Digital Filter Design

Supplement to Lecture Notes on FIR Filters

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Frequency Response of Digital Filters

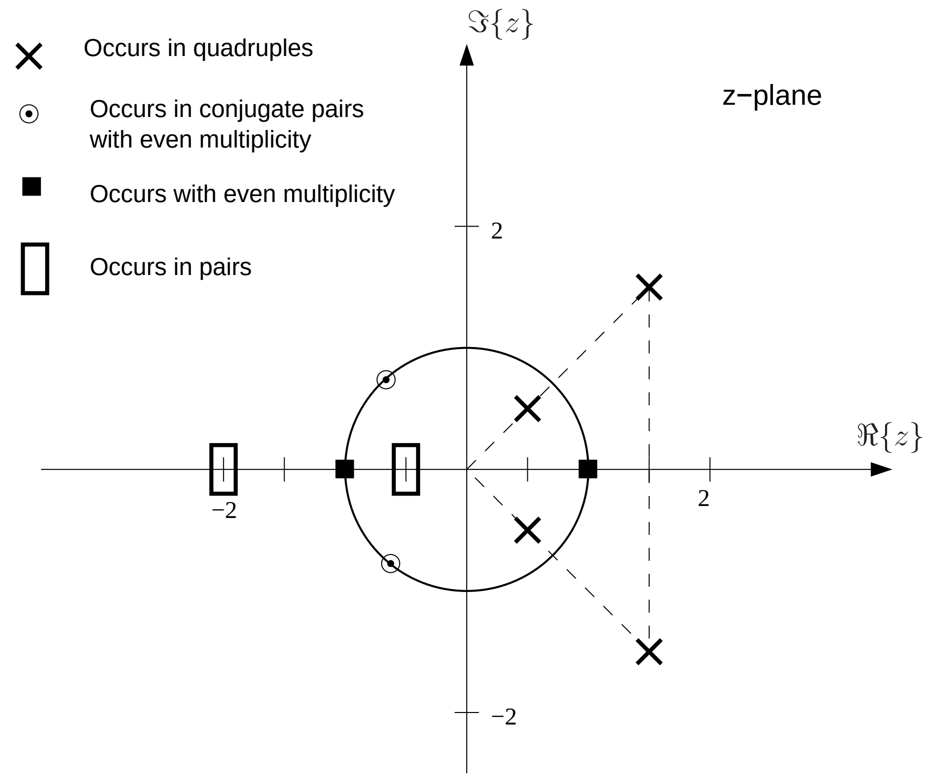
- Frequency response of digital Filter: $H(e^{\jmath\theta}) = |H(e^{\jmath\theta})|e^{-\jmath\phi(\theta)}$
  - continuous function of θ with period $2\pi \Rightarrow H(e^{\jmath\theta}) = H[e^{\jmath(\theta+m2\pi)}]$
- $|H(e^{\jmath\theta})|$ is the called the **Magnitude function**.

  $\rightarrow$ Magnitude functions are *even functions* $\Rightarrow |H(e^{\jmath\theta})| = |H(e^{-\jmath\theta})|$
- $\phi(\theta)$ is called the **Phase (lag) angle**, $\phi(\theta) \triangleq \angle H(e^{\jmath\theta})$.

  $\rightarrow$ Phase functions are *odd functions* $\Rightarrow \phi(\theta) = -\phi(-\theta)$
- More convenient to use the magnitude squared and group delay functions than $|H(e^{\jmath\theta})|$ and $\phi(\theta)$.
  - Magnitude squared function: $|H(e^{\jmath\theta})|^2 = H(z)H(z^{-1})\Big|_{z=e^{\jmath\theta}}$
  - It is assumed that $H(z)$ has real coefficients only.
  - Group delay function $\tau(\theta) = \frac{d\phi(\theta)}{d\theta}$. Measure of the delay of the filter response.

Digital Filter Frequency Response: Poles & Zeros

- Complex zeros z_k and poles p_k occur in conjugate pairs.
- If $z_k = a$ is a real zero/pole of $|H(e^{\jmath\theta})|^2 \Rightarrow z_k^{-1} = a^{-1}$ is also a real zero/pole.
- If $z_k = r_k e^{\jmath\theta}$ is a zero/pole of $|H(e^{\jmath\theta})|^2 \Rightarrow r_k e^{-\jmath\theta}, (\frac{1}{r_k})e^{\jmath\theta}$ and $(\frac{1}{r_k})e^{-\jmath\theta}$ are also zeros/poles.



Digital Filters: Transfer Functions

- The problem of finding the transfer function of a filter is the problem of universal function approximation. This is usually solved by involving some basis functions (Fourier, Chebyshev, ...). In our case, the basis functions will be polynomials or rational functions in z (or z^{-1}.

- **Finite Impulse Response** (FIR) filter: Digital filter characterised by transfer functions in the form of a polynomial

$$H(z) = a_0 + a_1 z^{-1} + \cdots + z_m z^{-M}$$

- **Infinite Impulse Response** (IIR) filter: characterised by transfer functions in the form of a rational function

$$H(z) = \frac{\sum_{i=0}^{M} a_i z^{-i}}{\sum_{j=0}^{N} b_j z^{-j}} = \frac{A(z^{-1})}{B(z^{-1})}$$

Digital Filters: Transfer Functions Properties

- FIR filters are stable and causal.

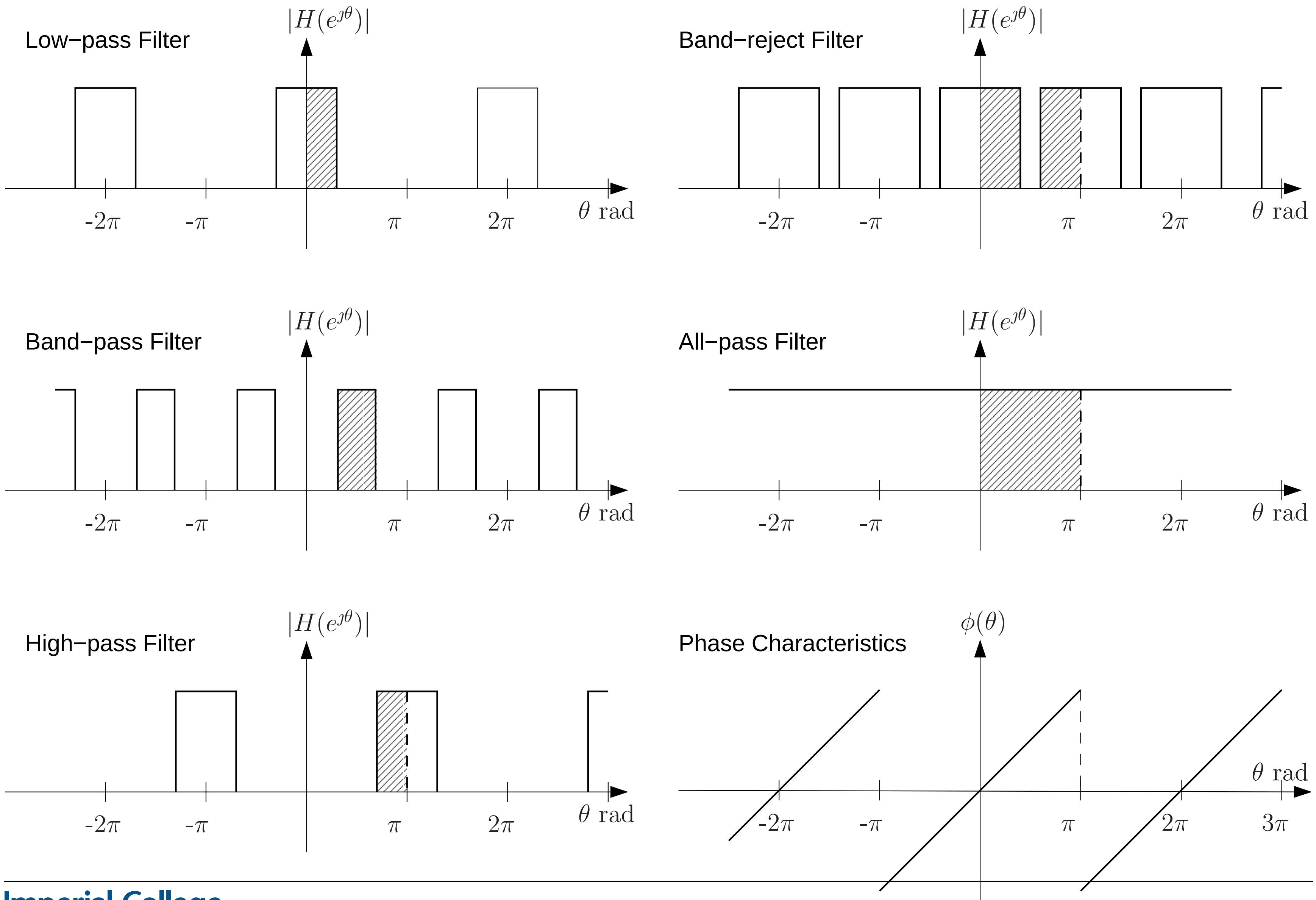
- IIR filters are:
  - Stable if all the poles of $H(z)$ are within the unit circle
  - Causal if b_L is the first non-zero coefficient in the denominator (i.e. $b_0 = b_1 = \cdots = b_{L-1} = 0$ and $a_0 = a_1 = \cdots = a_{L-1} = 0$.

- Causal filters are normally assumed, hence IIR filters are commonly written as:

$$H(z) = \frac{\sum_{i=0}^{M} a_i z^{-i}}{1 + \sum_{j=1}^{N} b_j z^{-j}} = \frac{A(z^{-1})}{B(z^{-1})}, \qquad b_0 = 1$$

- **We would ideally like to design filters with linear phase in the passband** - **what about the phase in the stopband?**

Digital Filters: Magnitude and Phase Characteristics



Design of All-pass Digital Filters

- An all-pass filter is an IIR filter with a constant magnitude function for all digital frequency values.

- For a transfer function $H(z)$ to represent an all-pass filter is that for every pole $p_k = r_k e^{j\theta}$, there is a corresponding zero $z_k = \frac{1}{r_k} e^{j\theta}$. The poles and zeros will occur in conjugate pairs if $\theta_k \neq 0$ or π.

- A digital filter $H(z)$ obtained by cascade connection of multiple all-pass filters $H_1(z), H_2(z) \cdots H_N(z)$ sections is itself an all-pass filter, and can be represented by

$$H(z) = H_1(z) H_2(z) \cdots H_N(z)$$

◦ So why do we need all-pass filters? **They are phase-selective (as opposed to frequency selective) and are extremely useful in the design of DSP systems**.

First order All-pass Digital Filter

- A typical first-order section of an all-pass digital filter has a transfer function

$$H_1(z) = \frac{z^{-1} - a}{1 - az^{-1}} \tag{1}$$

where a is real and to be stable, we must have $|a| < 1$.

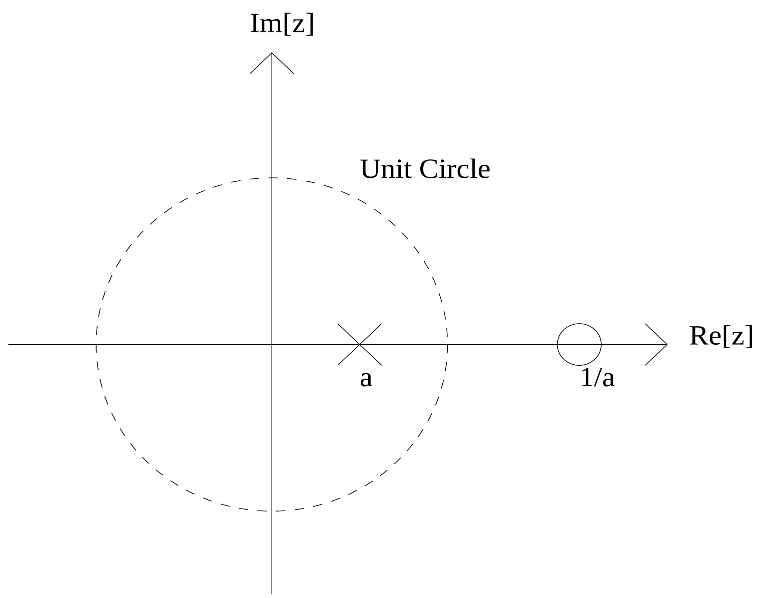


Figure 1: Pole-zero pattern of first order all-pass digital filter.

First- and Second-Order All-pass Digital Filter

- The magnitude function is unity for all frequencies, as given by

$$|H_1(e^{j\theta})|^2 = \left|\frac{e^{-j\theta} - a}{1 - ae^{-j\theta}}\right|^2 = \left|\frac{\cos\theta - a - j\sin\theta}{1 - a\cos\theta + aj\sin\theta}\right|^2 = \frac{1 - 2a\cos\theta + a^2}{1 - 2a\cos\theta + a^2} = 1$$

- A typical second-order section of an all-pass digital filter

$$H_2(z) = \frac{1 - (\frac{2}{r_k})\cos\theta_k z^{-1} + (\frac{1}{r_k^2})z^{-2}}{1 - 2r_k\cos\theta z^{-1} + r_k^2 z^{-2}} = \frac{[1 - (\frac{1}{r_k})z^{-1}e^{j\theta}][1 - (\frac{1}{r_k})z^{-1}e^{-j\theta}]}{[1 - r_k z^{-1}e^{j\theta}][1 - r_k z^{-1}e^{-j\theta}]}$$

⊛ The poles are at $p_{1,2} = r_k e^{\pm j\theta_k}$ and the zeros at $z_{1,2} = \frac{1}{r_k}e^{\pm j\theta_k}$

- **For filter to be stable, $|r_k| < 1$.**

First- and Second-Order All-pass Digital Filter

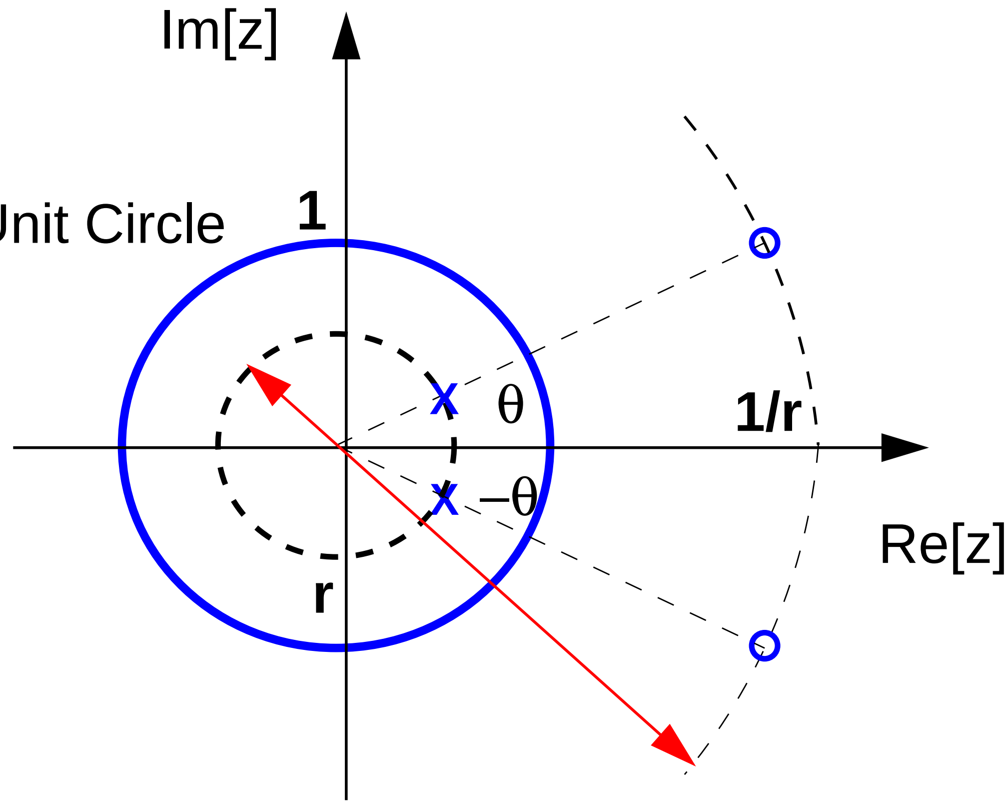


Figure 2: Pole-zero pattern of a second order all-pass digital filter.

First order All-pass Digital Filter

The magnitude function is given by

$$|H_2(e^{j\theta})|^2 = |\frac{e^{j\theta} - (\frac{1}{r_k})e^{j\theta_k}}{e^{j\theta} - r_k e^{j\theta_k}}|^2 |\frac{e^{j\theta} - (\frac{1}{r_k})e^{-j\theta_k}}{e^{j\theta} - r_k e^{-j\theta_k}}|^2 \qquad (2)$$

where $|\frac{e^{j\theta} - (\frac{1}{r_k})e^{j\theta_k}}{e^{j\theta} - r_k e^{j\theta_k}}|^2 = |\frac{e^{j\theta} - (\frac{1}{r_k})e^{-j\theta_k}}{e^{j\theta} - r_k e^{-j\theta_k}}|^2 = r_k^{-2}$

Hence

$$|H_2(e^{j\theta})|^2 = r_k^{-4} = c \qquad (3)$$

where c is a constant, implying that it represents an all-pass filter.

Design of FIR Digital Filter

The transfer function of FIR digital filter is in the form of

$$H(z) = \sum_{n=0}^{N-1} h(n) z^{-n} \tag{4}$$

where the impulse response is of length N.

The filter will have linear phase response if the FIR digital filter satisfies

$$h(n) = h(N-1-n) \tag{5}$$

Design of FIR Digital Filter

for $n = 0, 1, \ldots, (N/2) - 1$ if N is even, and for $n = 0, 1, \ldots, (N-1)/2$ if N is odd. Indeed if N is odd, then (4) and (5) give

$$\begin{aligned}
H(e^{j\theta}) &= \sum_{n=0}^{N-1} h(n)e^{-jn\theta} \\
&= \sum_{n=0}^{\frac{N-3}{2}} [h(n)e^{-jn\theta} + h(N-1-n)e^{-j(N-1-n)\theta}] + h\Big(\frac{N-1}{2}\Big)e^{-j\left\{n-[\frac{(N-1)}{2}]\right\}\theta} \\
&= \sum_{n=0}^{\frac{N-3}{2}} h(n)[e^{-jn\theta} + e^{-j(N-1-n)\theta}] + h\Big(\frac{N-1}{2}\Big)e^{-j\left\{n-[\frac{(N-1)}{2}]\right\}\theta} && (6) \\
&= e^{-j[(N-1)/2]\theta}\Big\{h\Big(\frac{N-1}{2}\Big) + \sum_{n=0}^{\frac{N-3}{2}} h(n)[e^{-j\left\{n-[\frac{(N-1)}{2}]\right\}\theta} + e^{j\left\{n-[\frac{(N-1)}{2}]\right\}\theta}]\Big\} \\
&= e^{-j[(N-1)/2]\theta}\Big\{h\Big(\frac{N-1}{2}\Big) + \sum_{n=0}^{\frac{N-3}{2}} 2h(n)\cos\Big[\Big(n - \frac{N-1}{2}\Big)\theta\Big]\Big\} && (7)
\end{aligned}$$

Design of FIR Digital Filter

In similar way, (4) and (5), for even values of N, give

$$H(e^{j\theta}) = e^{-j[(N-1)/2]\theta} \{ \sum_{n=0}^{(\frac{N}{2}-1)} 2h(n) \cos [(n - \frac{N-1}{2})\theta] \} \tag{8}$$

In both cases, the phase $\phi(\theta)$ of the FIR digital filter is given by

$$\phi(\theta) = \frac{N-1}{2}\theta \tag{9}$$

which is linear for $\pi < \theta \leq \pi$.

The group delay function is

$$\tau(\theta) = \phi'(\theta) = \frac{N-1}{2} \tag{10}$$

which is constant for $\pi < \theta \leq \pi$.

Constraints on zero-phase FIR filters

The zero locations of FIR filter are restricted to meet certain symmetry requirements due to constraints imposed by (5). To see this, (4) is written as

$$H(z) = z^{-(N-1)} \sum_{n=0}^{N-1} h(n) z^{N-n-1}$$

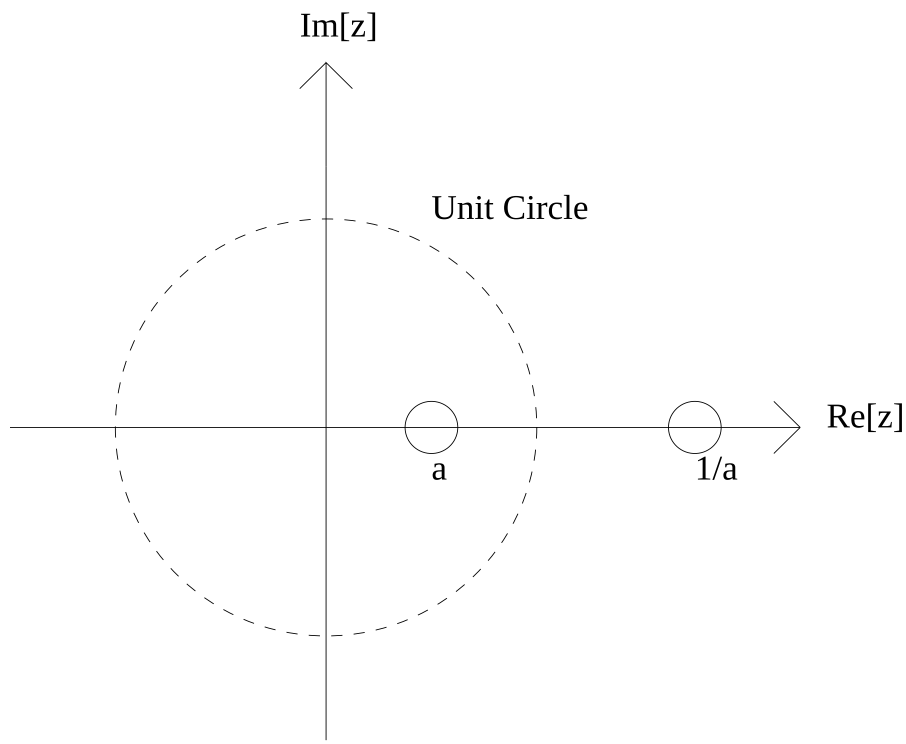
Let $m = N - n - 1$ be a new dummy variable, then (12) can be written as

$$\begin{aligned} H(z) &= z^{-(N-1)} \sum_{n=0}^{N-1} h(N-m-1) z^{m} \\ &= z^{-(N-1)} \sum_{n=0}^{N-1} h(m) (z^{-1})^{-m} \\ &= z^{-(N-1)} H(z^{-1}) \end{aligned} \tag{11}$$

This means that zeros of $H(z)$ are the zeros of $H(z^{-1})$ except, perhaps, for the zeros at origin.

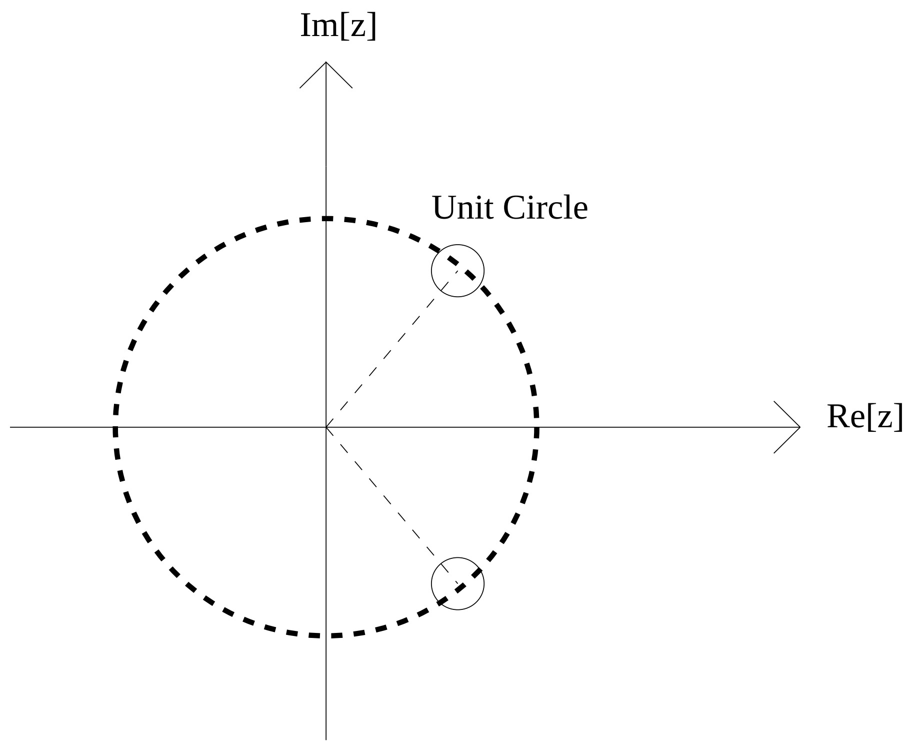
Symmetry properties of digital FIR filters

- If $z_i = a$ is a real zero of $H(z)$, then $z_i^{-1} = a^{-1}$ is also a zero of $H(z)$.



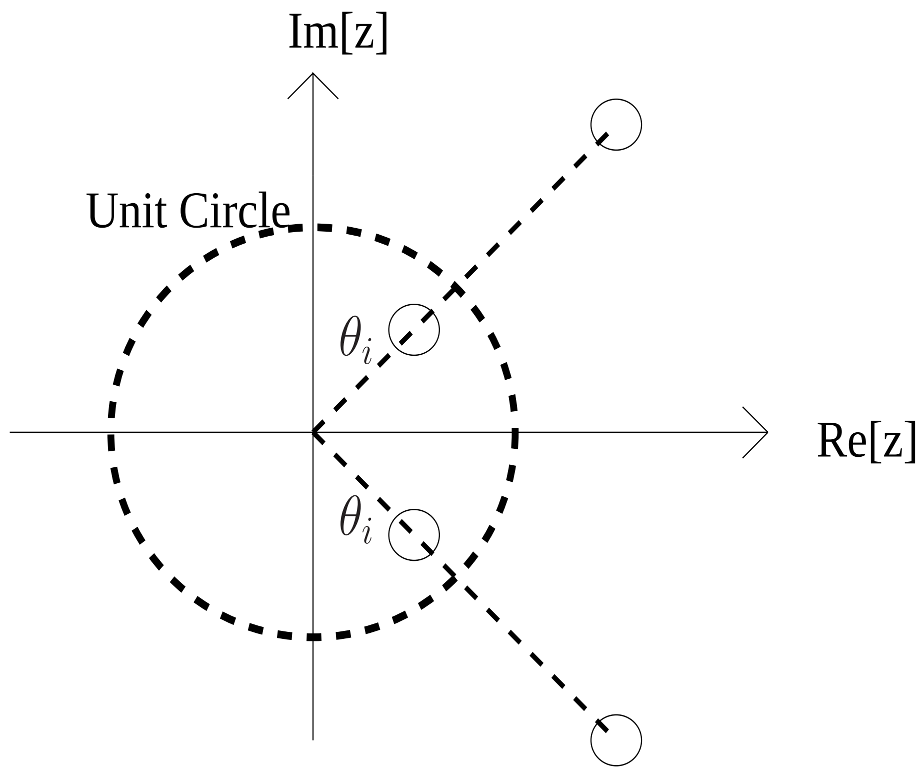
Symmetry properties of digital FIR filters

- If $z_i = e^{j\theta_i}$ is a zero of $H(z)$, where $\theta_i \neq 0$ and $\theta_i \neq \pi$, then $z_i^{-1} = \overline{z}_i = e^{-j\theta_i}$ is also a zero of $H(z)$.



Symmetry properties of digital FIR filters

- If $z_i = r_i e^{j\theta_i}$ is a zero of $H(z)$, where $r_i \neq 1$, $\theta_i \neq 0$ and $\theta_i \neq \pi$, then $\overline{z}_i = r_i e^{-j\theta_i}$ and $z_i^{-1} = \frac{1}{r_i} e^{-j\theta_i}$ and $\overline{z}_i^{-1} = \frac{1}{r_i} e^{j\theta_i}$ are also zeros of $H(z)$.



Frequency sampling method

An FIR filter has equivalent DFT representation, given by

$$\widetilde{H}(k) = \sum_{n=0}^{N-1} h(n) e^{[-\frac{j2\pi nk}{N}]} \tag{12}$$

where $\widetilde{H}(k)$ is actually the uniformly spaced N-point sample sequence of the frequency response of the digital filter. As a consequence, the impulse response sequence $h(n)$ and transfer function $H(z)$ are given by

$$h(n) = \frac{1}{N} \sum_{k=0}^{N-1} \widetilde{H}(k) e^{[\frac{j2\pi nk}{N}]} \tag{13}$$

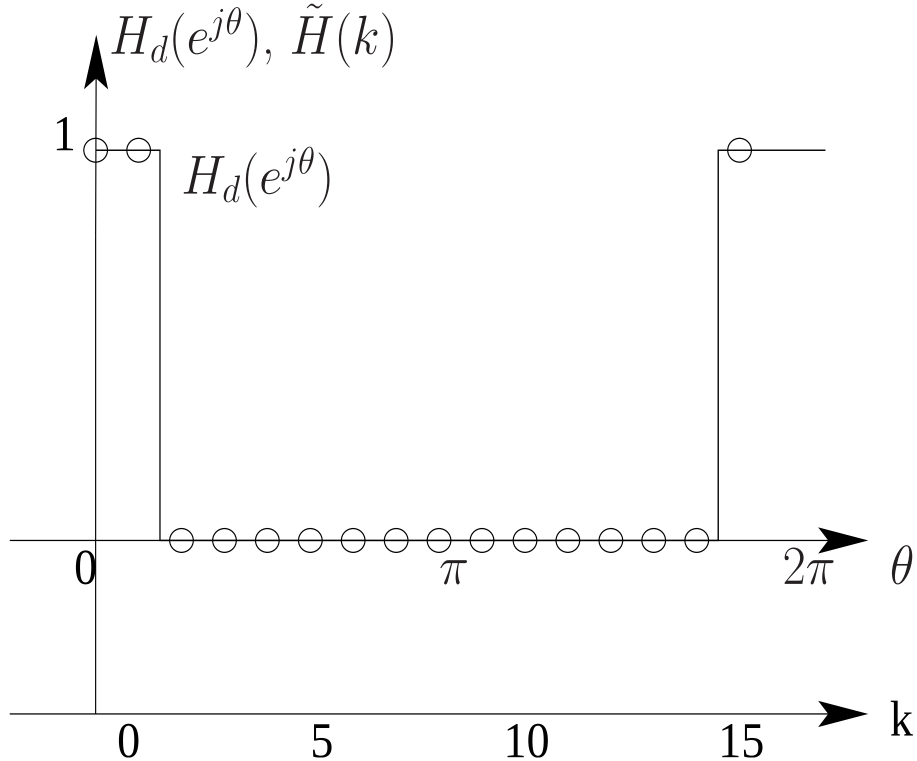
and

$$H(z) = \frac{1}{N} \sum_{k=0}^{N-1} \widetilde{H}(k) \frac{1 - z^{-N}}{1 - z^{-1} e^{[\frac{j2\pi k}{N}]}} \tag{14}$$

where equation (14) is the key to the design of FIR digital filter.

Example

Design a low-pass digital filter whose magnitude characteristics are shown in Figure. Find an appropriate transfer function via a 16-point frequency sampling method.



Solution: In this case, the DFT sequence is given by

Example

$$\widetilde{H}(0) = \widetilde{H}(1) = \widetilde{H}(15) = 1$$
$$\widetilde{H}(k) = 0 \text{ for } k = 2, 3, 4, \ldots, 14 \qquad (15)$$

By using (14), the desired transfer function can be found

$$\begin{aligned} H(z) &= \frac{1}{16}\Big[\sum_{k=0}^{15} \frac{(1-z^{-16})\widetilde{H}(k)}{1-z^{-1}e^{\frac{jk\pi}{8}}}\Big] \\ &= \frac{1-z^{-16}}{16}\Big[\frac{1}{1-z^{-1}e^{\frac{j0\pi}{4}}} + \frac{1}{1-z^{-1}e^{\frac{j\pi}{8}}} + \frac{1}{1-z^{-1}e^{\frac{j15\pi}{8}}}\Big] \qquad (16) \\ &= \frac{1-z^{-16}}{16}\Big[\frac{1}{1-z^{-1}} + \frac{2(1-z^{-1}\cos(\pi/8))}{1-2z^{-1}\cos(\pi/8)+z^{-2}}\Big] \end{aligned}$$

It can be be shown that the frequency response of (17) will be equal to the specifications of (15) at the sampling frequencies $\theta = \frac{k\pi}{8}$ for $k = 0, 1, 2, \ldots, 15$.

The Windowing Method

- The Fourier series expansion of the frequency response of a digital filter, $H(e^{j\theta})$, is given by

$$H(e^{j\theta}) = \sum_{n=-\infty}^{\infty} h(n)e^{-j\theta n} \tag{17}$$

where

$$h(n) = \frac{1}{2\pi}\int_{n=-\pi}^{\pi} H(e^{j\theta})e^{j\theta n} \tag{18}$$

where $h(n)$ is the impulse response of the digital filter.

- While the infinite series in (17) can be truncated to obtain the digital filter, the *Gibbs phenomenon* states that the truncation will cause overshoots and ripples in the desired frequency response.

- In the method of windowing, a finite weighting sequence $w(n)$, called *windows*, is used to obtain the finite impulse response $h_D(n)$, where

$$h_D(n) = h(n)w(n)$$

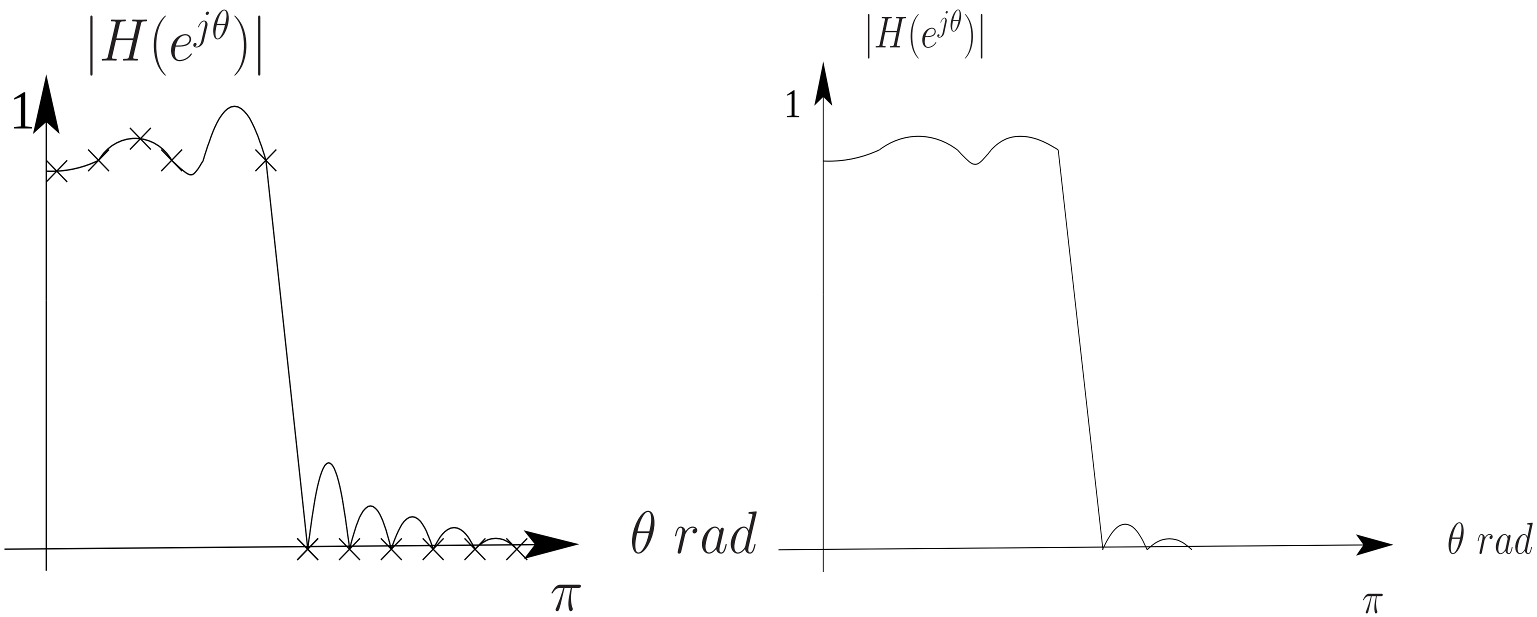
where $w(n)$ is $w(n) = 0$ for $n > N$ and $n < 0$.

The Windowing Method

- Given the desired frequency response $H(e^{j\theta})$, which may be obtained by the frequency sampling method.
- Find the associated impulse response sequence $h(n)$ from 17 or by inverse z-transform of $H(z)$, where $H(z)$ is obtained from $H(e^{j\theta})$ by replacing $e^{j\theta}$ with z.
- Employ an appropriate window function $w(n)$ to modify the sequence $h(n)$ to obtain the FIR digital filter's impulse response sequence $h_D(n) = h(n)w(n)$.

The windowing method has the effect of smoothing out the ripples and overshoots in the original frequency response as shown in the figure for a simple window function

The Windowing Method



$$
\begin{aligned}
w(n) &= 1 + \cos\frac{2\pi n}{N} \text{ for } 0 \le n \le N-1 \\
&= 0 \text{ otherwise} \qquad (19)
\end{aligned}
$$

The Windowing Method: Some common window functions

- Rectangular Window

$$\begin{aligned} w(n) &= 1 \text{ for } 0 \le n \le N-1 \\ &= 0 \text{ otherwise} \end{aligned} \tag{20}$$

- Bartlett Window or Triangular Window

$$\begin{aligned} w(n) &= \frac{2n}{N-1} \text{ for } 0 \le n \le (N-1)/2 \\ &= 2 - \frac{2n}{N-1} \text{ for } (N-2)/2 \le n \le N-1 \\ &= 0 \text{ elsewhere} \end{aligned} \tag{21}$$

where N is even.

The Windowing Method: Some common window functions

- Hann Window

$$
\begin{aligned}
w(n) &= \frac{1}{2}\left[1 - \cos\frac{2\pi n}{N-1}\right] \text{ for } 0 \le n \le N-1 \\
&= 0 \text{ elsewhere}
\end{aligned} \tag{22}
$$

- Hamming Window

$$
\begin{aligned}
w(n) &= 0.54 - 0.46\cos\left[\frac{2\pi n}{N-1}\right] \text{ for } 0 \le n \le N-1 \\
&= 0 \text{ elsewhere}
\end{aligned} \tag{23}
$$

The Windowing Method: Some common window functions

- Blackman Window

$$w(n) = 0.42 - 0.5\cos\left[\frac{2\pi n}{N-1}\right] + 0.008\cos\left[\frac{4\pi n}{N-1}\right] \text{ for } 0 \le n \le N-1$$
$$= 0 \text{ elsewhere} \qquad (24)$$

- Kaiser Window

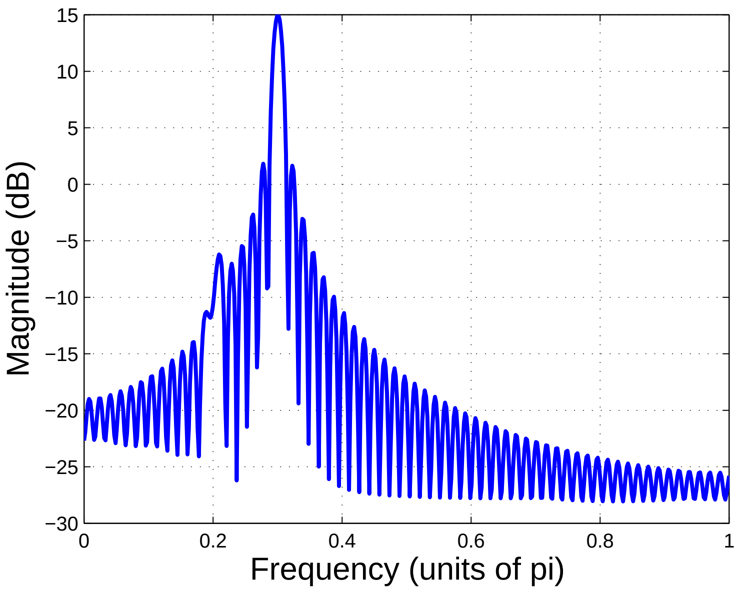
$$w(n) = \frac{I_0\Big[w_a\sqrt{\left(\frac{N-1}{2}\right)^2 - \left(n - \frac{N-1}{2}\right)^2}\Big]}{I_0\Big[w_a\left(\frac{N-1}{2}\right)\Big]} \text{ for } 0 \le n \le N-1$$
$$= 0 \text{ elsewhere} \qquad (25)$$

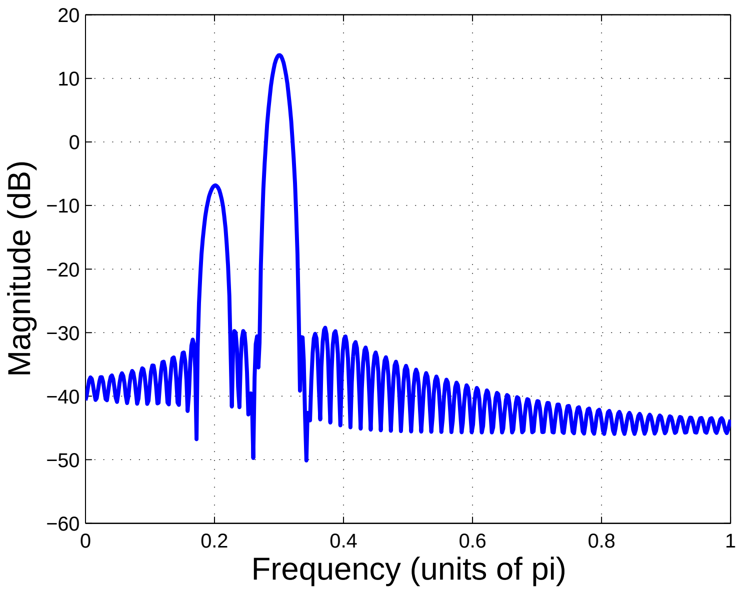
where $I_0(.)$ is a modified zeroth order Bassel function of the first kind and w_a is a window shaper parameter.

Two Sinusoids in WGN:- Hamming window

$$x[n] = 0.1\sin(n * 0.2\pi + \Phi_1) + \sin(n * 0.3\pi + \Phi_2) + w[n] \qquad N = 128$$

$$Hamming \quad window \qquad w[n] = 0.54 - 0.46\cos\left(2\pi\frac{n}{N}\right)$$





Expexted value of periodogram **Periodogram Using Hamming window**

The Modified Periodogram

The periodogram of a process that is windowed with a general window $w[n]$ is called a **modified periodogram** and is given by:-

$$\hat{P}_M(\omega) = \frac{1}{NU} \left| \sum_{n=-\infty}^{\infty} x[n]w[n]e^{-\jmath n\omega} \right|^2$$

where N is the window length and $U = \frac{1}{N}\sum_{n=0}^{N-1} |w[n]|^2$ is a constant, and is defined so that $\hat{P}_M(\omega)$ is asymptotically unbiased.

In Matlab:-

```
xw=x(n1:n2).*w/norm(w);
Pm=N * periodogram(xw);
```

where, for different windows

```
w=hanning(N); w=bartlett(N);w=blackman(n);
```

“Cosine–type windows”

Idea:- suppress sidelobes, perhaps sacrify the width of mainlobe

- **Hann** window

```
w = 0.5 * (1 - cos(2*pi*(0:m-1)'/(n-1)));
```

- **Hamming** window

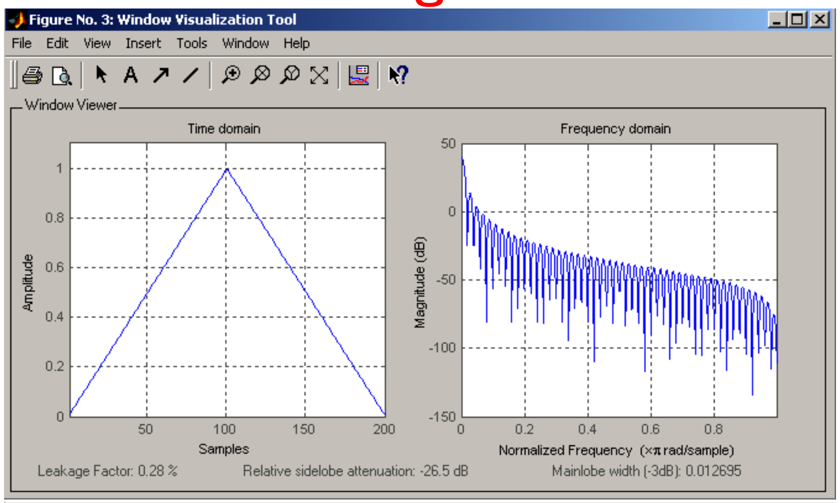
```
w = (54 - 46*cos(2*pi*(0:m-1)'/(n-1)))/100;
```

- **Blackman** window

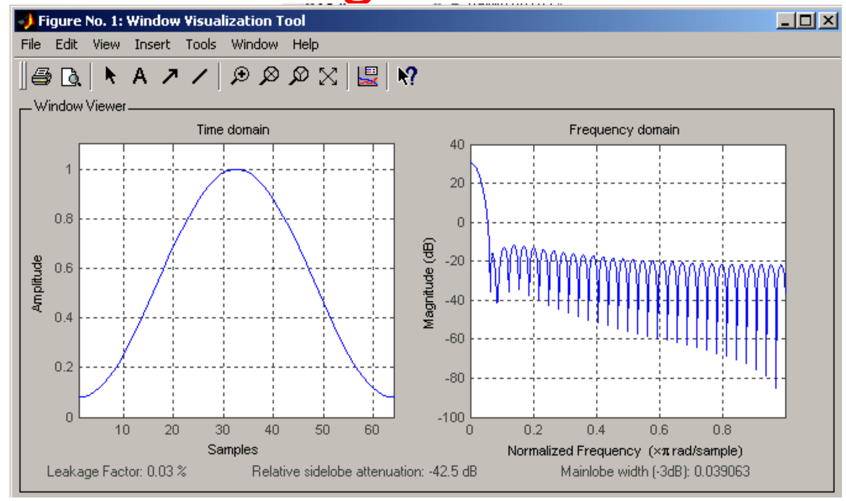
```
w = (42 - 50*cos(2*pi*(0:m-1)/(n-1)) +
                   + 8*cos(4*pi*(0:m-1)/(n-1)))'/100;
```

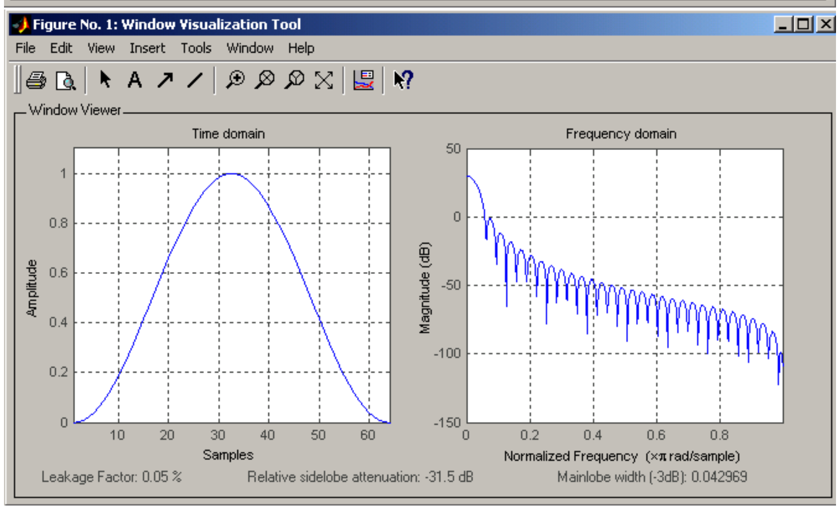
Standard Window Functions:- Properties

Triangular window

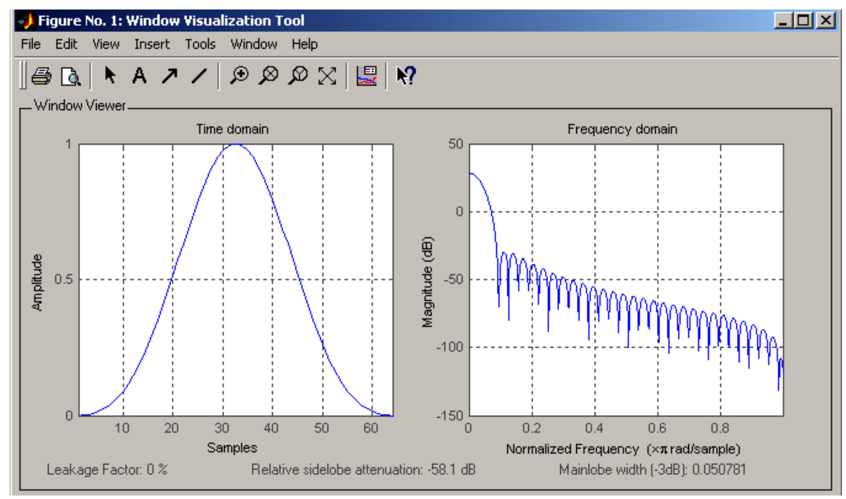


Hamming window





Hann window



Blackman window

Some Comments on FIR digital Filter

- Unlike IIR filters, FIR filters can be designed to have linear phase characteristics.
- FIR filters are always stable.
- FIR filters are, however, computationally more expensive than IIR filters and hence are called for to perform tasks not possible/or not practical by IIR filters such as linear phase, and multirate filters.