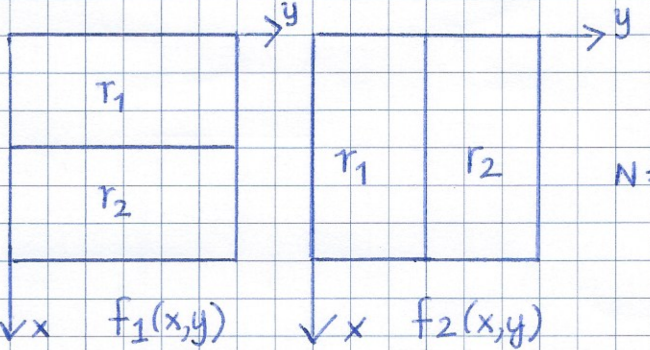
(ii) $$F(0,0) = \frac{1}{(\text{size})} \sum_x \sum_y f(x,y)$$

Therefore: E, H

(iii) $F(u,v)$ has circular symmetry if $f(x,y)$ has circular symmetry, which occurs only for F.

(iv) The real part will be zero for any antisymmetric image such that $f(x,y) = -f(-x,-y)$ which is true only for E.

(c)



$N = M^2$ number of pixels

(i) For $f_1(x,y)$:

mean $m_1 = \frac{r_1 + r_2}{2} = E\{f_1\}$

$$E\{f_1^2\} = \frac{\frac{N}{2} r_1^2 + \frac{N}{2} r_2^2}{N} = \frac{r_1^2 + r_2^2}{2}$$

$$\sigma_{f_1} = \frac{r_1^2 + r_2^2}{2} - \left(\frac{r_1 + r_2}{2}\right)^2 = E\{f_1^2\} - (E\{f\})^2 = \frac{(r_1 - r_2)^2}{4}$$

variance of $f_1 = \sigma$

For $f_2(x,y)$

$m_2 = m_1$ and $\sigma^2_{f_2} = \sigma^2_{f_1}$

$$E\{f_1 f_2\} = \frac{r_1^2}{4} + \frac{r_2^2}{4} + \frac{r_1 r_2}{2} = \frac{(r_1 + r_2)^2}{4}$$

$f_1 f_2$

r_1^2	$r_1 r_2$
$r_1 r_2$	r_2^2

We must calculate the covariance matrix of the two images.

$$C = \begin{bmatrix} C_{11} = \sigma & C_{12} \\ C_{12} & \sigma = C_{22} \end{bmatrix}$$

$$C_{12} = E\{f_1 f_2\} - m^2 = 0 \Rightarrow C = \begin{bmatrix} \sigma & 0 \\ 0 & \sigma \end{bmatrix}$$

A is the matrix which contains the eigenvectors of C.
These are $[1 \;\; 0]^T$, $[0, 1]^T$. Eigenvalues are $\lambda_{1,2} = \sigma$
The Karhunen Loeve transform converts the set $\begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$

to the set $\begin{bmatrix} g_1 \\ g_2 \end{bmatrix} = A \cdot \begin{bmatrix} f_1 - m \\ f_2 - m \end{bmatrix} = I \cdot \begin{bmatrix} f_1 - m \\ f_2 - m \end{bmatrix} = \begin{bmatrix} f_1 - m \\ f_2 - m \end{bmatrix}$

As seen the KLT is not beneficial since the new images are the zero mean versions of the previous images.

(ii) The above result is obvious since the previous images consist of a horizontal and a vertical edge, therefore there isn't any redundancy between the two images.

2. (a)

(i) The pdf $p_r(r)$ is $p_r(r) = 2 - 2r$

By equalizing r we obtain s through the transformation

$$s = \int_0^r p_r(w)dw = \int_0^r (2-2w)dw = 2r - r^2$$

The pdf $p_z(z)$ is $p_z(z) = 2z$

(ii) By equalizing z we obtain g through the tranformation

$$g = \int_0^z p_z(w)dw = \int_0^z 2w\,dw = z^2$$

We assume that $s = g \Rightarrow 2r - r^2 = z^2 \Rightarrow z = \sqrt{2r - r^2}$

(b)

(i) Two images have the same total power if their histograms are the same. This is because the total power depends only on the pixel values and not their order.

(ii) Same comment as in (i) is valid for the entropy.

$$\text{Power} = \sum_r H(r)\, r^2 \qquad \text{Entropy} = \sum_r -H(r)\, r \ln_2 r$$

where $H(r)$ is the histogram and r is the grey level.

(iii) The inter-pixel covariance function is not necessarily the same. One could take the pixels in an image of a face (high inter-pixel covariance or similarity between adjacent pixels) and move the randomly around, the image histogram would be the same but the covariance between pixels would be very small.

(c)

(i) F is a weighted averaging filter. It is mainly used for noise removal. It emphasizes more in the central pixel. It is obviously a lowpass filter.

L is the Laplacian operator which computes differences among pixels. It highlights abrupt changes (edges) in an image. It is a highpass filter.

(ii)

Corner pixels will become:

$$\begin{array}{ccc} & 1 & \\ 1 & \boxed{1} & 1 \\ & 1 & 8 \end{array} \quad .\times \quad \begin{array}{ccc} & 1/8 & \\ 1/8 & 1/2 & 1/8 \\ & 1/8 & 0 \end{array} \rightarrow \frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{2}=1$$

point by point multiplication

Middle outer pixels will become:

$$\begin{array}{ccc} 1 & 1 & 1 \\ 1 & \boxed{1} & 1 \\ 1 & 8 & 1 \end{array} \rightsquigarrow \frac{3}{8}+\frac{1}{2}+\frac{8}{8}=\frac{7}{8}+1 \text{ rounded to } 1$$

central pixel will become

$$\begin{array}{ccc} 1 & 1 & 1 \\ 1 & \boxed{8} & 1 \\ 1 & 1 & 1 \end{array} \rightsquigarrow \frac{4}{8}+4 \text{ rounded to } 4$$

Therefore, convolution with the F mask will be

$$\begin{array}{ccc} & 1 & \\ 1 & 4 & 1 \\ & 1 & \end{array}$$

(iii) The result is quite obvious

$$\begin{matrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{matrix}$$

(iv) The median filter is good for removing isolated pixel values very different from the rest, like the 8 in the centre of the I image.

(v)

$$\begin{matrix} & 1 & \\ 1 & -4 & 1 \\ & 1 & \end{matrix} = \underbrace{\begin{matrix} & 1 & \\ 1 & 1 & 1 \\ & 1 & \end{matrix}}_{\text{5-point local mean mask}} - \begin{matrix} & 0 & \\ 0 & 5 & 0 \\ & 0 & \end{matrix}$$

The Laplacian is a high pass filter, therefore it should be written as the difference between an allpass and a low pass.

3. (a)

$$H_1(u,v) = \frac{1}{(2M+1)^2}\sum_{-M}^{M}\sum_{-M}^{M} h_1(x,y)\, e^{-j\frac{2\pi}{(2M+1)}(ux+vy)}$$

$$= \underset{x=y=0}{1} + \underset{x=0,\,y=1}{e^{-j\frac{2\pi}{(2M+1)}v}} + \underset{y=0,\,x=1}{e^{-j\frac{2\pi}{(2M+1)}u}} + \underset{x=0,\,y=-1}{e^{j\frac{2\pi}{(2M+1)}v}} + \underset{y=0,\,x=-1}{e^{j\frac{2\pi}{(2M+1)}u}}$$

$$= \frac{1}{(2M+1)^2}\left(1 + 2\cos\left(\frac{2\pi}{2M+1}u\right) + 2\cos\left(\frac{2\pi}{2M+1}v\right)\right)$$

To recover the effect of the above distortion we require Pseudo-Inverse filtering. This is because for certain frequencies $H_1(u,v)$ can become 0.

$$H_2(u,v) = 5 + 2\cos\left(\frac{2\pi}{2M+1}u\right) + 2\cos\left(\frac{2\pi}{2M+1}v\right)$$

In that case, due to the term 5, $H_2(u,v)$ never becomes 0, and therefore, Inverse Filtering is efficient.

(b)

(i) As previously Inverse Filtering is not suitable because $H(u,v)$ contains zeros.

(ii)

$$W(u,v) = \frac{H^* S_{ff}}{|H|^2 S_{ff} + S_{nn}}$$

Inputing the expressions for $H(u,v)$ and S_{ff}, S_{nn} we get:

$$W(u,v) = \frac{\frac{\sin(\pi u/u_0)}{\pi s \Delta t\, u}\cdot\frac{1}{u}}{\frac{\sin^2(\pi u/u_0)}{\pi^2 s^2(\Delta t)^2 u^2}\cdot\frac{1}{u} + s\Delta t}$$

$$= \frac{\frac{\sin(\pi u/u_0)}{\cancel{\pi s \Delta t\, u^2}}\cdot \pi^{\cancel{2}} s^{\cancel{2}} (\Delta t)^{\cancel{2}}\cdot u^{\cancel{3}}}{\sin^2(\pi u/u_0) + \pi^2 s^3 (\Delta t)^3 u^3}$$

$$= \frac{(\pi s \Delta t\, u)\sin(\pi u/u_0)}{\sin^2(\pi u/u_0) + \pi^2(u/u_0)^3}$$

$$= \frac{(\pi u/u_0)\sin(\pi u/u_0)}{\sin^2(\pi u/u_0) + \pi^2(u/u_0)^3}$$

For $u = u_0$, $W(u,v) = 0$

(c) Due to the rotational symmetries presented in the distorted images the distortion mask is of the form:

$$\begin{bmatrix} a & b & a \\ b & c & b \\ a & b & a \end{bmatrix}$$

If the original image is of the form

$$\begin{matrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & {}^{1}0 & {}^{2}0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & {}^{3}0 & {}^{4}r & r & r & r & r & \cdots \\ 0 & 0 & r & r & r & r & r & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \end{matrix}$$

The ticked pixels are changing after the convolution with the distortion filters. Furthermore, $r = 10$.

Pixel 1 changes to $r \cdot a = 0 \Rightarrow a = 0$

Pixel 2 changes to $b \cdot r = 2 \Rightarrow b = 0.2$

Pixel 4 changes to $c \cdot r + 2 \cdot r \cdot b = 6 \Rightarrow$
$c \cdot r + 0.4 \cdot r = 6 \Rightarrow c = 0.2$

The above values work for the rest.

Therefore, the mask is:

$$\begin{matrix} 0 & 0.2 & 0 \\ 0.2 & 0.2 & 0.2 \\ 0 & 0.2 & 0 \end{matrix}$$

4. (a)

(i) The minimum number of bits per symbol for lossless compression is given by the entropy

$$H = -\sum_i p_i \log_2 p_i$$

M = 1. In this case we code each pixel separately

Symbol	Probability	Huffman Code
0	0.95	0
1	0.05	1

$H_1 = 0.286$ bits/symbol

Average lenght of Huffman code $l_{avg} = 1$ bit/symbol

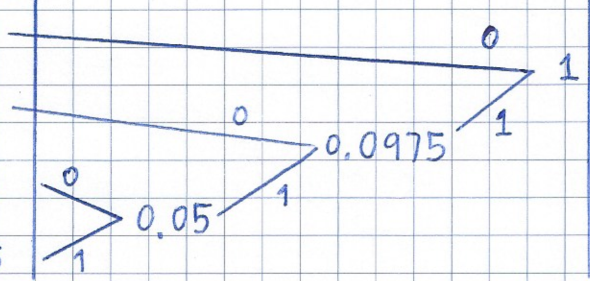
Efficiency: $H_1 / l_{avg} = 0.286$

Redundancy: $l_{avg} - H_1 = 0.714$ b/s

(ii) M = 2

In this case we encode pairs of pixels. Since we are told that successive pixels are independent we can easily calculate the probability of each possible combination of two pixels.

Symbol	Probability	HC
00	0.9025	0
01	0.0475	10
10	0.0475	110
11	0.0025	111



$H_2 = 0.572$ b/s, $l_{avg} = 1.147$ b/s

$= 0.286$ b/pixel

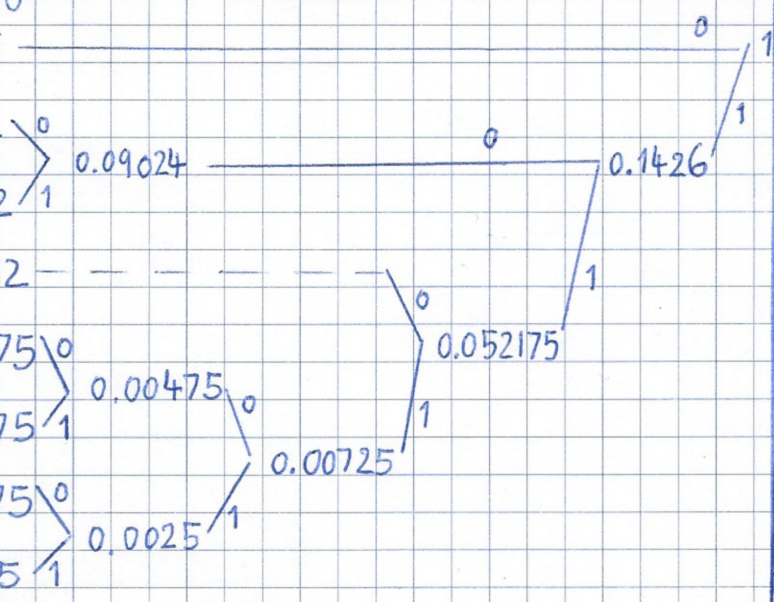
Efficiency: $0.573 / 1.147 \cong 0.5$

Redundancy: $1.147 - 0.573 = 0.574$ b/s

$= 0.287$ b/pixel

(iii) M=3

Symbol	Probability	HC
000	0.87375	0
001	0.04512	100
010	0.04512	101
100	0.04512	110
011	0.002375	11100
101	0.002375	11101
110	0.002375	11110
111	0.000125	11111



$H_3 = 0.859$ b/s or 0.286 b/pixel

$l_{avg} = 1.3$ bits per symbol

Efficiency $= 0.859 / 1.3$

Redundancy $= 0.286 - 1.3/3 = 0.147$ b/p

(b)

(i) The residuals are represented with binary strings of variable length.
Positive residuals have MSB 1.
Negative residuals are the complements of positive.

For example:

residual in decimal	binary
1	1
−1	0
2	10
3	11
−2	01
−3	00
⋮	⋮

The term category refes to the number of bits required to represent the residual.

Obviously the binary representation of the residual is not enough for decoding. Category is also required.

What is transmitted is the category of a residual Huffman coded followed by the binary representation of the residual.

(ii)

$r = y - x = a + b - c - x = 0$

In case that $r=0$, only codeword of category is transmitted. Therefore, the required codeword is 00.