Sampling Moments and Reconstructing Signals with Finite Rate of Innovation: Shannon meets Strang-Fix*

Pier Luigi Dragotti

Communications and Signal Processing Group
Imperial College London

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**Problem Statement and Motivation**

Given the samples \( y_n = \langle \varphi(t/T - n), x(t) \rangle \), we want to reconstruct \( x(t) \).

Natural questions:

- What signals can be sampled?
- What kernels \( \varphi(t) \) can be used?
- What reconstruction algorithm?
Problem Statement and Motivation

Why sampling?

- Related to the notion of sparsity of a signal which is important in signal processing applications.
- Important for hybrid digital/analog processing.
- Crucial in spread-spectrum communications (Nyquist is not good enough!).
- Related to resolution enhancement and super-resolution (very important in image processing).
Problem Statement and Motivation

Classical Sampling Formulation:

- Sampling of $x(t)$ is equivalent to projecting $x(t)$ into the shift-invariant subspace $V = \text{span}\{\varphi(t/T - n)\}_{n \in \mathbb{Z}}$.
- If $x(t) \in V$, perfect reconstruction is possible.
- Reconstruction process is linear: $\hat{x}(t) = \sum_n y_n \varphi(t/T - n)$.
- For bandlimited signals $\varphi(t) = \text{sinc}(t)$.
Signals with Finite Rate of Innovation

What is so special about those signals?
The signal $x(t) = \sum_n y_n \varphi(t/T - n)$ has a finite number $\rho = 1/T$ of degrees of freedom per unit of time.

**Definition** [VetterliMB:02] The number $\rho$ of degrees of freedom per unit of time is called rate of innovation. A signal with a finite $\rho$ is called signal with **finite rate of innovation**.

**Intuition** If the number of samples $y_n$ per unit of time is greater or equal to the number of degrees of freedom, we can reconstruct $x(t)$ from the $y_n$s.
Signals with Finite Rate of Innovation

Notice: Many signals that do not belong to a shift-invariant subspace have finite rate of innovation.

Streams of Diracs or piecewise polynomials are signals with finite rate of innovation (e.g., a stream of Diracs with $K$ Diracs has $2K$ degrees of freedom).

See for instance [VetterliMB:02], where either the sinc or the Gaussian kernel is used.
Possible classes of kernels (we want to be as general as possible):

1. Any kernel \( \varphi(t) \) that can reproduce polynomials:

\[
\sum_n c_{m,n} \varphi(t - n) = t^m \quad m = 0, 1, \ldots, N
\]

2. Any kernel \( \varphi(t) \) that can reproduce exponentials:

\[
\sum_n c_{m,n} \varphi(t - n) = e^{a_mt} \quad a_m = \alpha_0 + m\lambda \text{ and } m = 0, 1, \ldots, N.
\]

3. Any kernel with rational Fourier transform:

\[
\hat{\varphi}(\omega) = \frac{\prod_i (j\omega - b_i)}{\prod_m (j\omega - a_m)} \quad a_m = \alpha_0 + m\lambda \text{ and } m = 0, 1, \ldots, N.
\]
**Sampling Kernels**

**Class 1** is made of all functions satisfying Strang-Fix conditions. This includes any scaling function generating a wavelet with $N + 1$ vanishing moments (e.g., Splines and Daubechies scaling functions).

**Class 2** contains any composite kernel of the form $\varphi(t) \ast \beta_{\alpha}(t)$ where $\beta_{\alpha}(t) = \beta_{\alpha_0}(t) \ast \beta_{\alpha_1}(t) \ast \ldots \ast \beta_{\alpha_N}(t)$ and $\beta_{\alpha_i}(t)$ is an Exponential Spline of first order [UnserB:05].

\[ \beta_{\alpha}(t) \Leftrightarrow \hat{\beta}(\omega) = \frac{1 - e^{\alpha-j\omega}}{j\omega - \alpha} \]

Notice:
- $\alpha$ can be complex.
- E-Spline is of compact support.
- E-Spline reduces to the classical polynomial spline when $\alpha = 0$
Sampling Kernels

Class 3 includes any linear differential acquisition device. That is, any device for which the input and output are related by linear differential equations. This includes most electrical, mechanical or acoustic systems.

A class 3 kernel $\hat{\phi}(\omega) = \frac{\prod_i (j\omega - b_i)}{\prod_m (j\omega - a_m)}$ can be converted into the class 2 kernel $\prod_i (j\omega - b_i) \hat{\beta}_\alpha(\omega)$ by filtering the samples $y_n = \langle x(t), \phi(t - n) \rangle$ with an FIR filter $H(z) = \prod_{m=0}^N (1 - e^{a_m} z)$.

Example:
Assume $\hat{\phi}(\omega) = \frac{1}{j\omega - \alpha}$ and $y_n = \langle x(t), \phi(t - n) \rangle$. Then

$$z_n = h_n \ast y_n = y_n - e^{\alpha} y_{n+1} = \langle x(t), \varphi(t - n) - e^{\alpha} \varphi(t - n - 1) \rangle \stackrel{(a)}{=} \langle x(t), \beta_\alpha(t - n) \rangle,$$

where (a) follows from: $\varphi(t - n) - e^{\alpha} \varphi(t - n - 1) \iff e^{-j\omega n (1 - e^{\alpha - j\omega})} = e^{-j\omega n \hat{\beta}_\alpha(\omega)}$. 
Sampling Streams of Diracs

Assume that $x(t)$ is a stream of $K$ Diracs:
$$x(t) = \sum_{k=0}^{K-1} a_k \delta(t - t_k)$$
and assume for simplicity $T = 1$.

Problem: Given the samples $y_n = \langle x(t), \varphi(t - n) \rangle$, how can we find the locations and the amplitudes of the Diracs?

Assume that $\varphi(t)$ can reproduce polynomials of maximum degree $N \geq 2K - 1$:
$$\sum_n c_{m,n} \varphi(t - n) = t^m \quad m = 0, 1, \ldots, N$$
Sampling Streams of Diracs

Call $s[m] = \sum_n c_{m,n} y_n$, $m = 0, 1, \ldots, N$, we have that

$$s[m] = \sum_n c_{m,n} y_n$$

$$= \langle x(t), \sum_n c_{m,n} \phi(t - n) \rangle$$

$$= \int_{-\infty}^{\infty} x(t) t^m dt$$

$$= \sum_{k=0}^{K-1} a_k t_k^m \quad m = 0, 1, \ldots, N$$

(1)

We thus observe $s[m] = \sum_{k=0}^{K-1} a_k t_k^m$, $m = 0, 1, \ldots, N$.

It is possible to retrieve the amplitudes $a_k$ and the locations $t_k$ of the Diracs using the annihilating filter method.
Sampling Streams of Diracs

\[ \sum_n y_n = \langle a_0 \delta(t-t_0), \sum_n \varphi(t-n) \rangle = \int_{-\infty}^{\infty} a_0 \delta(t-t_0) \sum_n \varphi(t-n) dt = a_0 \sum_n \varphi(t_0-n) = a_0 \]

\[ \sum_n c_{1,n} y_n = \langle a_0 \delta(t-t_0), \sum_n c_{1,n} \varphi(t-n) \rangle = a_0 \sum_n c_{1,n} \varphi(t_0-n) = a_0 t_0 \]
Sampling Streams of Diracs

\[ s[0] = \sum_n y_n = a_0 + a_1 \]

\[ s[1] = \sum_n c_{1,n} y_n = a_0 t_0 + a_1 t_1 \]

\[ s[2] = \sum_n c_{2,n} y_n = a_0 t_0^2 + a_1 t_1^2 \]

\[ s[3] = \sum_n c_{3,n} y_n = a_0 t_0^3 + a_1 t_1^3 \]
The Annihilating Filter Method

1. Find the filter $h[m]$ such that $h[m] \ast s[m] = 0$.

Call $h[m]$ the filter with $z$-transform $H(z) = \prod_{k=0}^{K-1} (1 - t_k z^{-1})$.

We have that $h[m] \ast s[m] = \sum_{i=0}^{K} h[i] s[m - i] = 0$. In matrix/vector form

\[
\begin{bmatrix}
    s[K - 1] & s[K - 2] & \cdots & s[0] \\
    s[K] & s[K - 1] & \cdots & s[1] \\
    \vdots & \vdots & \ddots & \vdots \\
    s[N - 1] & s[N - 2] & \cdots & s[N - K]
\end{bmatrix}
\begin{bmatrix}
    h[1] \\
    h[2] \\
    \vdots \\
    h[K]
\end{bmatrix}
\end{bmatrix}
= -
\begin{bmatrix}
    s[K] \\
    s[K + 1] \\
    \vdots \\
    s[N]
\end{bmatrix}.
\]

Classic Yule-Walker system. Unique solution for distinct Diracs.
The Annihilating Filter Method

2. Find the roots of $H(z)$, this gives us the locations $t_k$.

3. Solve the first $K$ equations in $s[m] = \sum_{k=0}^{K-1} a_k t_k^m$, this gives us the amplitudes $a_k$. In matrix/vector form

$$
\begin{bmatrix}
1 & 1 & \cdots & 1 \\
t_0 & t_1 & \cdots & t_{K-1} \\
\vdots & \vdots & \ddots & \vdots \\
t_0^{K-1} & t_1^{K-1} & \cdots & t_{K-1}^{K-1}
\end{bmatrix}
\begin{pmatrix}
a_0 \\
a_1 \\
\vdots \\
a_{K-1}
\end{pmatrix}
= 
\begin{pmatrix}
s[0] \\
s[1] \\
\vdots \\
s[K-1]
\end{pmatrix}.
$$

(2)

Classic Vandermonde system. Unique solution for distinct $t_k$s.
Sampling Streams of Diracs

Notice:

- The kernels we consider have, in general, compact support. Thus, $x(t)$ can have more than $K$ Diracs as far as groups of $K$ Diracs are far away so that they do not influence each other.
- Interesting trade-off between local rate of innovation and complexity in the reconstruction process.
Sampling Streams of Diracs

Proposition 1. Assume a sampling kernel $\varphi(t)$ that can reproduce polynomials of maximum degree $N \geq 2K - 1$ and of compact support $L$. An infinite-length stream of Diracs $x(t) = \sum_n a_n \delta(t - t_n)$ is uniquely determined from the samples defined by $y_n = \langle \varphi(t/T - n), x(t) \rangle$ if there are at most $K$ Diracs in an interval of size $KLT$. 
**Sampling Piecewise Constant Signals**

**Insight:** the derivative of a piecewise constant signal is a stream of Diracs. Thus, if we can compute the derivative of piecewise constant signals, we can sample them.

Given the samples $y_n$ compute the finite difference $z_n^{(1)} = y_{n+1} - y_n$, we have that

$$z_n^{(1)} = y_{n+1} - y_n = \langle x(t), \varphi(t - n - 1) - \varphi(t - n) \rangle$$

$$= \frac{1}{2\pi} \langle X(\omega), \hat{\varphi}(\omega) e^{-j\omega n} (e^{-j\omega} - 1) \rangle$$

$$= \frac{1}{2\pi} \langle X(\omega), -j\omega \hat{\varphi}(\omega) e^{-j\omega n} \left( \frac{1 - e^{-j\omega}}{j\omega} \right) \rangle$$

$$= -\langle x(t), \frac{d}{dt} [\varphi(t - n) \ast \beta_0(t - n)] \rangle$$

$$= \langle \frac{dx(t)}{dt}, \varphi(t - n) \ast \beta_0(t - n) \rangle.$$ 

Thus the samples $z_n^{(1)}$ are related to the derivative of $x(t)$. 

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Sampling Piecewise Constant Signals

\[ T \beta \phi(t) \]

\[ x(t) \]

\[ \varphi(t) \]

\[ \frac{dx(t)}{dt} \]

\[ \varphi(t) \ast \beta_0(t) \]

\[ y_n \rightarrow z_n = y_n - y_{n-1} \]
The same argument can be extended to piecewise polynomial signals and this leads to the following result.

**Proposition 2.** Assume a sampling kernel \( \varphi(t) \) of compact support \( L \) and that can reproduce polynomials of maximum degree \( N \). An infinite-length piecewise polynomial signal with pieces of maximum degree \( M - 1 \) is uniquely defined by the samples \( y_n = \langle \varphi(t/T - n), x(t) \rangle \) if there are at most \( K + 1 \) polynomials in an interval of size \( (L + M)KT \) and \( 2KM - 1 \leq (M + N) \).
Coarse Approximations and Infinite Resolutions

We can sample piecewise polynomial functions using scaling functions. Recall that

$$x(t) = \sum_{n=-\infty}^{\infty} y_{J,n} \varphi_{J,n}(t) + \sum_{m=-\infty}^{J} \sum_{n=-\infty}^{\infty} d_{m,n} \psi_{m,n}(t).$$

Given a finite resolution version of $x(t)$: $x_J(t) = \sum_{n=-\infty}^{\infty} y_{J,n} \varphi_{J,n}(t)$, we can arbitrarily increase the resolution of $x_J(t)$ to eventually recover $x(t)$. 
Coarse Approximations and Infinite Resolutions

(a) Original discrete-time piecewise linear signal (128 samples). (b) Sample values obtained with a Daubechies filter with two vanishing moments (16 samples). (c) Coarse reconstruction. (d) Reconstruction with annihilating filter method using the 16 samples of (b).
**Footprints**

**Insight:** all the wavelet coefficients across scales generated by a Dirac are dependent. They have only one degree of freedom.

Footprints [DragottiV:03] model this dependency precisely.

**Definition 1.** Given is a stream of Diracs with only one Dirac at location $t_k$, we call footprint $f_{t_k}$ the scale-space vector obtained by gathering together all the wavelet coefficients in the cone of influence of $t_k$ and then imposing $\|f_{t_k}(t)\| = 1$. 
Expressed in terms of wavelets, this footprint can be written as

\[ f_{t_k}(t) = \sum_{j=-\infty}^{J} \sum_{n \in I_k} u_{j,n} \psi_{j,n}(t) \]

and the original signal

\[ x(t) = \sum_{n=-\infty}^{\infty} y_{J,n} \varphi_{J,n}(t) + \sum_{m=-\infty}^{J} \sum_{n=-\infty}^{\infty} d_{m,n} \psi_{m,n}(t) \]

becomes

\[ x(t) = \sum_{n=-\infty}^{\infty} y_{J,n} \varphi_{J,n}(t) + b_k f_{t_k}(t) \]
Resolution Enhancement with footprints

Increase resolution from $J_0$ to $J_1$

- Given a finite resolution version of $x(t)$:  
  $x_{J_0}(t) = \sum_{n=-\infty}^{\infty} y_{J,n} \varphi_{J,n}(t) + b_k \hat{f}_{t_k}(t)$ with $\hat{f}_{t_k}(t) = \sum_{j=J_0}^{J} \sum_{n \in I_k} u_{j,n} \psi_{j,n}(t)$

- Consider all the possible finite resolution footprints $f_{t_n}$ at locations $t_n = n2^{J_1}$

- Compute the inner products $\langle x_{J_0}(t), \hat{f}_{t_n}(t) \rangle$

- Choose the maximum inner product

- This provides the correct location and amplitude of the Dirac
Simulation Results. Noisy Case

(a) Original Signal (N=128)
(b) Original Signal (dashed) and reconstructed signal (SNR=22.9dB)

<table>
<thead>
<tr>
<th>Samples</th>
<th>8</th>
<th>16</th>
<th>32</th>
<th>64</th>
</tr>
</thead>
<tbody>
<tr>
<td>SNR</td>
<td>20.7dB</td>
<td>22.2dB</td>
<td>23.2dB</td>
<td>24.5dB</td>
</tr>
</tbody>
</table>

Effect of the oversampling in the reconstruction of a piecewise constant signal.
The Exponential Case

Assume that $x(t) = \sum_{k=0}^{K-1} a_k \delta(t - t_k)$ and that $\varphi(t)$ reproduces exponential:

$$\sum_n c_{m,n} \varphi(t - n) = e^{a_mt} \quad a_m = \alpha_0 + m\lambda \quad \text{and} \quad m = 0, 1, \ldots, N.$$

We have that

$$s[m] = \sum_n c_{m,n} y_n$$

$$= \langle x(t), \sum_n c_{m,n} \varphi(t - n) \rangle$$

$$= \int_{-\infty}^{\infty} x(t) e^{\alpha mt} dt$$

$$= \sum_{k=0}^{K-1} a_k e^{\alpha mt_k} \quad m = 0, 1, \ldots, N$$

And we can again use the annihilating filter method to reconstruct the Diracs.

Notice: if $\alpha_m = jm\omega_0$ with $m = 0, 1, \ldots, N$, then $s[m] = X(m\omega_0)$. 
We want to sample and reconstruct this signal. Why is it hard?
This signal contains innovation in both spectral and temporal domains.
We thus need both exponentials and polynomials to reconstruct it.
Sampling Piecewise Sinusoidal Signals

Call $y_n = \langle x(t), \varphi(t - n) \rangle$ the observed samples and assume $\omega_0$ is the frequency of the sinusoid.

- The samples generated by the sinusoid and that are not influenced by the discontinuities can be annihilated by $H(z) = (1 - e^{-j\omega_0 z^{-1}})(1 - e^{j\omega_0 z^{-1}})$.
- The coefficients $h_n$ of $H(z)$ can be found by solving a Yule-Walker system. We need five consecutive ‘clean’ samples $y_n$ to design the Yule-Walker system.
- All the parameters of the sinusoid are then obtained using the annihilating filter method.
- Apply $H(z)$ to the entire sequence of samples $y_n$, we have that $z_n = y_n \ast h_n = \langle x_\delta(t), \varphi(t - n) \ast \beta_\alpha(t - n) \rangle$.
- $x_\delta(t)$ is a stream of differentiated Diracs located at $t_0$ and $t_1$. If $\varphi(t)$ satisfied the hypotheses of the previous propositions, we can retrieve $t_0$ and $t_1$ from $z_n$. 
Sampling Piecewise Sinusoidal Signals

Notice

- We can reconstruct signals with more than one sinusoid per piece, but then we need a larger number of ‘clean’, consecutive samples.

- We thus lose resolution in time.

- This leads to a ‘parametric’ uncertainty principle.

- The size $I_k = t_k - t_{k-1}$ of the piece must satisfy $I_k \geq T(4N_d + L)$ where $N_d$ is the total number of sinusoids in two consecutive pieces, $L$ is the support of $\varphi(t)$ and $T$ is the sampling period.
Reconstruction of FRI signals at the output of an RC circuit

- Input: \( x(t) = Au(t - t_0) \)
- Output before sampling: \( y(t) = Au(t - t_0) - Ae^{-\alpha(t-t_0)}u(t - t_0) \) with \( \alpha = 1/RC \).
- Samples: \( y_n = \langle x(t), \varphi(t - n) \rangle = Au(n - t_0) - Ae^{-\alpha(n-t_0)}u(n - t_0) \) and \( \hat{\varphi}(\omega) = \alpha/(\alpha - j\omega) \).

We want to reconstruct \( x(t) \) from the samples \( y_n \).
Reconstruction of FRI signals at the output of an RC circuit

First, compute the following difference

\[ z_n = e^{\alpha} y_{n+1} - y_n = \langle x(t), e^{\alpha} \varphi(t - n - 1) - \varphi(t - n) \rangle \]

\[ = \frac{1}{2\pi} \langle X(\omega), \alpha e^{-j\omega n} \frac{1 - e^{\alpha - j\omega}}{(j\omega - \alpha)} \rangle = \langle x(t), \alpha \beta_\alpha(t - n) \rangle. \]

Then compute the first order difference

\[ z_n^{(1)} = z_{n+1} - z_n = \left\langle \frac{dx(t)}{dt}, \alpha \beta_\alpha(t - n) * \beta_0(t - n) \right\rangle. \]

The signal \( dx(t)/dt \) is a Dirac centered at \( t_0 \) and with amplitude \( A \). The new kernel \( \varphi_\alpha(t) = \alpha \beta_\alpha(t) * \beta_0(t) \) is of compact support and can reproduce a constant function or the exponential \( e^{\alpha t} \).

Thus \( \frac{1}{e^{\alpha} - 1} \sum_n z_n^{(1)} = A \) and \( \frac{1}{1 - e^{-\alpha}} \sum_n e^{\alpha n} z_n^{(1)} = A e^{\alpha t_0} \).
Reconstruction of FRI signals at the output of an RC circuit

Let us verify the above analysis for our specific example. In our case $y_n = A u(n - t_0) - Ae^{-\alpha(n-t_0)}u(n - t_0)$ and assume for simplicity that $t_0 \in [0, 1]$, then

$$z_n = e^\alpha y_{n+1} - y_n = \left\{ \begin{array}{ll}
0 & \text{for } n < 0 \\
A(e^\alpha - e^{\alpha t_0}) & \text{for } n = 0 \\
A(e^\alpha - 1) & \text{for } n > 0
\end{array} \right.$$ 

and

$$z_{n}^{(1)} = z_{n+1} - z_n = \left\{ \begin{array}{ll}
A(e^\alpha - e^{\alpha t_0}) & \text{for } n = -1 \\
A(e^{\alpha t_0} - 1) & \text{for } n = 0 \\
0 & \text{otherwise}
\end{array} \right.$$ 

Therefore, it is clearly true that $1/(e^\alpha - 1) \sum_n z_n^{(1)} = A$ and that $1/(1 - e^{-\alpha}) \sum_n e^{\alpha n} z_n^{(1)} = Ae^{\alpha t_0}$. 

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The 2-D case

The 2-D sampling kernel is a separable kernel given by the tensor product of two 1-D functions that can reproduce polynomials: \( \varphi_{x,y}(x, y) = \varphi(x)\varphi(y) \). Clearly,

\[
\sum_m \sum_n c_{m,n} \varphi_{x,y}(x - n, y - m) = x^k y^l.
\]

We observe the samples \( y_{n,m} = \langle f(x, y), \varphi_{x,y}(x - n, y - m) \rangle \).
2-D Sets of Diracs

We want to reconstruct $f(x, y) = \sum_{k=0}^{K-1} \alpha_k \delta(x - x_k, y - y_k)$ from the observed samples $y_{n,m} = \langle f(x, y), \varphi_{x,y}(x - n, y - m) \rangle$.

- In 1-D we used the ability of $\varphi(t)$ to reproduce polynomials to retrieve the moments

$$\tau_m = \int_{-\infty}^{\infty} x(t) t^m dt = \sum_{k=0}^{K-1} \alpha_k t_k^m$$

of the signal $x(t)$. Then used the annihilating filter method.

- In 2-D we simply need to retrieve the complex moments

$$\tau_m = \int \int f(x, y) z^m dx dy = \int \int f(x, y)(x + jy)^m dx dy = \sum_{k=0}^{K-1} \alpha_k z_k^m$$

Then use the annihilating filter method to retrieve the locations $z_k = (x_k + jy_k)$ and the amplitudes $\alpha_k$. 
The same applies to polygonal images. Assume the polygon has $N$ vertices. It follows that [Davis:64],[MilanfarKVW:95]:

$$\tau_m = m(m-1) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) z^{m-2} dxdy = \int \int_{\Omega} z^{m-2} dxdy = \sum_{k=1}^{N} \rho_k z_k^m$$

Thus, from the samples $y_{n,m}$ we can estimate the complex moments and, from the moments, using the annihilating filter, the locations $z_k$ of the vertices.
Sampling Polygonal Images. Example

(a) (b) (c)

Notice: The polygon must be convex to guarantee uniqueness in the reconstruction.
Sampling Polygonal Images.

Since the kernel is of compact support, more polygons can be reconstructed independently.
Application: Resolution Enhancement of ‘Real’ Images

We can enhance the resolution of some real images.
Application: Image Super-Resolution

One hundred low-resolution and shifted versions of the original. Accurate registration is achieved by retrieving the continuous moments of the earth from the samples. The registered images are interpolated to achieve super-resolution.
Conclusions

Good News:

• We now have a large class of signals to sample: Signals with Finite Rate of Innovations.
• We now have a rich choice of kernels.
• Huge potential impact in signal processing and communications. But more work needs to be done!

But also

• At the moment, we are not able to sample any FRI signal, more research needs to be done!
• Noisy scenarios needs to be investigated.
• Wavelet community likes polynomials, Fourier community likes exponentials. There is room for research at the intersection.
• These results go beyond sampling. The discrete and analog worlds are not so distant. ‘Act Digital, but think analog!’
• On sampling

• On Image Super-Resolution
  – L. Baboulaz and P.L. Dragotti ‘Distributed Acquisition and Image Super-Resolution based on Continuous Moments from Samples’, International Conference of Image Processing (ICIP), Atlanta, USA, October 2006.

• On Wavelet Footprints