SHAPES FROM SAMPLES USING MOMENTS AND RADON PROJECTIONS

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ABSTRACT

In this paper, we extend the results on sampling the signals with finite rate of innovation (FRI) [1] in *higher dimensions* using kernels that reproduce polynomials [2]. In particular, we show that convex-bilevel polygons, 2-D Diracs, and quadrature domains (e.g. ellipses and cardioids) can be perfectly reconstructed from their samples using complex-moments based approach [3], while more general shapes (signals) such as piecewise polynomials with polygonal boundaries, and n-dimensional Diracs and bilevel polytopes can be reconstructed from their samples by integrating the Radon-moment connection [3] with the annihilating-filter-based-back-projection (AFBP) reconstruction [4].

1. INTRODUCTION

Shannon's sampling theory allows perfect reconstruction of bandlimited signals. However, the recent developments in sampling of signals with finite rate of innovation (FRI) suggest that many non-bandlimited signals such as Diracs and piecewise polynomials are perfectly reconstructed from their samples using a range of sampling kernels [1, 2].

In particular, a compactly supported kernel that satisfies the Strang-Fix conditions can reproduce polynomials, and therefore, the samples obtained with such kernel are sufficient to retrieve the moments of the original signal. Using a finite number of these moments in the annihilating filter method, one can perfectly reconstruct the original FRI signal [2]. In this paper, we extend the sampling results of [2] in 2-D and above. In the first part of the following section, we show how to obtain the (geometric and complex) moments of a signal from its samples. Then, in the following parts, using complex-moments and Radon transform, we present a method to recover FRI signals from their samples.

2. MATERIALS AND METHODS

Consider a 2-D generic sampling setup, where the original signal is first prefiltered with a lowpass sampling kernel and then sampled uniformly to obtain a set of 2-D samples. Note that this setup is valid in n-D as well. Since we assume that the sampling kernel satisfies the Strang-Fix conditions, it can reproduce polynomials (e.g. B-splines, scaling functions). The polynomial reproduction property of the kernel allows us to obtain higher order moments of the original signal from the linear combinations of its samples. To be more precise, a kernel that reproduces polynomials up to degree N allows us to reproduce the (geometric) moments up to order N. Moreover, it is straightforward to obtain various types of complex-moments, e.g. weighted complex-moments up to order $N+2$ using binomial combinations of the geometric moments up to order N.

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The results of [3] state that the convex-bilevel polygons and quadrature domains (e.g. ellipses and cardioids) are uniquely determined from a finite number of complex-moments. Therefore, we show that these signals are also uniquely reconstructed from their samples. For example, we show that a bilevel and convex polygon with N corner points (i.e. with a rate of innovation equal to $2N$) can be reconstructed from its samples using a kernel that can reproduce polynomials up to degree $2N-3$ (or in other worlds, using $2N$ complex-moments). In a similar manner, we show that 2-D Diracs and quadrature domains can be reconstructed from their samples using kernels that reproduce higher order moments.

From the Radon-moment connection of [3], we know that the higher order moments of certain signals (e.g. 2-D Diracs and bilevel polygons) are sufficient to retrieve their Radon projections at various angles. Moreover, from [4], we know that by back-projecting a finite number of such projections, it is possible to uniquely determine the original signals. By integrating these results with the fundamentals of lattice theory and bivariate polynomial interpolation, we show that a more general signal such as piecewise polynomial of degree $R-1$ within an N sided convex polygonal closure can be uniquely reconstructed from its samples using only $N+1$ Radon-projections, given that the sampling kernel can reproduce polynomials up to degree $(2N-1)(R+1)-1$ and that the condition $2R-1 \leq N+1$ is satisfied. Finally, by exploiting the multidimensional property of the Radon transform, we show that an n-dimensional bilevel polytope with N corner points (or a set of N n-D Diracs) can be reconstructed from its samples using $(N+1)^{n-1}$ Radon projections in 1-D.

3. CONCLUSION

In this paper, we have presented multidimensional sampling of FRI signals using kernels that reproduce polynomials. Implicitly, we have also provided a sampling perspective to the complex-moments based approach of [3], and an n-D extension to the AFBP reconstruction of [4].

4. REFERENCES

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