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The Magic of Complex Numbers

The notion of complex number is intimately related to the *Fundamental Theorem of Algebra* and is therefore at the very foundation of mathematical analysis. The development of complex algebra, however, has been far from straightforward.[1]

The human idea of 'number' has evolved together with human society. The *natural* numbers ($1, 2, \ldots \in \mathbb{N}$) are straightforward to accept, and they have been used for *counting* in many cultures, irrespective of the actual base of the number system used. At a later stage, for *sharing*, people introduced fractions in order to answer a simple problem such as 'if we catch $\mathcal{U}$ fish, I will have two parts $\frac{2}{5}\,\mathcal{U}$ and you will have three parts $\frac{3}{5}\,\mathcal{U}$ of the whole catch'. The acceptance of negative numbers and zero has been motivated by the emergence of economy, for dealing with profit and loss. It is rather impressive that ancient civilisations were aware of the need for irrational numbers such as $\sqrt{2}$ in the case of the Babylonians [77] and π in the case of the ancient Greeks.[2]

The concept of a new 'number' often came from the need to solve a specific practical problem. For instance, in the above example of sharing $\mathcal{U}$ number of fish caught, we need to solve for $2\mathcal{U} = 5$ and hence to introduce fractions, whereas to solve $x^2 = 2$ (diagonal of a square) irrational numbers needed to be introduced. Complex numbers came from the necessity to solve equations such as $x^2 = -1$.

[1] A classic reference which provides a comprehensive account of the development of numbers is *Number: The Language of Science* by Tobias Dantzig [57].

[2] The Babylonians have actually left us the basics of Fixed Point Theory (see Appendix P), which in terms of modern mathematics was introduced by Stefan Banach in 1922. On a clay tablet (YBC 7289) from the Yale Babylonian Collection, the Mesopotamian scribes explain how to calculate the diagonal of a square with base 30. This was achieved using a fixed point iteration around the initial guess. The ancient Greeks used π in geometry, although the irrationality of π was only proved in the 1700s. More information on the history of mathematics can be found in [34] whereas P. Nahin's book is dedicated to the history of complex numbers [215].

Complex Valued Nonlinear Adaptive Filters: Noncircularity, Widely Linear and Neural Models
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1.1 History of Complex Numbers

Perhaps the earliest reference to square roots of negative numbers occurred in the work of Heron of Alexandria[3], around 60 AD, who encountered them while calculating volumes of geometric bodies. Some 200 years later, Diophantus (about 275 AD) posed a simple problem in geometry,

Find the sides of a right–angled triangle of perimeter 12 units and area 7 squared units.

which is illustrated in Figure 1.1. To solve this, let the side $|AB| = x$, and the height $|BC| = h$, to yield

$$area = \frac{1}{2}\, x\, h$$

$$perimeter = x + h + \sqrt{x^2 + h^2}$$

In order to solve for x we need to find the roots of

$$6x^2 - 43x + 84 = 0$$

however this equation does not have real roots.

A similar problem was posed by Cardan[4] in 1545. He attempted to find two numbers a and b such that

$$a + b = 10$$
$$a\,b = 40$$

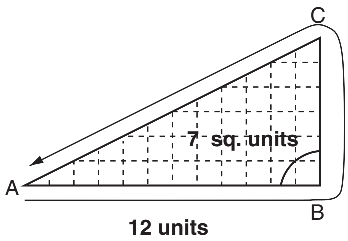


Figure 1.1 Problem posed by Diophantus (third century AD)

[3]Heron (or Hero) of Alexandria was a Greek mathematician and inventor. He is credited with finding a formula for the area of a triangle (as a function of the perimeter). He invented many gadgets operated by fluids; these include a fountain, fire engine and siphons. The aeolipile, his engine in which the recoil of steam revolves a ball or a wheel, is the forerunner of the steam engine (and the jet engine). In his method for approximating the square root of a number he effectively found a way round the complex number. It is fascinating to realise that complex numbers have been used, implicitly, long before their introduction in the 16th century.

[4]Girolamo or Hieronimo Cardano (1501–1576). His name in Latin was Hieronymus Cardanus and he is also known by the English version of his name Jerome Cardan. For more detail on Cardano's life, see [1].

These equations are satisfied for

$$a = 5 + \sqrt{-15} \quad \text{and} \quad b = 5 - \sqrt{-15} \tag{1.1}$$

which are clearly not real.

The need to introduce the complex number became rather urgent in the 16th century. Several mathematicians were working on what is today known as the *Fundamental Theorem of Algebra* (FTA) which states that

Every nth order polynomial with real[5] *coefficients has exactly n roots in* $\mathbb{C}$.

Earlier attempts to find the roots of an arbitrary polynomial include the work by Al-Khwarizmi (ca 800 AD), which only allowed for positive roots, hence being only a special case of FTA. In the 16th century Niccolo Tartaglia[6] and Girolamo Cardano (see Equation 1.1) considered closed formulas for the roots of third- and fourth-order polynomials. Girolamo Cardano first introduced complex numbers in his *Ars Magna* in 1545 as a tool for finding *real* roots of the 'depressed' cubic equation $x^3 + ax + b = 0$. He needed this result to provide algebraic solutions to the general cubic equation

$$ay^3 + by^2 + cy + d = 0$$

By substituting $y = x - \frac{1}{3}b$, the cubic equation is transformed into a depressed cubic (without the square term), given by

$$x^3 + \beta x + \gamma = 0$$

Scipione del Ferro of Bologna and Tartaglia showed that the depressed cubic can be solved as[7]

$$x = \sqrt[3]{-\frac{\gamma}{2} + \sqrt{\frac{\gamma^2}{4} + \frac{\beta^3}{27}}} + \sqrt[3]{-\frac{\gamma}{2} - \sqrt{\frac{\gamma^2}{4} + \frac{\beta^3}{27}}} \tag{1.2}$$

For certain problem settings (for instance $a = 1, b = 9, c = 24, d = 20$), and using the substitution $y = x - 3$, Tartaglia could show that, by symmetry, there exists $\sqrt{-1}$ which has mathematical meaning. For example, Tartaglia's formula for the roots of $x^3 - x = 0$ is given by

$$\frac{1}{\sqrt{3}}\left((\sqrt{-1})^{\frac{1}{3}} + \frac{1}{(\sqrt{-1})^{\frac{1}{3}}}\right)$$

[5] In fact, it states that every nth order polynomial with complex coefficients has n roots in $\mathbb{C}$, but for historical reasons we adopt the above variant.

[6] Real name Niccolo Fontana, who is known as Tartaglia (the stammerer) due to a speaking disorder.

[7] In modern notation this can be written as $x = (q + w)^{\frac{1}{3}} + (q - w)^{\frac{1}{3}}$.

Rafael Bombelli also analysed the roots of cubic polynomials by the 'depressed cubic' transformations and by applying the Ferro–Tartaglia formula (1.2). While solving for the roots of

$$x^3 - 15x - 4 = 0$$

he was able to show that

$$\left(2 + \sqrt{-1}\right) + \left(2 - \sqrt{-1}\right) = 4$$

Indeed $x = 4$ is a correct solution, however, in order to solve for the real roots, it was necessary to perform calculations in $\mathbb{C}$. In 1572, in his *Algebra*, Bombelli introduced the symbol $\sqrt{-1}$ and established rules for manipulating 'complex numbers'.

The term 'imaginary' number was coined by Descartes in the 1630s to reflect his observation that '*For every equation of degree n, we can imagine n roots which do not correspond to any real quantity*'. In 1629, Flemish mathematician[8] Albert Girard in his *L'Invention Nouvelle en l'Algèbre* asserts that there are n roots to an nth order polynomial, however this was accepted as self-evident, but with no guarantee that the actual solution has the form $a + \jmath b$, $a, b \in \mathbb{R}$.

It was only after their geometric representation (John Wallis[9] in 1685 in *De Algebra Tractatus* and Caspar Wessel[10] in 1797 in the *Proceedings of the Copenhagen Academy*) that the complex numbers were finally accepted. In 1673, while investigating geometric representations of the roots of polynomials, John Wallis realised that for a general quadratic polynomial of the form

$$x^2 + 2bx + c^2 = 0$$

for which the solution is

$$x = -b \pm \sqrt{b^2 - c^2} \tag{1.3}$$

a geometric interpretation was only possible for $b^2 - c^2 \geq 0$. Wallis visualised this solution as displacements from the point $-b$, as shown in Figure 1.2(a) [206]. He interpreted each solution as a vertex (A and B in Figure 1.2) of a right triangle with height c and side $\sqrt{b^2 - c^2}$. Whereas this geometric interpretation is clearly correct for $b^2 - c^2 \geq 0$, Wallis argued that for $b^2 - c^2 < 0$, since b is shorter than c, we will have the situation shown in Figure 1.2(b); this

[8]Albert Girard was born in France in 1595, but his family later moved to the Netherlands as religious refugees. He attended the University of Leiden where he studied music. Girard was the first to propose the fundamental theorem of algebra, and in 1626, in his first book on trigonometry, he introduced the abbreviations sin, cos, and tan. This book also contains the formula for the area of a spherical triangle.

[9]In his *Treatise on Algebra* Wallis accepts negative and complex roots. He also shows that equation $x^3 - 7x = 6$ has exactly three roots in $\mathbb{R}$.

[10]Within his work on geodesy Caspar Wessel (1745–1818) used complex numbers to represent directions in a plane as early as in 1787. His article from 1797 entitled 'On the Analytical Representation of Direction: An Attempt Applied Chiefly to Solving Plane and Spherical Polygons' (in Danish) is perhaps the first to contain a well-thought-out geometrical interpretation of complex numbers.

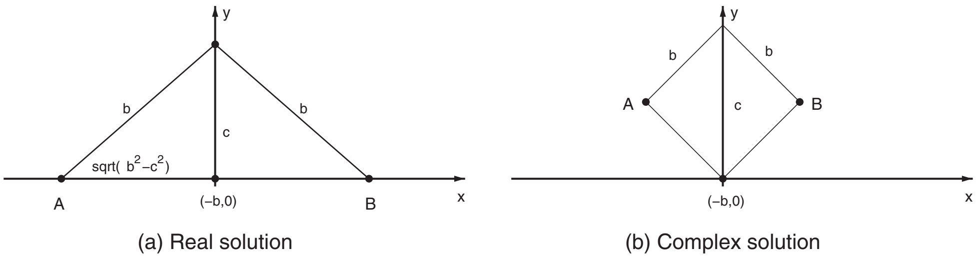


Figure 1.2 Geometric representation of the roots of a quadratic equation

way we can think of a complex number as a *point on the plane*.[11] In 1732 Leonhard Euler calculated the solutions to the equation

$$x^n - 1 = 0$$

in the form of

$$\cos\theta + \sqrt{-1}\sin\theta$$

and tried to visualise them as the vertices of a planar polygon. Further breakthroughs came with the work of Abraham de Moivre (1730) and again Euler (1748), who introduced the famous formulas

$$(\cos\theta + \jmath\sin\theta)^n = \cos n\theta + \jmath\sin n\theta$$
$$\cos\theta + \jmath\sin\theta = e^{\jmath\theta}$$

Based on these results, in 1749 Euler attempted to prove FTA for real polynomials in *Recherches Sur Les Racines Imaginaires des Équations*. This was achieved based on a decomposition a monic polynomials and by using Cardano's technique from *Ars Magna* to remove the second largest degree term of a polynomial.

In 1806 the Swiss accountant and amateur mathematician Jean Robert Argand published a proof of the FTA which was based on an idea by d'Alembert from 1746. Argand's initial idea was published as *Essai Sur Une Manière de Représenter les Quantités Imaginaires Dans les Constructions Géométriques* [60, 305]. He simply interpreted $\jmath$ as a rotation by 90° and introduced the Argand plane (or Argand diagram) as a geometric representation of complex numbers. In Argand's diagram, $\pm\sqrt{-1}$ represents a unit line, perpendicular to the real axis. The notation and terminology we use today is pretty much the same. A complex number

$$z = x + \jmath y$$

[11] In his interpretation $-\sqrt{-1}$ is the same point as $\sqrt{-1}$, but nevertheless this was an important step towards the geometric representation of complex numbers.

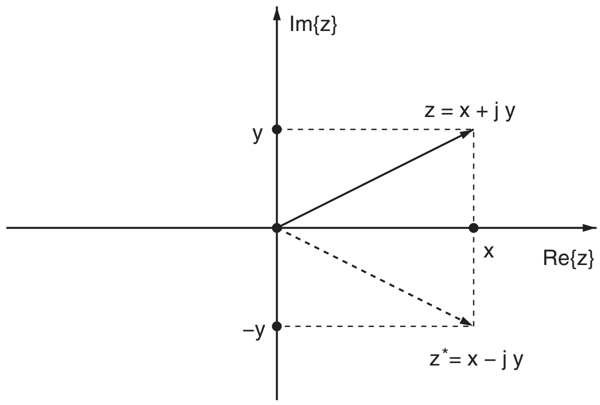


Figure 1.3 Argand's diagram for a complex number z and its conjugate z^*

is simply represented as a vector in the complex plane, as shown in Figure 1.3. Argand called $\sqrt{x^2 + y^2}$ the *modulus*, and Gauss introduced the term *complex number* and notation[12] $\imath = \sqrt{-1}$ (in signal processing we use $\jmath = \imath = \sqrt{-1}$). Karl Friedrich Gauss used complex numbers in his several proofs of the fundamental theorem of algebra, and in 1831 he not only associated the complex number $z = x + \jmath y$ with a point (x, y) on a plane, but also introduced the rules for the addition[13] and multiplication of such numbers. Much of the terminology used today comes from Gauss, Cauchy[14] who introduced the term 'conjugate', and Hankel who in 1867 introduced the term *direction coefficient* for $\cos\theta + \jmath \sin\theta$, whereas Weierstrass (1815–1897) introduced the term *absolute value* for the modulus.

Some analytical aspects of complex numbers were also developed by Georg Friedrich Bernhard Riemann (1826–1866), and those principles are nowadays the basics behind what is known as manifold signal processing.[15] To illustrate the potential of complex numbers in this context, consider the stereographic[16] projection [242] of the Riemann sphere, shown in Figure 1.4(a). In a way analogous to Cardano's 'depressed cubic', we can perform dimensionality reduction by embedding $\mathbb{C}$ in $\mathbb{R}^3$, and rewriting

$$Z = a + \jmath b, \quad (a, b, 0) \in \mathbb{R}^3$$

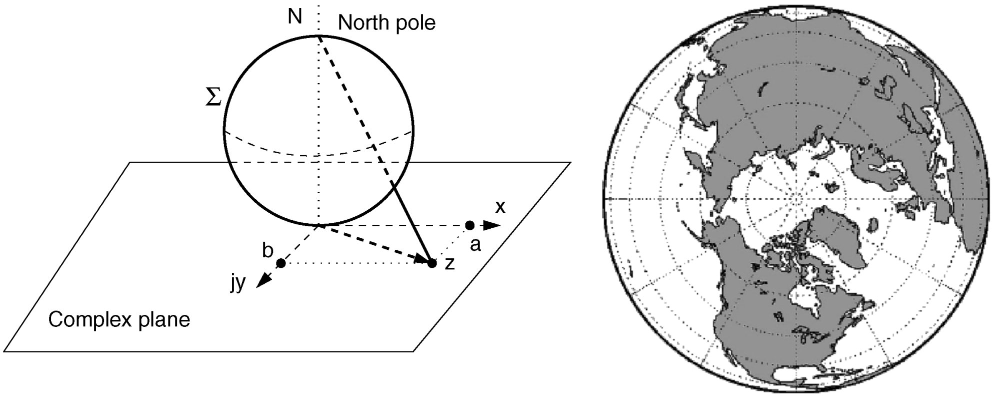
[12] There is a simple trap, that is, we cannot apply the identity of the type $\sqrt{ab} = \sqrt{a}\sqrt{b}$ to the 'imaginary' numbers, this would lead to the wrong conclusion $1 = \sqrt{(-1)(-1)} = \sqrt{-1}\sqrt{-1}$, however $\sqrt{-1}^2 = \sqrt{-1}\sqrt{-1} = -1$.

[13] So much so that, for instance, 3 remains a prime number whereas 5 does not, since it can be written as $(1 - 2\jmath)(1 + 2\jmath)$.

[14] Augustin Louis Cauchy (1789–1867) formulated many of the classic theorems in complex analysis.

[15] Examples include the Natural Gradient algorithm used in blind source separation [10, 49].

[16] The stereographic projection is a mapping that projects a sphere onto a plane. The mapping is smooth, bijective and conformal (preserves relationships between angles).



(a) Riemann sphere (b) Earth projection

Figure 1.4 Stereographic projection and Riemann sphere: (a) the principle of the stereographic projection; (b) stereographic projection of the Earth (seen from the south pole S)

Consider a sphere Σ defined by

$$\Sigma = \left\{ (x, y, u) \in \mathbb{R}^3 \; : \; x^2 + y^2 + (u - d)^2 = r^2 \right\}, \quad d, r \in \mathbb{R}$$

There is a one-to-one correspondence between the points of $\mathbb{C}$ and the points of Σ, excluding N (the north pole of Σ), since the line from any point $z \in \mathbb{C}$ cuts $\Sigma \setminus \{N\}$ in precisely one point. If we include the point ∞, so as to have the *extended complex plane* $\mathbb{C} \cup \{\infty\}$, then the north pole N from sphere Σ is also included and we have a mapping of the Riemann sphere onto the extended complex plane. A stereographic projection of the Earth onto a plane tangential to the north pole N is shown in Figure 1.4(b).

1.1.1 Hypercomplex Numbers

Generalisation of complex numbers (generally termed 'hypercomplex numbers') includes the work of Sir William Rowan Hamilton (1805–1865), who introduced the quaternions in 1843. A quaternion $\vec{q}$ is defined as [103]

$$\vec{q} = q_0 + q_1\imath + q_2\jmath + q_3 k \tag{1.4}$$

where the variables $\imath$, $\jmath$, k are all defined as $\sqrt{-1}$, but their multiplication is not commutative.[17]

Pivotal figures in the development of the theory of complex numbers are Hermann Günther Grassmann (1809–1877), who introduced multidimensional vector calculus, and James Cockle,

[17]That is: $\imath\jmath = -\jmath\imath = k$, $\jmath k = -k\jmath = \imath$, and $k\imath = -\imath k = \jmath$.

who in 1848 introduced split-complex numbers.[18] A split-complex number (also known as motors, dual numbers, hyperbolic numbers, tessarines, and Lorenz numbers) is defined as [51]

$$z = x + \jmath y, \qquad \jmath^2 = 1$$

In 1876, in order to model spins, William Kingdon Clifford introduced a system of hypercomplex numbers (Clifford algebra). This was achieved by conveniently combining the quaternion algebra and split-complex numbers. Both Hamilton and Clifford are credited with the introduction of *biquaternions*, that is, quaternions for which the coefficients are complex numbers. A comprehensive account of *hypercomplex* numbers can be found in [143]; in general a hypercomplex number system has at least one non-real axis and is closed under addition and multiplication. Other members of the family of hypercomplex numbers include McFarlane's hyperbolic quaternion, hyper-numbers, multicomplex numbers, and twistors (developed by Roger Penrose in 1967 [233]).

1.2 History of Mathematical Notation

It is also interesting to look at the development of 'symbols' and abbreviations in mathematics. For books copied by hand the choice of mathematical symbols was not an issue, whereas for printed books this choice was largely determined by the availability of fonts of the early printers. Thus, for instance, in the 9th century in Al-Khwarizmi's *Algebra* solutions were descriptive rather than in the form of equations, while in Cardano's *Ars Magna* in the 16th century the unknowns were denoted by single roman letters to facilitate the printing process.

It was arguably Descartes who first established some general rules for the use of mathematical symbols. He used lowercase italic letters at the beginning of the alphabet to denote unknown constants (a, b, c, d), whereas letters at the end of the alphabet were used for unknown variables (x, y, z, w). Using Descartes' recommendations, the expression for a quadratic equation becomes

$$a\,x^2 + b\,x + c = 0$$

which is exactly the way we use it in modern mathematics.

As already mentioned, the symbol for imaginary unit $\imath = \sqrt{-1}$ was introduced by Gauss, whereas boldface letters for vectors were first introduced by Oliver Heaviside [115]. More details on the history of mathematical notation can be found in the two–volume book *A History of Mathematical Notations* [39], written by Florian Cajori in 1929.

In the modern era, the introduction of mathematical symbols has been closely related with the developments in computing and programming languages.[19] The relationship between computers and typography is explored in *Digital Typography* by Donald E. Knuth [153], who also developed the TeX typesetting language.

[18] Notice the difference between the split-complex *numbers* and split-complex *activation functions* of neurons [152, 190]. The term split-complex number relates to an alternative hypercomplex *number* defined by $x + \jmath y$ where $\jmath^2 = 1$, whereas the term split-complex function refers to *functions* $g : \mathbb{C} \to \mathbb{C}$ for which the real and imaginary part of the 'net' function are processed separately by a real function of real argument f, to give $g(net) = f(\Re(net)) + \jmath f(\Im(net))$.

[19] Apart from the various new symbols used, e.g. in computing, one such symbol is © for 'copyright'.

1.3 Development of Complex Valued Adaptive Signal Processing

The distingushing characteristics of complex valued nonlinear adaptive filtering are related to the character of complex nonlinearity, the associated learning algorithms, and some recent developments in complex statistics. It is also important to notice that the universal function approximation property of some complex nonlinearities does not guarantee fast and efficient learning.

Complex nonlinearities. In 1992, Georgiou and Koutsougeras [88] proposed a list of requirements that a complex valued activation function should satisfy in order to qualify for the nonlinearity at the neuron. The calculation of complex gradients and Hessians has been detailed in work by Van Den Bos [30]. In 1995 Arena *et al.* [18] proved the *universal approximation property*[20] of a Complex Multilayer Perceptron (CMLP), based on the split-complex approach. This also gave theoretical justification for the use of complex neural networks (NNs) in time series modelling tasks, and thus gave rise to temporal neural networks. The split-complex approach has been shown to yield reasonable performance in channel equalisation applications [27, 147, 166], and in applications where there is no strong coupling between the real and imaginary part within the complex signal. However, for the common case where the inphase (I) and quadrature (Q) components have the same variance and are uncorrelated, algorithms employing split-complex activation functions tend to yield poor performance.[21] In addition, split-complex based algorithms do not have a generic form of their real-valued counterparts, and hence their signal flow-graphs are fundamentally different [220]. In the classification context, early results on Boolean threshold functions and the notion of multiple-valued threshold function can be found in [7, 8].

The problems associated with the choice of complex nonlinearities suitable for nonlinear adaptive filtering in $\mathbb{C}$ have been largely unlocked by Kim and Adali in 2003 [152]. They have identified a class of 'fully complex' activation functions (differentiable and bounded almost everywhere in $\mathbb{C}$ such as tanh), as a suitable choice, and have derived the fully complex back-propagation algorithm [150, 151], which is a generic extension of its real-valued counterpart. They also provide an insight into the character of singularities of fully complex nonlinearities, together with their universal function approximation properties. Uncini *et al.* have introduced a 2D splitting complex activation function [298], and have also applied complex neural networks in the context of blind equalisation [278] and complex blind source separation [259].

Learning algorithms. The first adaptive signal processing algorithm operating completely in $\mathbb{C}$ was the complex least mean square (CLMS), introduced in 1975 by Widrow, Mc Cool and Ball [307] as a natural extension of the real LMS. Work on complex nonlinear architectures, such as complex neural networks (NNs) started much later. Whereas the extension from real LMS to CLMS was fairly straightforward, the extensions of algorithms for nonlinear adaptive filtering from $\mathbb{R}$ into $\mathbb{C}$ have not been trivial. This is largely due to problems associated with the

[20]This is the famous 13th problem of Hilbert, which has been the basis for the development of adaptive models for universal function approximation [56, 125, 126, 155].

[21]Split-complex algorithms cannot calculate the true gradient unless the real and imaginary weight updates are mutually independent. This proves useful, e.g. in communications applications where the data symbols are made orthogonal by design.

choice of complex nonlinear activation function.[22] One of the first results on complex valued NNs is the 1990 paper by Clarke [50]. Soon afterwards, the complex backpropagation (CBP) algorithm was introduced [25, 166]. This was achieved based on the so called split-complex[23] nonlinear activation function of a neuron [26], where the real and imaginary parts of the *net* input are processed separately by two real-valued nonlinear functions, and then combined together into a complex quantity. This approach produced bounded outputs at the expense of closed and generic formulas for complex gradients. Fully complex algorithms for nonlinear adaptive filters and recurrent neural networks (RNNs) were subsequently introduced by Goh and Mandic in 2004 [93, 98]. As for nonlinear sequential state estimation, an extended Kalman filter (EKF) algorithm for the training of complex valued neural networks was proposed in [129].

Augmented complex statistics. In the early 1990s, with the emergence of new applications in communications and elsewhere, the lack of general theory for complex-valued statistical signal processing was brought to light by several authors. It was also realised that the statistics in $\mathbb{C}$ are not an analytical continuation of the corresponding statistics in $\mathbb{R}$. Thus for instance, so called 'conjugate linear' (also known as widely linear [240]) filtering was introduced by Brown and Crane in 1969 [38], generalised complex Gaussian models were introduced by Van Den Bos in 1995 [31], whereas the notions of 'proper complex random process' (closely related[24] to the notion of 'circularity') and 'improper complex random process' were introduced by Neeser and Massey in 1993 [219]. Other important results on 'augmented complex statistics' include work by Schreier and Scharf [266, 268, 271], and Picinbono, Chevalier and Bondon [237–240]. This work has given rise to the application of augmented statistics in adaptive filtering, both supervised and blind. For supervised learning, EKF based training in the framework of complex-valued recurrent neural networks was introduced by Goh and Mandic in 2007 [95], whereas augmented learning algorithms in the stochastic gradient setting were proposed by the same authors in [96]. Algorithms for complex-valued blind separation problems in biomedicine were introduced by Calhoun and Adali [40–42], whereas Eriksson and Koivunen focused on communications applications [67, 252]. Notice that properties of complex signals are not only varying in terms of their statistical nature, but also in terms of their 'dual univariate', 'bivariate', or 'complex' nature. A statistical test for this purpose based on hypothesis testing was developed by Gautama, Mandic and Van Hulle [85], whereas a test for complex circularity was developed by Schreier, Scharf and Hanssen [270]. The recent book by Schreier and Scharf gives an overview of complex statistics [269].

Hypercomplex nonlinear adaptive filters. A comprehensive introduction to hypercomplex neural networks was provided by Arena, Fortuna, Muscato and Xibilia in 1998 [17], where special attention was given to quaternion MLPs. Extensions of complex neural networks include

[22]We need to make a choice between boundedness for differentiability, since by Liouville's theorem the only continuously differentiable function on $\mathbb{C}$ is a constant.

[23]The reader should not mistake split-complex numbers for split-complex nonlinearities.

[24]Terms *proper random process* and *circular random process* are often used interchangeably, although strictly speaking, 'properness' is a second-order concept, whereas 'circularity' is a property of the probability density function, and the two terms are not completely equivalent. For more detail see Chapter 12.

neural networks whose operation is based on the geometric (Clifford) algebra, proposed by Pearson [230]. The Clifford MLPs are based on a variant of the complex activation function from [88], where the standard product of two scalar variables is replaced by a special product of two multidimensional quantities [17, 18].

A comprehensive account of standard linear and nonlinear adaptive filtering algorithms in $\mathbb{C}$, which are based on the assumption of second-order circularity of complex processes, can be found in *Adaptive Filter Theory* by Simon Haykin [113]. Complex-valued NNs in the context of classification and pattern recognition have been addressed in an edited book and a monograph by Akira Hirose [119, 120], and in work by Naum Aizenberg [6, 7], Igor Aizenberg [4, 5] and Tohru Nitta [221].

The existing statistical signal processing algorithms are based on standard complex statistics, which is a direct extension of real statistics, and boils down to exactly the same expressions as those in $\mathbb{R}$, if we

- remove complex conjugation whenever it occurs in the algorithm;
- replace the Hermitian transpose operator with the ordinary transpose operator.

This, however, applies only to the rather limited class of circular complex signals, and such solutions when applied to general complex data are suboptimal.

This book provides a comprehensive account of so-called augmented complex statistics, and offers solutions to a general adaptive filtering problem in $\mathbb{C}$.