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Digital Image Procesing

Unitary Transforms
Discrete Fourier Trasform (DFT) in Image Processing

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1-D Signal Transforms

Scalar form

$$\{ f(x), 0 \le x \le N-1 \}$$

$$g(u) = \sum_{x=0}^{N-1} T(u,x) f(x) \qquad 0 \le u \le N-1$$

Matrix form

$$\underline{g} = \underline{T} \cdot \underline{f}$$

1-D Signal Transforms-Remember the 1-D DFT

General form

$$\{f(x), 0 \le x \le N-1\}$$

$$g(u) = \sum_{x=0}^{N-1} T(u,x) f(x) \qquad 0 \le u \le N-1$$

DFT

$$g(u) = \sum_{x=0}^{N-1} \frac{1}{N} e^{-j2\pi \frac{ux}{N}} f(x)$$

1-D Inverse Signal Transforms-General Form

Scalar form

$$f(x) = \sum_{u=0}^{N-1} I(x,u) g(u)$$

Matrix form

$$\underline{f} = \underline{T}^{-1} \cdot \underline{g}$$

1-D Inverse Signal Transforms-Remember the 1D DFT

General form

$$\{f(x), 0 \le x \le N-1\}$$

$$f(x) = \sum_{u=0}^{N-1} I(x,u) g(u) \qquad 0 \le u \le N-1$$

Inverse DFT

$$f(x) = \sum_{x=0}^{N-1} e^{j2\pi \frac{u\,x}{N}} g(u)$$

1-D Unitary Transforms

Matrix form

$$\underline{g} = \underline{T} \cdot \underline{f}$$

$$\underline{T}^{-1} = \underline{T}^{*T}$$

Signal Reconstruction

$$\underline{T}^{-1} = \underline{T}^{*T}$$

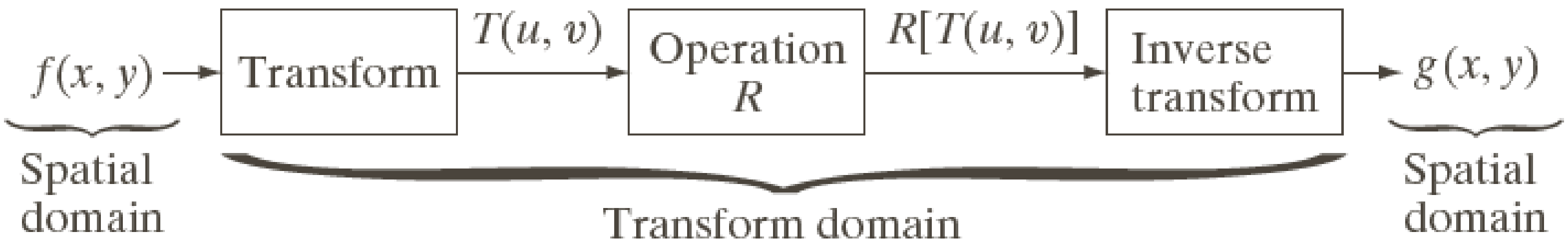
$$\underline{f} = \underline{T}^{*T} \cdot \underline{g} \Rightarrow f(x) = \sum_{u=0}^{N-1} T^{*}(u,x) g(u)$$

Why do we use Image Transforms?

Often, image processing tasks are best performed in a domain other than the *spatial domain.*

Key steps:

- Transform the image
- Carry the task(s) of interest in the *transformed domain*.
- Apply *inverse transform* to return to the spatial domain.



2-D (Image) Transforms-General Form

$$g(u,v)=\sum_{x=0}^{N-1}\sum_{y=0}^{N-1}T(u,v,x,y)f(x,y)$$

$$f(x,y)=\sum_{u=0}^{N-1}\sum_{v=0}^{N-1}I(x,y,u,v)g(u,v)$$

2-D (Image) Transforms-Special Cases

Separable

$$T(u,v,x,y) = T_1(u,x)T_2(v,y)$$

Symmetric

$$T(u,v,x,y) = T_1(u,x)T_1(v,y)$$

2-D (Image) Transforms-Special Cases (cont.)

Separable and Symmetric

$$\underline{g} = \underline{T}_1 \cdot \underline{f} \cdot \underline{T}_1^{\,T}$$

Separable, Symmetric and Unitary

$$\underline{f} = \underline{T}_1^{*T} \cdot \underline{g} \cdot \underline{T}_1^{*}$$

Energy Preservation

1-D

$$\left\| \underline{g} \right\|^2 = \left\| \underline{f} \right\|^2$$

2-D

$$\sum_{x=0}^{M-1} \sum_{y=0}^{N-1} \left| f(x,y) \right|^2 = \sum_{u=0}^{M-1} \sum_{v=0}^{N-1} \left| g(u,v) \right|^2$$

Energy Compaction

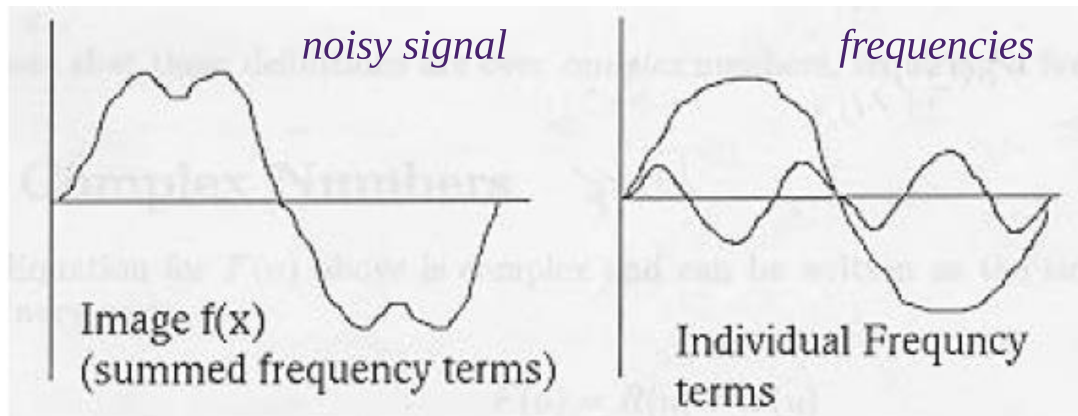
- **Most of the energy of the original data is concentrated in only a few transform coefficients, which are placed close to the origin; remaining coefficients have small values.**
- **This property facilitate compression of the original image.**

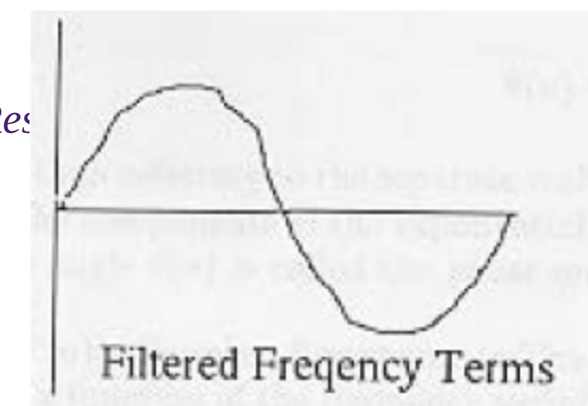
Let's talk about DFT in images: Why is it useful?

- It is easier for removing undesirable frequencies.
- It is faster to perform certain operations in the frequency domain than in the spatial domain.
- The transform is independent of the signal.

Example: Removing undesirable frequencies

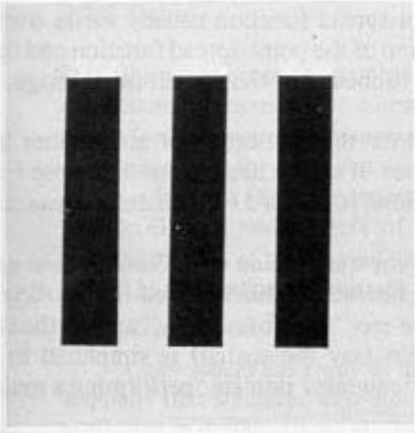
Example of removing a high frequency using the transform domain



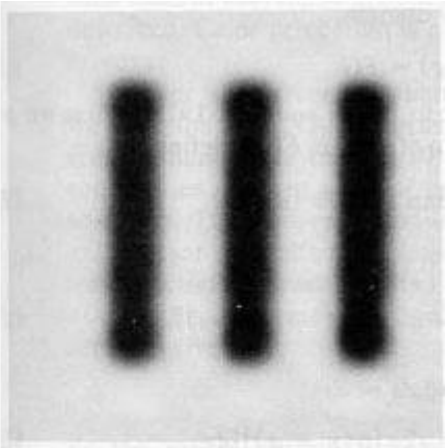


How do frequencies show up in an image?

- Low frequencies correspond to slowly varying information (e.g., continuous surface).
- High frequencies correspond to quickly varying information (e.g., edges)



Original Image



Low-passed version

2-D Discrete Fourier Transform

$$F(u,v) = \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x,y) e^{-j2\pi(ux/M + vy/N)}$$

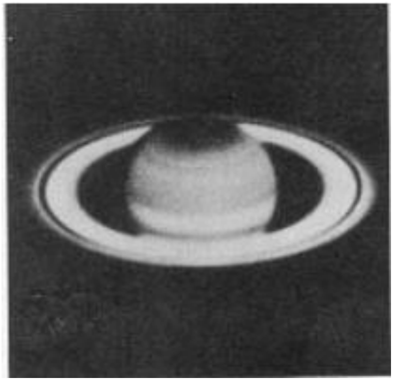
$$f(x,y) = \frac{1}{MN} \sum_{u=0}^{M-1} \sum_{v=0}^{N-1} F(u,v) e^{j2\pi(ux/M + vy/N)}$$

- It is separable, symmetric and unitary
- It results in a sequence of two 1-D DFT operations (prove this)

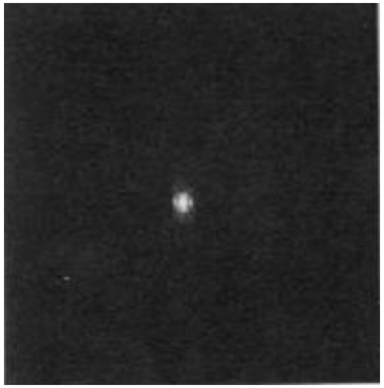
Visualizing DFT

- The dynamic range of $F(u,v)$ is typically very large
- Small values are not distinguishable
- We apply a logarithmic transformation to enhance small values.

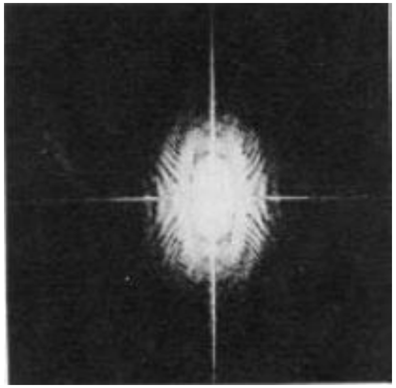
$$D(u,v) = c\log[1+|F(u,v)|], c \text{ is a constant}$$



original image



before transformation



after transformation

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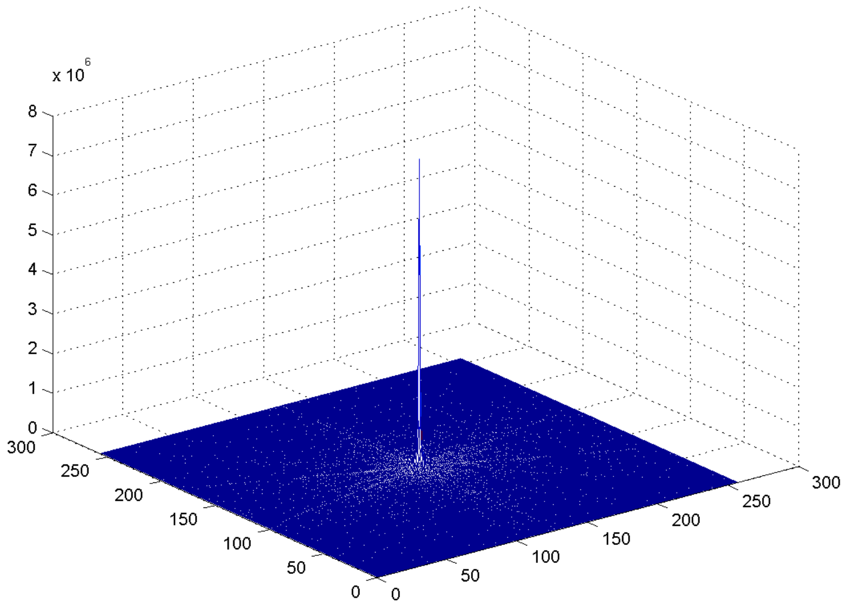
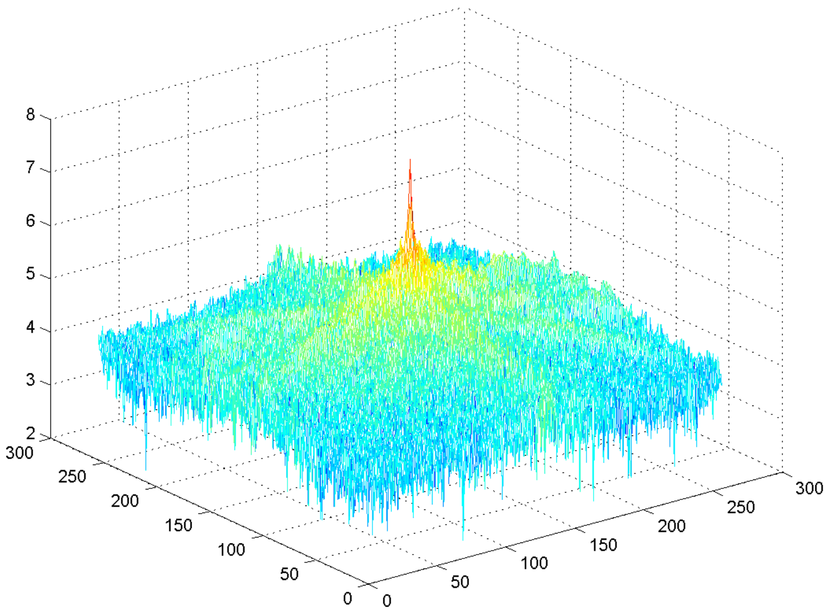
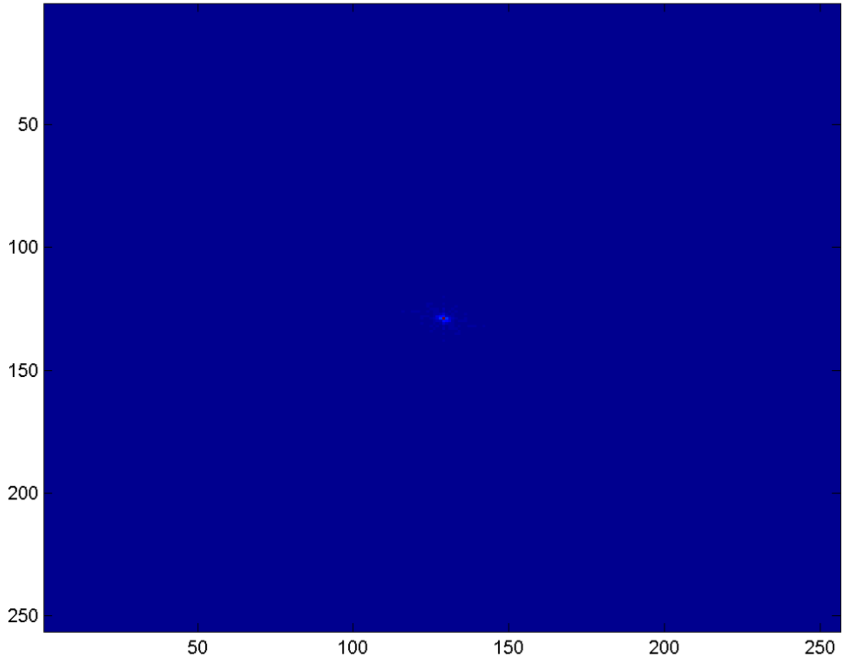
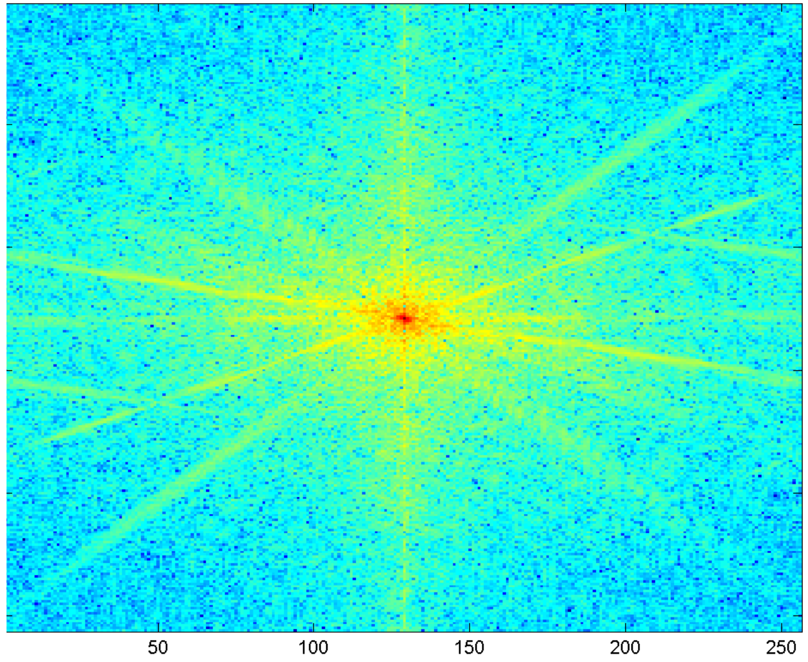
Amplitude and Log of the Amplitude



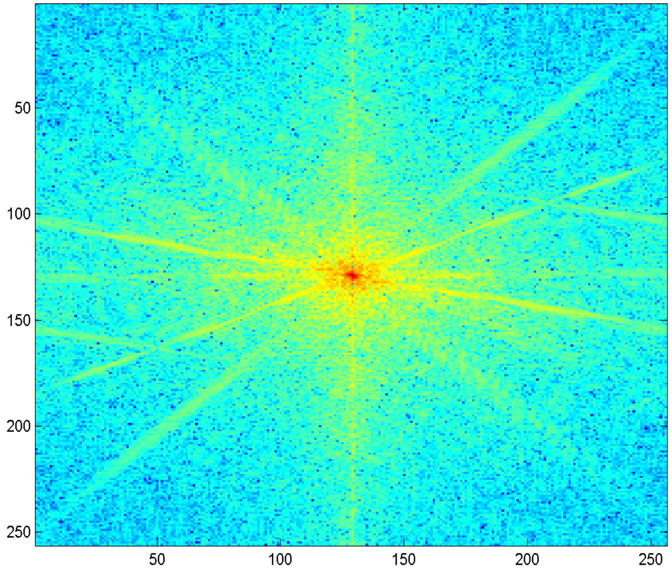


Amplitude and Log of the Amplitude





Original Image and Log of the Amplitude



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DFT properties: Separability

$$F(u,v) = \sum_{x=0}^{M-1} e^{-j2\pi ux/M} \sum_{y=0}^{N-1} f(x,y) e^{-j2\pi vy/N} \Rightarrow$$

$$G(x,v) = \sum_{y=0}^{N-1} f(x,y) e^{-j2\pi vy/N}$$

$$F(u,v) = \sum_{x=0}^{M-1} e^{-j2\pi ux/M} G(x,v)$$

DFT properties: Separability

$$G(x,v) = \sum_{y=0}^{N-1} f(x,y) e^{-j2\pi vy/N}$$

$$= N \cdot \left(\frac{1}{N} \sum_{y=0}^{N-1} f(x,y) e^{-j2\pi vy/N} \right)$$

$N \times$ DFT of rows of $f(x.y)$

$$F(u,v) = M \left(\frac{1}{M} \sum_{x=0}^{M-1} G(x,v) e^{-j2\pi ux/M} \right)$$

$M \times$ DFT of columns of $G(x,v)$

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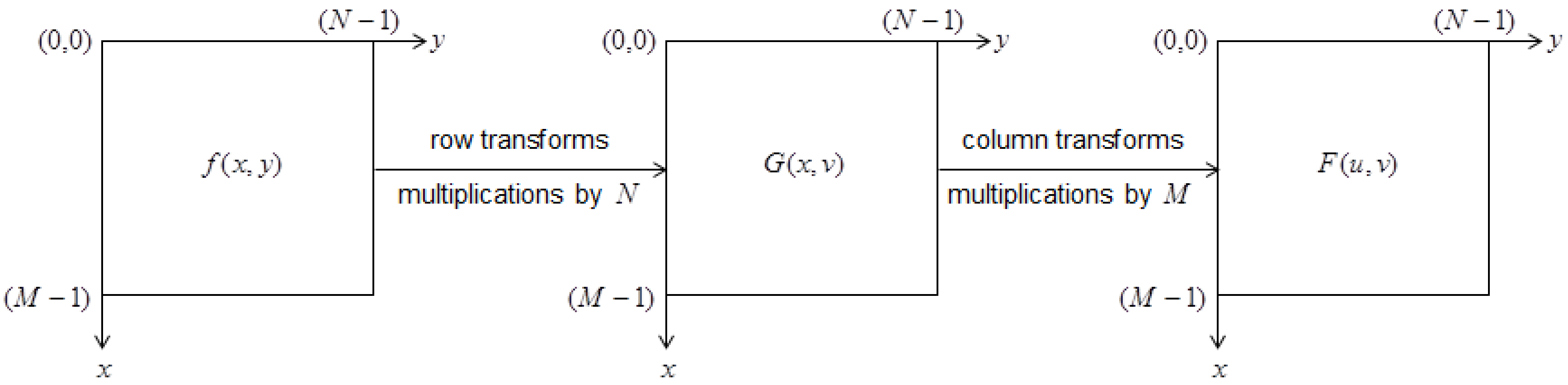
DFT properties: Separability



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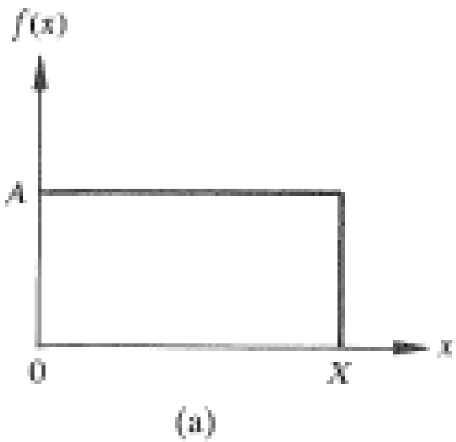
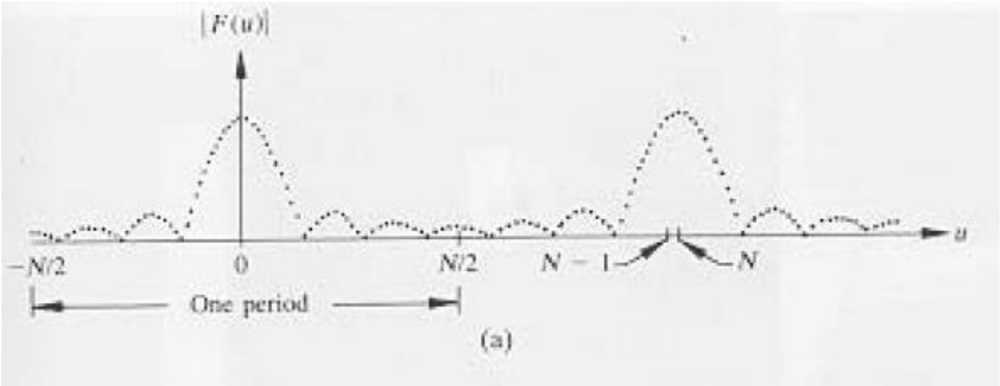
DFT properties: Separability

The DFT and its inverse are periodic.

$$F(u,v) = F(u+M,v) = F(u,v+N) = F(u+M,v+N)$$





DFT properties: Conjugate Symmetry

If $F(m,n)$ is an image of size $M \times N$, then:

$$F(u,v) = F^*(-u + pM, -v + qN), \text{ with } p,q \text{ any integers}$$

$$\Rightarrow |F(u,v)| = |F(-u,-v)|$$

$$f(x,y) \text{ real and even} \Leftrightarrow F(u,v) \text{ real and even}$$

$$f(x,y) \text{ real and odd} \Leftrightarrow F(u,v) \text{ imaginary and odd}$$

DFT properties: Translation

Translation in spatial domain:

$$f(x - x_0, y - y_0) \leftrightarrow F(u,v)e^{-j2\pi(u_0 x/M + v_0 y/N)}$$

Translation in frequency domain:

$$f(x,y)e^{j2\pi(u_0 x/M + v_0 y/N)} \leftrightarrow F(u - u_0, v - v_0)$$

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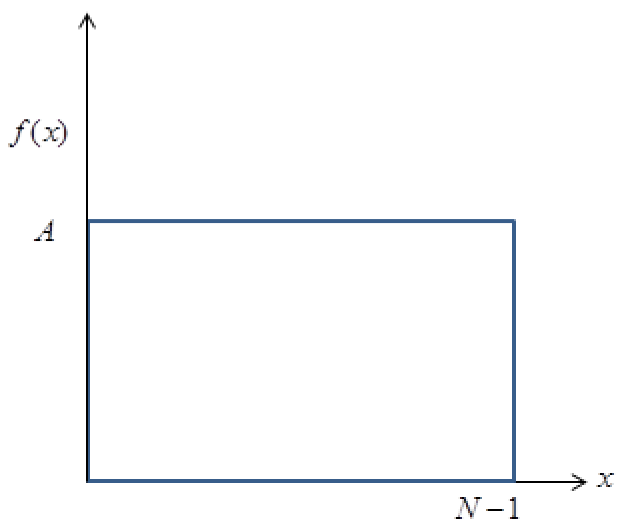
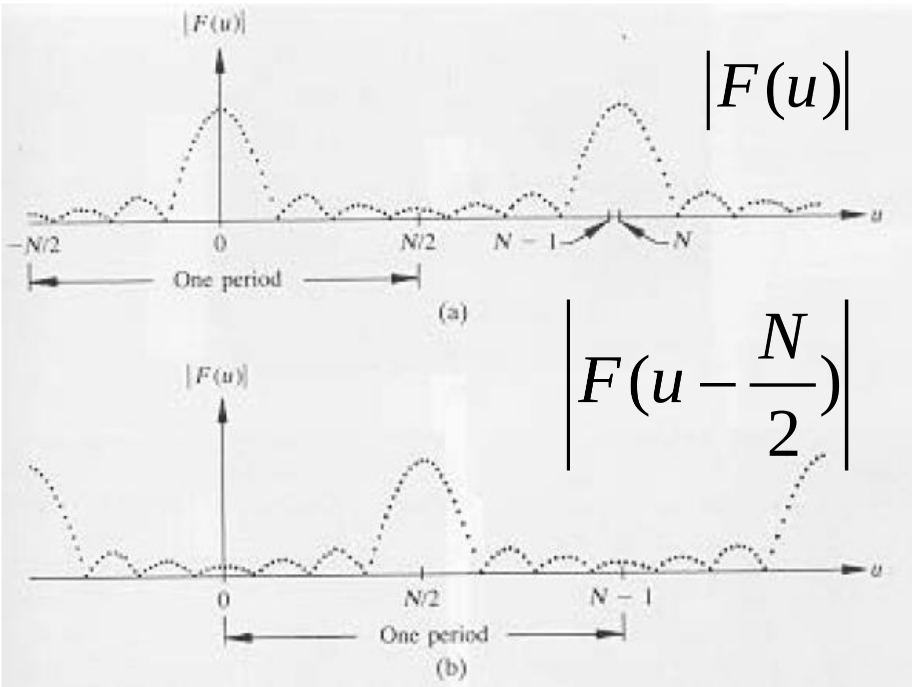
DFT properties: Translation

Warning: to show a full period, we need to translate the origin of the transform at $(u, v) = (M/2, N/2)$





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DFT properties: Translation

To move $F(u,v)$ at $(M/2, N/2)$

replace $u_0 = M/2$ and $v_0 = N/2$

In that case

$$e^{j2\pi(u_0 x/M + v_0 y/N)} = e^{j\pi(x+y)} = (-1)^{(x+y)}$$

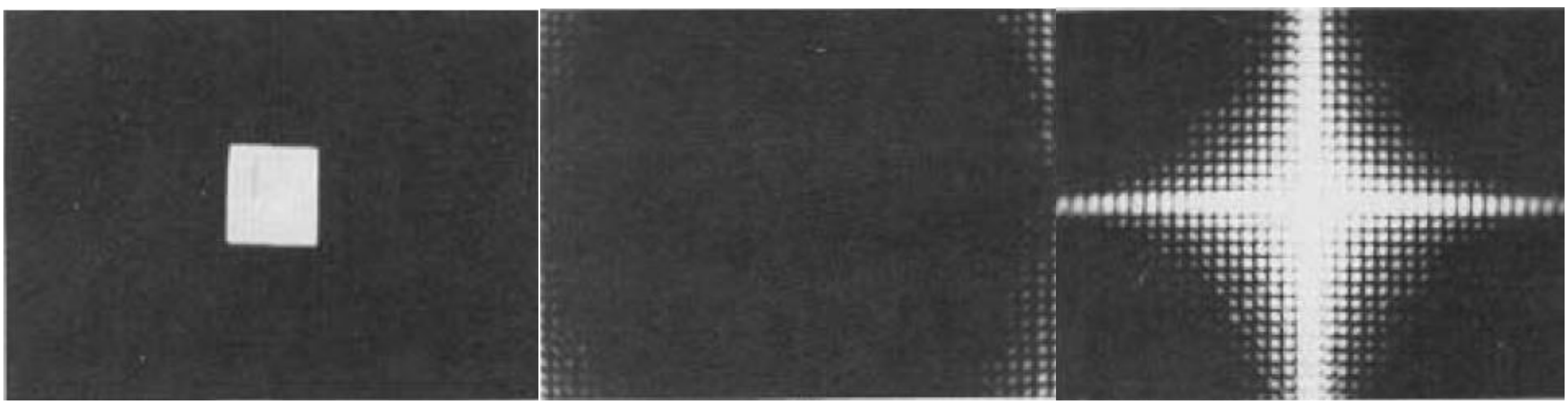
Using

$$f(x,y)e^{j2\pi(u_0 x/M + v_0 y/N)} \leftrightarrow F(u-u_0, v-v_0) \Rightarrow$$

$$f(x,y)(-1)^{(x+y)} \leftrightarrow F(u-M/2, v-n/2)$$

DFT properties: Translation

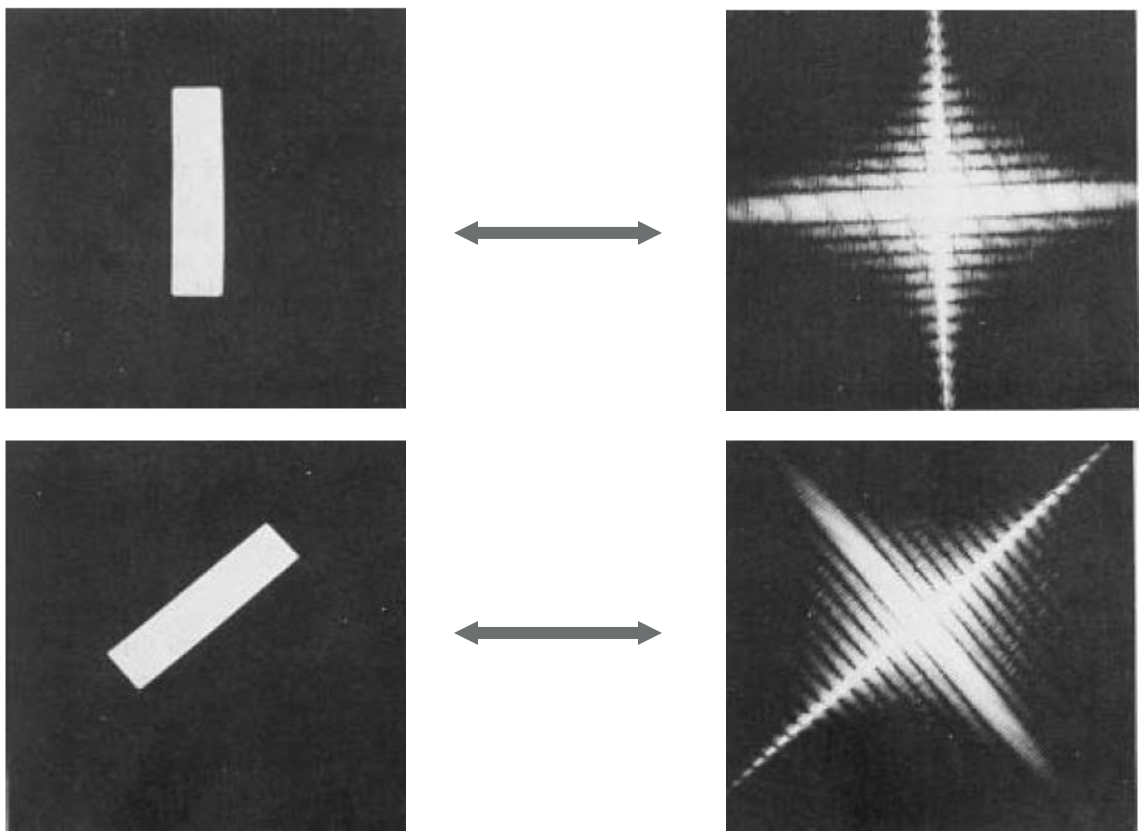
$$f(x,y)(-1)^{(x+y)} \leftrightarrow F(u-M/2, v-n/2)$$



without translation after translation

DFT properties: Rotation

rotating $f(x,y)$ by θ rotates $F(u,,v)$ by θ



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DFT properties: Addition and Multiplication

$$\Im[f(x,y)+g(x,y)]=\Im[f(x,y)]+\Im[g(x,y)]$$

$$\Im[f(x,y)\cdot g(x,y)]\neq\Im[f(x,y)]\cdot\Im[g(x,y)]$$

where $\Im[\cdot]$ is the Fourier transform

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DFT properties: Average value of the signal

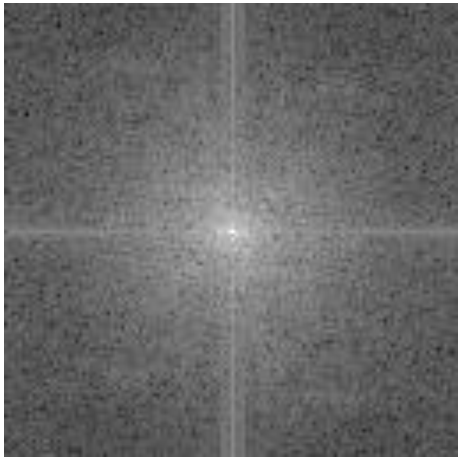
Average value of the image:

$$\bar{f}(x,y) = \frac{1}{MN}\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)$$

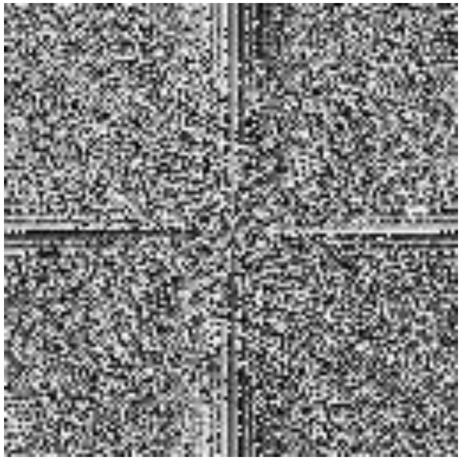
$$F(0,0) = \sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y) \Rightarrow$$

$$\bar{f}(x,y) = \frac{1}{MN} F(0,0)$$

Original Image



Fourier Amplitude



Fourier Phase

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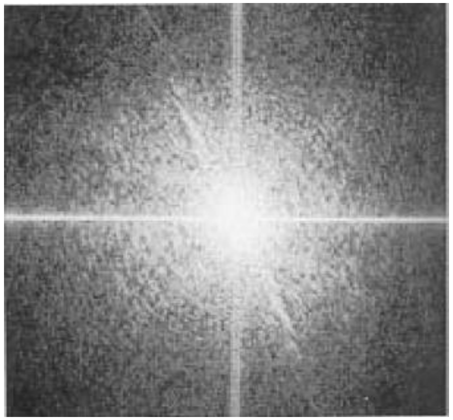
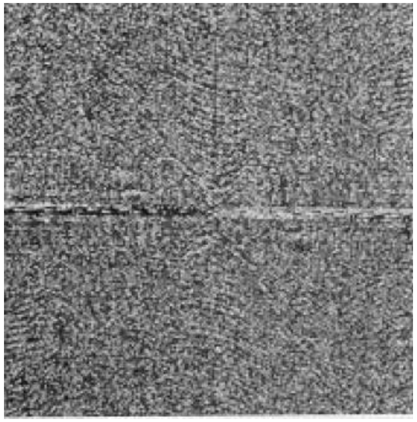
Magnitude and Phase of DFT

What is more important?



magnitude



phase

<u>Hint:</u> use inverse DFT to reconstruct the image using magnitude or phase only information

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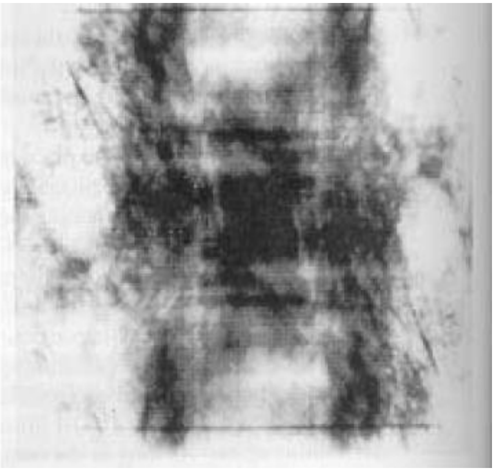
Magnitude and Phase of DFT



Reconstructed image using magnitude only

(i.e., magnitude determines the contribution of each component!)

Reconstructed image using phase only

(i.e., phase determines which components are present!)

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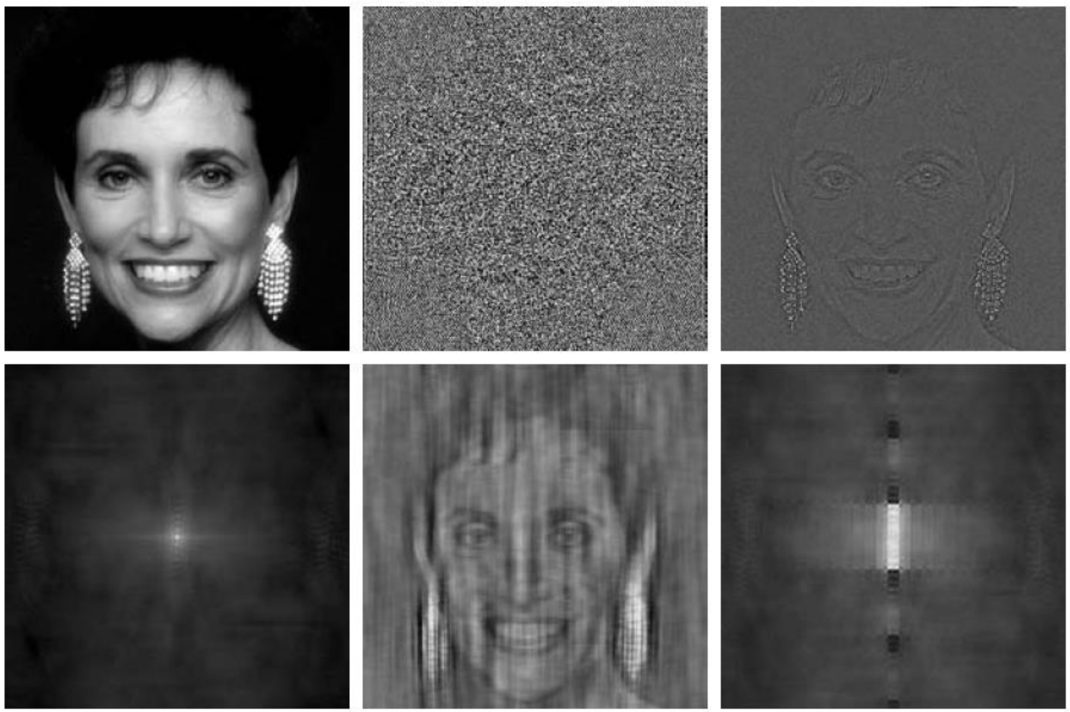
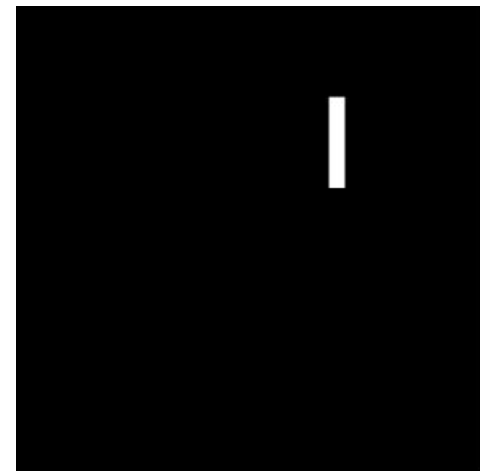
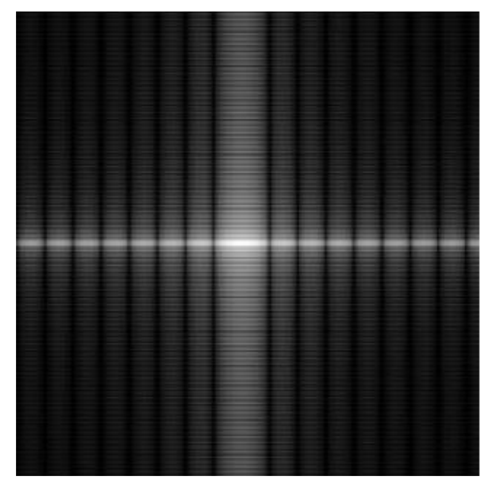
Magnitude and Phase of DFT



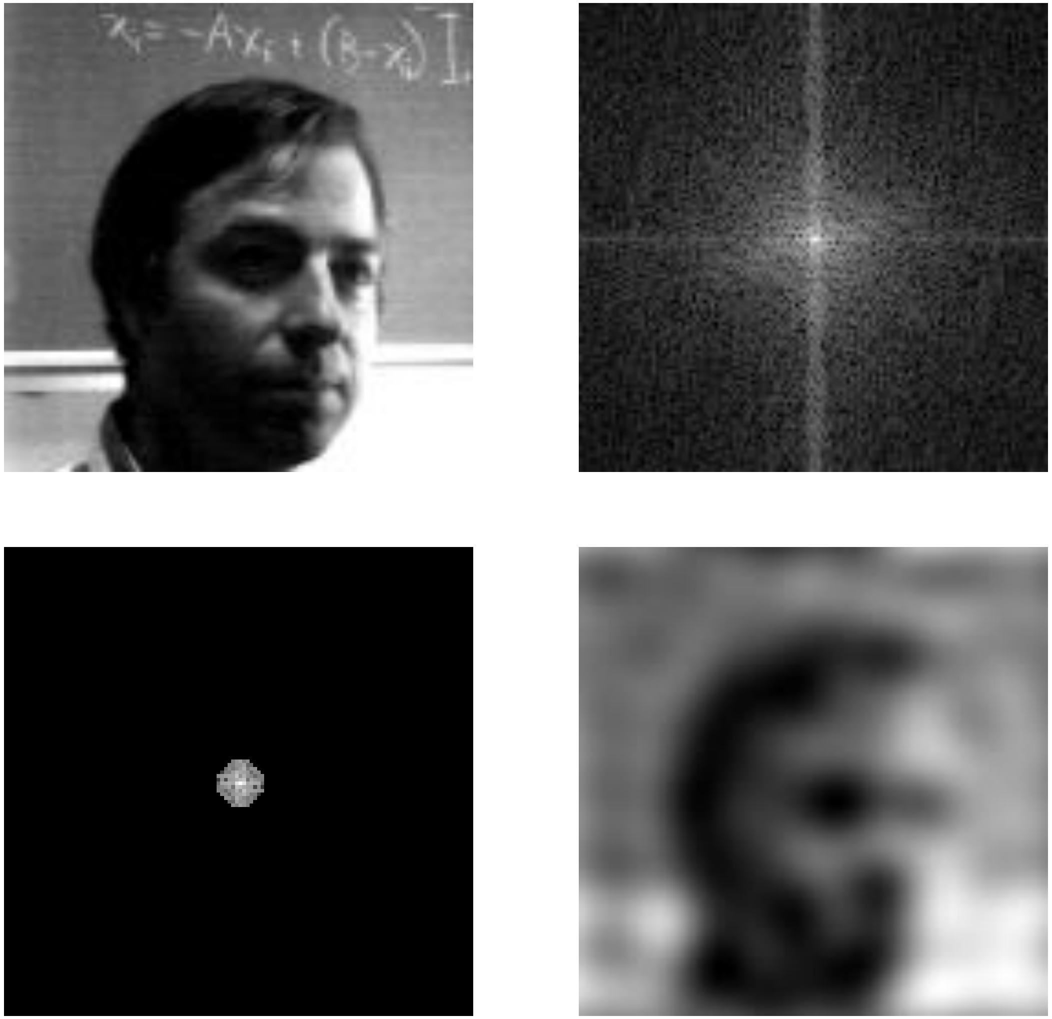
a b c
d e f

FIGURE 4.27 (a) Woman. (b) Phase angle. (c) Woman reconstructed using only the phase angle. (d) Woman reconstructed using only the spectrum. (e) Reconstruction using the phase angle corresponding to the woman and the spectrum corresponding to the rectangle in Fig. 4.24(a). (f) Reconstruction using the phase of the rectangle and the spectrum of the woman.





Low pass filtering using DFT



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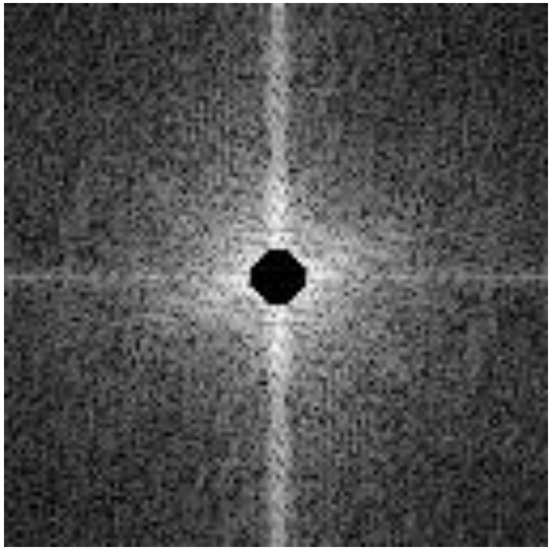
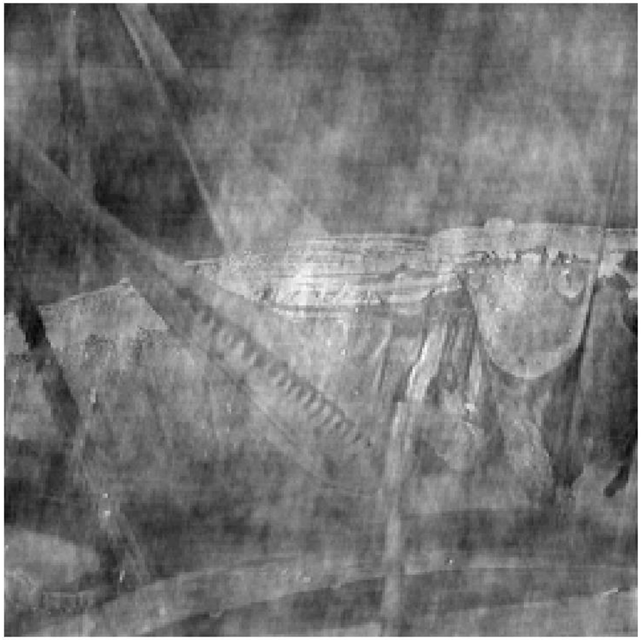
High pass filtering using DFT



Experiment: Verify the importance of phase in images

Reconstruction from phase of one image and amplitude of the other

phase of cameraman
amplitude of grasshopper



phase of grasshopper
amplitude of cameraman

Experiment: Verify the importance of phase in images

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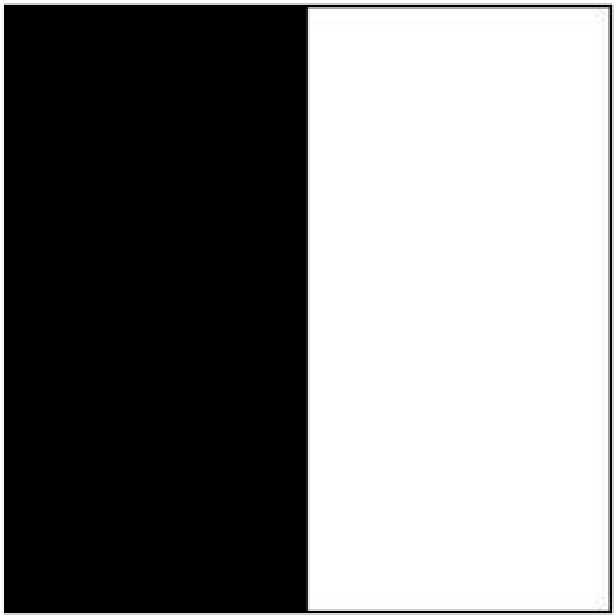
Reconstruction from phase of one image and amplitude of the other

phase of buffalo
amplitude of rocks

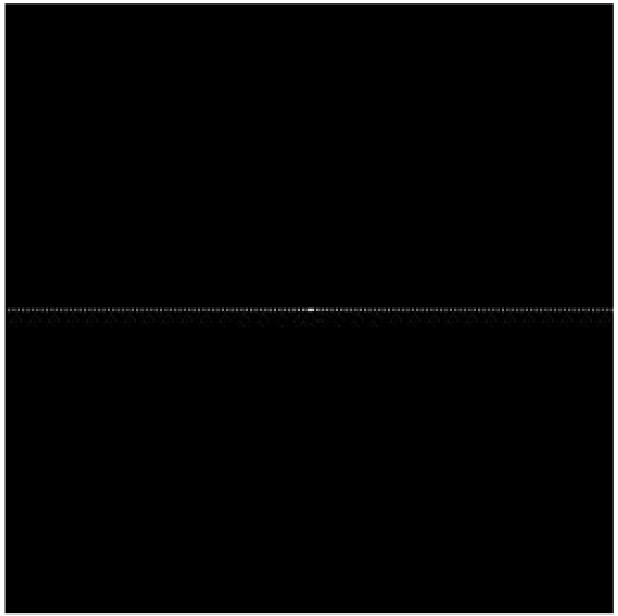
phase of rocks
amplitude of buffalo

DFT of a single edge

- Consider DFT of image with single edge.
- For display, DC component is shifted to the centre.
- Log of magnitude of Fourier Transform is displayed



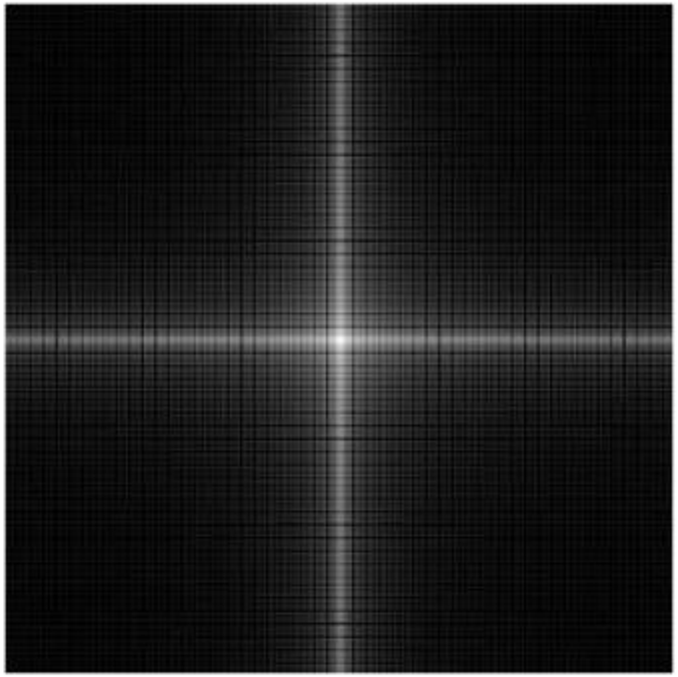
Image



DFT

DFT of a box

Box



DFT

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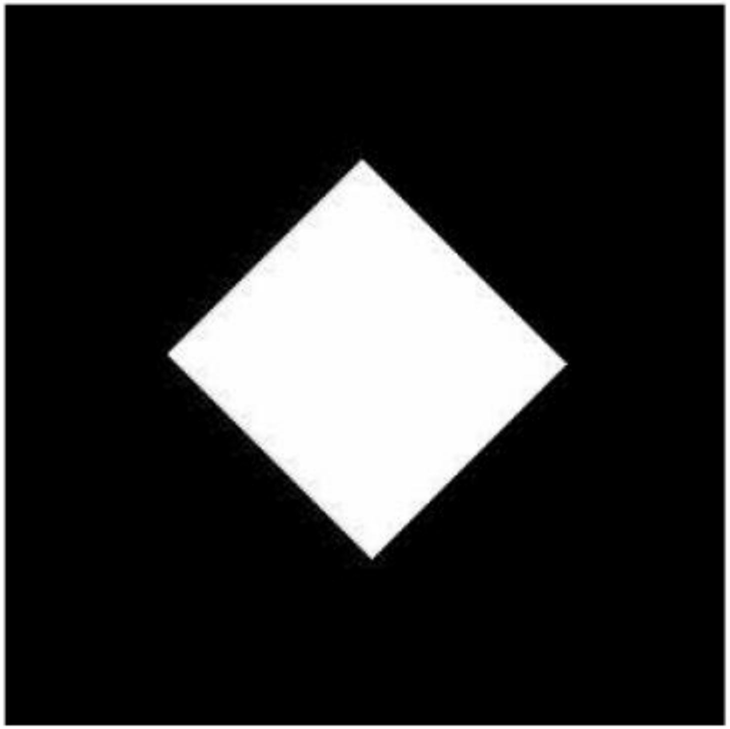
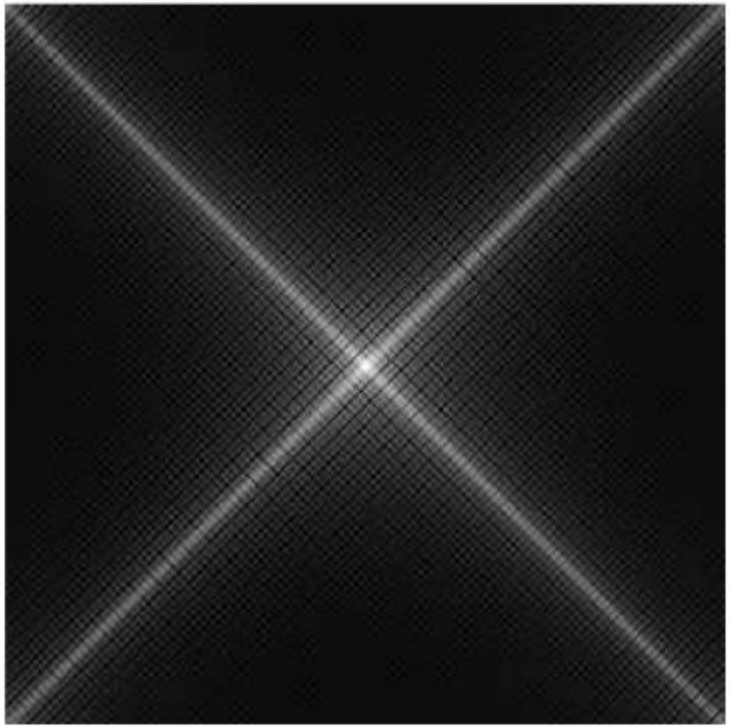
DFT of rotated box



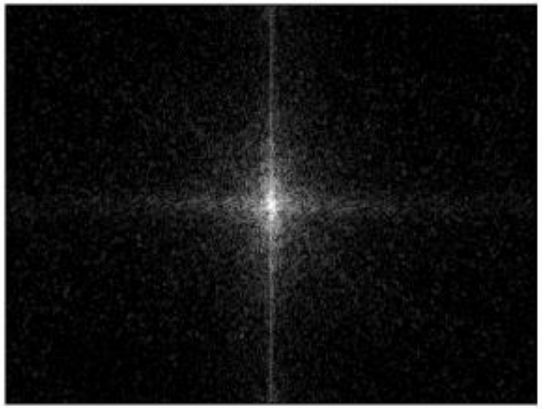
Rotated Box



DFT

DFT computation: Extended image

- DFT computation assumes image repeated horizontally and vertically.
- Large intensity transition at edges = vertical and horizontal line in middle of spectrum after shifting.



Windowing

- Can multiply image by windowing function before DFT to reduce sharp transitions between borders of repeated images.
- Ideally, causes image to drop off towards ends, reducing transitions