Adaptive SP & Machine Intelligence
Complex and Multi-D Learning Systems

Danilo Mandic

room 813, ext: 46271

Department of Electrical and Electronic Engineering
Imperial College London, UK

d.mandic@imperial.ac.uk, URL: www.commsp.ee.ic.ac.uk/∼mandic

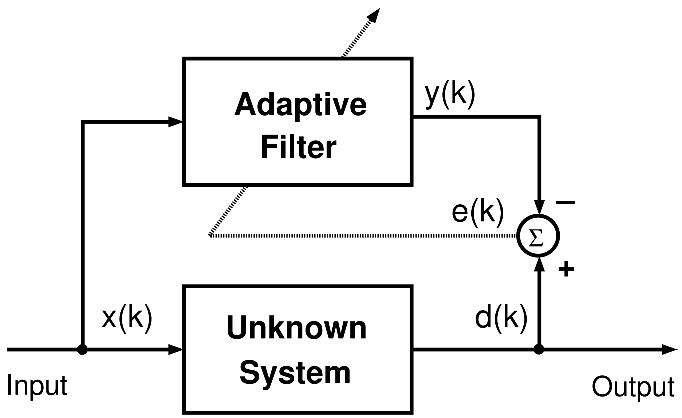
Outline:

Basically, to de-mystify multidimensional adaptive estimation

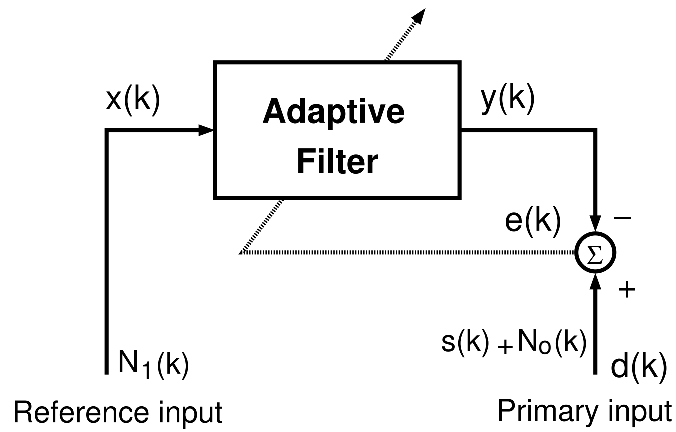
- Multidimensional and multichannel sensors – a unified approach to adaptive filtering of such signals
- Circularity – a unique signature of bivariate signals $\looparrowright$ second order circularity (properness) $\looparrowright$ augmented complex statistics
- Duality between the processing in $\mathbb{R}^2$ and $\mathbb{C}$ (isomorphism)
- The issue of complex gradient $\looparrowright$ $\mathbb{CR}$ calculus
- Covariance, pseudocovariance, and widely linear models
- Complex Wiener filter, complex least mean square (CLMS) and augmented CLMS (ACLMS)
- Magnitude-only, phase-only, and least mean magnitude phase (LMMP) approaches
- Multivariate adaptive filters (any number of data channels)
- Applications: communications, radar and sonar, target tracking, renewable energy, smart grid

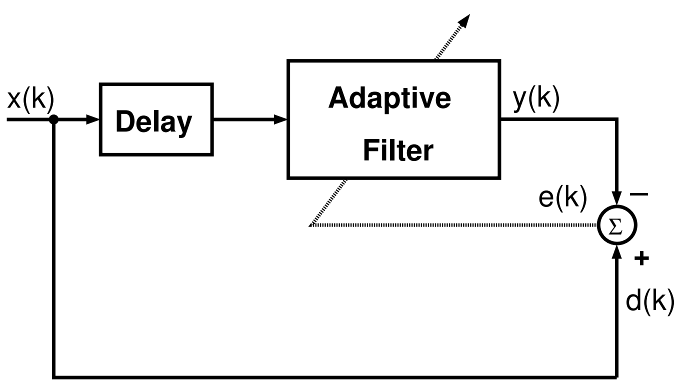
We can use multidimensional filters within our usual adaptive filtering configurations!

System Identification

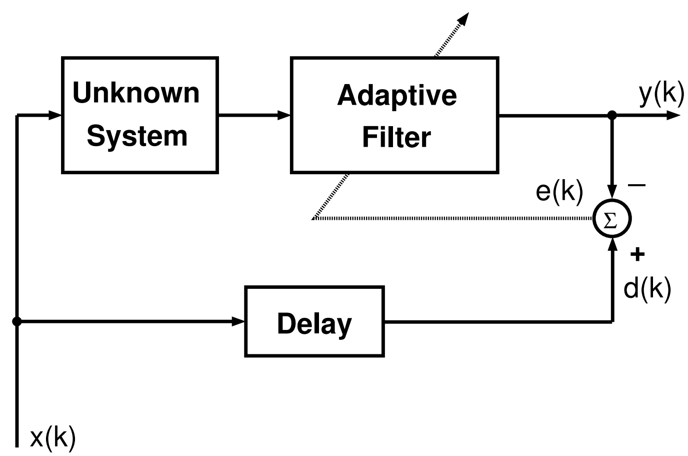


Noise Cancellation



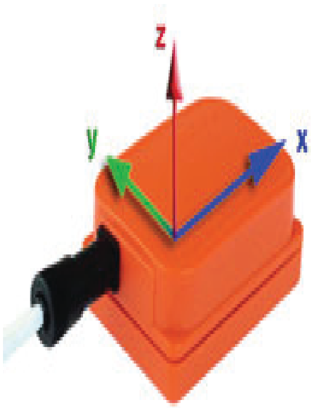


Adaptive Prediction



Inverse System Modelling

Multidimensional adaptive signal processing: Applications



Renewable Energy

2D and 3D anemometers

control of wind turbine

Body motion sensor

3D - position, gyroscope, speed

gait, biometrics

Wearable technologies

Biomechanics

virtual reality

Wind sensors - 2D and 3D anemometers

Why modelling in $\mathbb{C}$?

- Complex signals by design (communications, analytic signals, equivalent baseband represenation to eliminate spectral redundancy)
- By convenience of representation (radar, sonar, wind field)
- Problem: Different algebra (no ordering - operator "$\leq$" makes no sense!), and the notion of pdf has to be induced
- Problem: Special form of nonlinearity (the only continuously differentiable function in $\mathbb{C}$ is a constant (Liouville theorem)
- Solution: Special statistics – augmented complex statistics (started in mathematics in 1992)
- We can differentiate between several kinds of noises (doubly white circular with various distributions $n_r \perp n_i$ & $\sigma^2_{n_r} = \sigma^2_{n_i}$, doubly white noncircular $n_r \perp n_i$ & $\sigma^2_{n_r} > \sigma^2_{n_i}$, noncircular noise)

What is the right basis for real world data?

Back to Dennis Gabor $\looparrowright$ but careful: Bedrossian and Nutall theorems

Consider

- Amplitude modulated signal $x(t) = m(t)\cos(\omega_0 t) \rightarrow m(t) \looparrowright$ envelope
- Phase modulated signal $x(t) = a\cos(\Phi(t)) \rightarrow \Phi(t) \looparrowright$ phase

Problem: there is an infinite number of pairs $[a(t), \Phi(t)]$ s.t. $m(t)\cos(\omega_0 t) = a\cos(\Phi(t))$

Solution: an analytic transform $z(t) = x(t) + \jmath\mathcal{H}\big(x(t)\big) = a(t)e^{\jmath\Phi(t)}$

Remark#1: $z(t)$ **cannot** be real, as $\mathcal{F}\big(z(t)\big) = 0 \text{ for } \omega < 0$

Remark#2: Hilbert transform (analytic signal) makes it possible to associate a **unique** pair $\big[a(t), \Phi(t)\big]$ to any real $x(t) = \Re\big\{a(t)e^{\jmath\Phi(t)}\big\}$

Remark#3: For $x(t) = a(t)\cos\Phi(t) \quad \Rightarrow \quad \mathcal{H}\big\{x(t)\big\} = a(t)\sin\Phi(t)$

Remark#4: From instantaneous phase $\Phi(t) \quad \rightarrow \quad$ **instantaneous frequency**

$$f(t) = d\Phi(t)/dt$$

so we have an excellent resolution and do not depend on stationarity

Standard adaptive filtering algorithms in $\mathbb{C}$

Considered straightforward extensions of the corresponding algorithms in $\mathbb{R}$ – replace the vector transpose by the Hermitian transpose:

- Covariance

$$\mathcal{C} = E\{\mathbf{x}\mathbf{x}^T\} \qquad \rightsquigarrow \qquad \mathcal{C} = E\{\mathbf{z}\mathbf{z}^H\}$$

- Autoregressive model and Wiener solution

$$\mathbf{w} = \mathbf{R}^{-1}\mathbf{p} \qquad \rightsquigarrow \qquad \mathbf{w}^* = \mathbf{R}^{-1}\mathbf{p}$$

- Least mean square (LMS) $\rightsquigarrow$ complex LMS [Widrow *at al.* 1975]

$$\mathbf{w}(k+1) = \mathbf{w}(k) + \mu e(k)\mathbf{x}(k) \qquad \rightsquigarrow \qquad \mathbf{w}(k+1) = \mathbf{w}(k) + \mu e(k)\mathbf{z}^*(k)$$

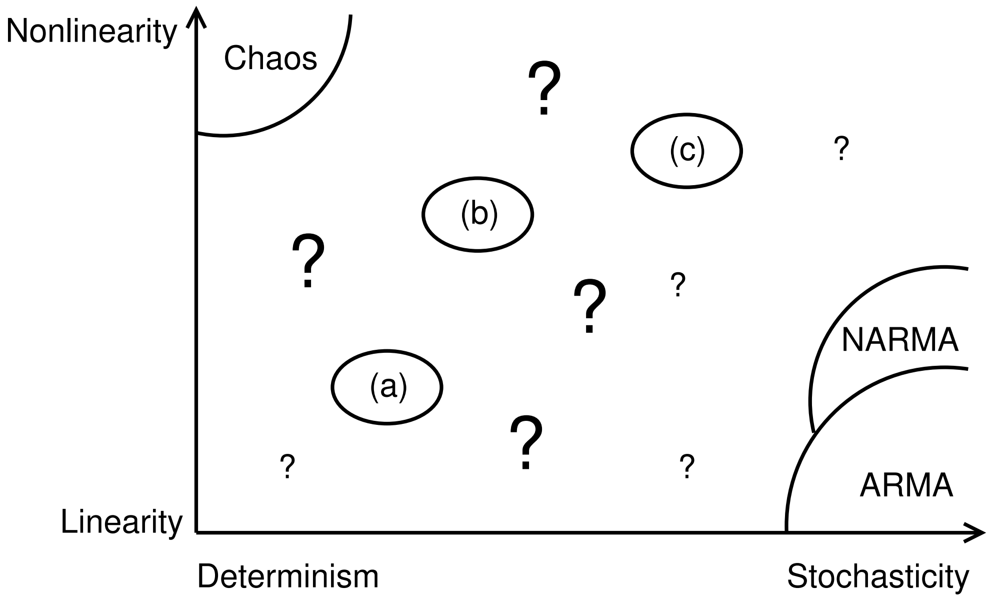
- Real time recurrent learning (RTRL) for recurrent neural networks (RNN) [Williams & Zipser 1989] $\rightsquigarrow$ complex RTRL [Goh & Mandic 2004]

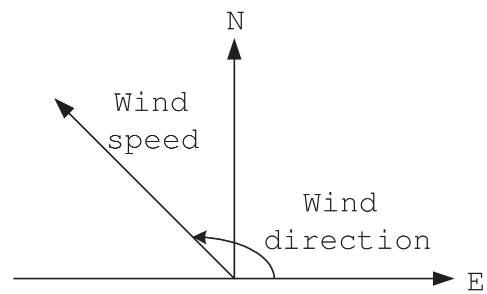
$$\mathbf{w}(k+1) = \mathbf{w}(k) + \mu e(k)\mathbf{\Pi}(k) \qquad \rightsquigarrow \qquad \mathbf{w}(k+1) = \mathbf{w}(k) + \mu e(k)\mathbf{\Pi}^*(k)$$

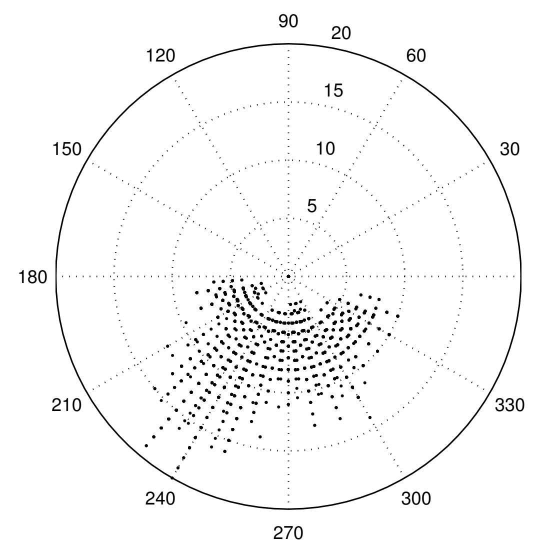
This is, however, valid only for circular complex random processes

Noncircularity of the wind distribution $\quad v(k) = |v(k)|e^{\jmath\Phi(k)}$

Deterministic vs. Stochastic nature
Linear vs. Nonlinear nature







Change in signal modality can indicate e.g. health hazard (fMRI, HRV)

Real world signals are denoted by '???'

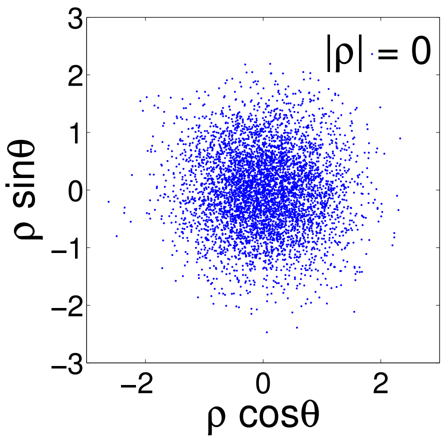
- $\exists$ a unique signature of complex signals?
- $\looparrowright$ **degree of noncircularity**

Circular vs noncircular complex random variables

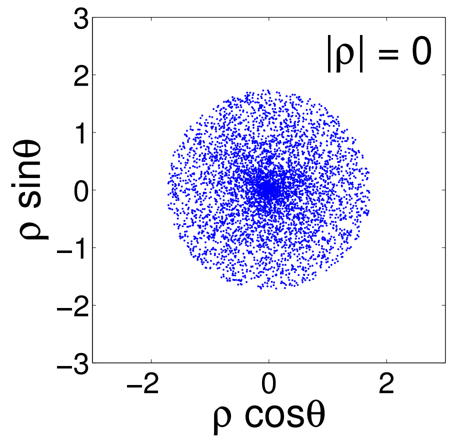
Circularity = Rotation invariant distribution $p(\rho, \theta) = p(\rho, \theta - \phi)$

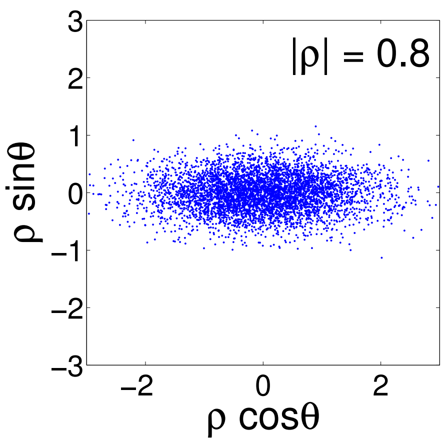
circular variable = construct $z = \rho\cos(\theta) + j\rho\sin(\theta)$, $\theta \sim \mathcal{U}[0, 2\pi)$, $\rho \sim$ any pdf

Gaussian circular

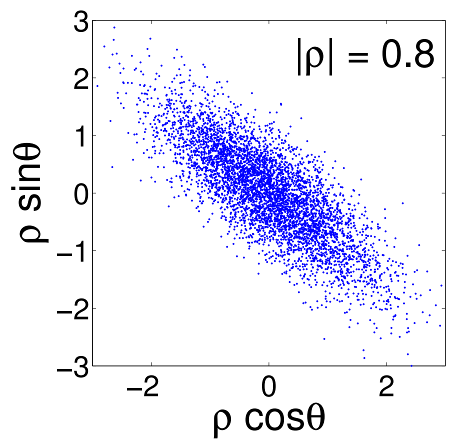


Uniform circular





Gaussian uncorrelated noncircular



Gaussian correlated noncircular

- Covariance of z is $c = E\{zz^*\} = E\{|z^2|\}$
- Pseudocovariance is $p = E\{zz^T\} = E\{z^2\}$
- Circularity quotient: $\rho = \frac{p}{c}$
- Circularity coefficient: $|\rho| = \frac{|p|}{c}$

Circularity quotient and coefficient quantify the **degree of non-circularity**

Some observations

Although a formalism similar to that used for real valued adaptive filters can also be used for complex valued adaptive filters, notice that in this case **the cost function** $J(k) = \frac{1}{2}|e(k)|^2 = \frac{1}{2}e(k)e^*(k) = \frac{1}{2}(e_x^2 + e_y^2)$ **is a real valued function of complex variable.**

- Standard complex differentiability is based on the Cauchy–Riemann (C–R) equations and imposes a stringent structure on complex holomorphic functions;

- Cost functions are real functions of complex variable, that is $J : \mathbb{C} \mapsto \mathbb{R}$, and so they are not differentiable in the complex sense, the Cauchy–Riemann equations do not apply, and we need to develop alternative, more general and relaxed, ways of calculating their gradients ($\mathbb{CR}$-calculus);

- It is also desired that these generalised gradients are formally equivalent to standard complex gradients (their generic extensions) when applied to holomorphic (analytic) functions.

Complex Wiener filter

Notice that we have used: $(\mathbf{x}^H\mathbf{y})^* = \mathbf{y}^H\mathbf{x}$

The **Wiener solution** can be obtained by minimising

$$J(\mathbf{w}) = E\big[e(k)e^*(k)\big] = E\Big[\big(d(k) - \mathbf{w}^H\mathbf{x}(k)\big)\big(d(k) - \mathbf{w}^H\mathbf{x}(k)\big)^*\Big]$$

$$\begin{aligned} J(\mathbf{w}) &= E\big[|d(k)|^2\big] - \mathbf{w}^H E\big[\mathbf{x}(k)d^*(k)\big] - \mathbf{w}^T E\big[\mathbf{x}^*(k)d(k)\big] + \mathbf{w}^H E\big[\mathbf{x}(k)\mathbf{x}^H(k)\big]\mathbf{w} \\ &= \sigma_d^2 - \mathbf{w}^H\mathbf{p} - \mathbf{p}^H\mathbf{w} + \mathbf{w}^H\mathbf{R}\mathbf{w} \quad \rightarrow \quad \nabla_{\mathbf{w}^*}J(\mathbf{w}) = \mathbf{R}\mathbf{w} - \mathbf{p} = \mathbf{0} \end{aligned}$$

The optimum solution $\mathbf{w}_o$ and the minimum mean square error $J_{\min}$

$$\mathbf{w}_o = arg\min_{\mathbf{w}} J(\mathbf{w}) = \mathbf{R}^{-1}\mathbf{p} \qquad\qquad J_{min} = J(\mathbf{w}_o) = \sigma_d^2 - \mathbf{p}^H\mathbf{R}^{-1}\mathbf{p}$$

Also, $J(\mathbf{w})$ can be expressed as
$J(\mathbf{w}) = J_{min} + (\mathbf{w} - \mathbf{w}_o)^H\,\mathbf{R}\,(\mathbf{w} - \mathbf{w}_o) = J_{min} + \mathbf{v}^H\mathbf{R}\mathbf{v}$ by noticing that $d(k) = \mathbf{w}_o^H\mathbf{x}(k) + q(k)$ and $e(k) = (\mathbf{w}_o - \mathbf{w}(k))^H\mathbf{x}_k + q(k) = \mathbf{v}^H(k)\mathbf{x}(k)$, where $q(k) \sim \mathcal{N}(0, \sigma_q^2)$.

Notice that $J(\mathbf{w})$ is quadratic in $\mathbf{w}$ and has a global minimum for $\mathbf{w} = \mathbf{w}_o$, with $J_{min} = \sigma_q^2$.

Derivative of the cost function $\frac{1}{2}e(k)e^*(k)$ and CLMS

As $\mathbb{C}$-derivatives are not defined for real functions of complex variable

$$\mathbb{R}-\text{der:}\quad \frac{\partial}{\partial \mathbf{z}} = \frac{1}{2}\left[\frac{\partial}{\partial \mathbf{x}} - \jmath\frac{\partial}{\partial \mathbf{y}}\right] \qquad \mathbb{R}^*-\text{der:}\quad \frac{\partial}{\partial \mathbf{z}^*} = \frac{1}{2}\left[\frac{\partial}{\partial \mathbf{x}} + \jmath\frac{\partial}{\partial \mathbf{y}}\right]$$

and the gradient

$$\nabla_{\mathbf{w}} J = \frac{\partial J(e,e^*)}{\partial \mathbf{w}} = \left[\frac{\partial J(e,e^*)}{\partial w_1}, \ldots, \frac{\partial J(e,e^*)}{\partial w_N}\right]^T = 2\frac{\partial J}{\partial \mathbf{w}^*} = \underbrace{\frac{\partial J}{\partial \mathbf{w}^r} + \jmath\frac{\partial J}{\partial \mathbf{w}^i}}_{pseudogradient}$$

The standard Complex Least Mean Square (CLMS) (Widrow *et al.* 1975)

$$y(k) \quad = \quad \mathbf{w}^H(k)\mathbf{x}(k)$$

$$e(k) = d(k) - y(k) = d(k) - \mathbf{w}^H(k)\mathbf{x}(k) \qquad e^*(k) = d^*(k) - \mathbf{x}^H(k)\mathbf{w}(k)$$

$$\text{and} \quad \nabla_{\mathbf{w}} J = \nabla_{\mathbf{w}^*} J$$ (conjugate gradient direction)

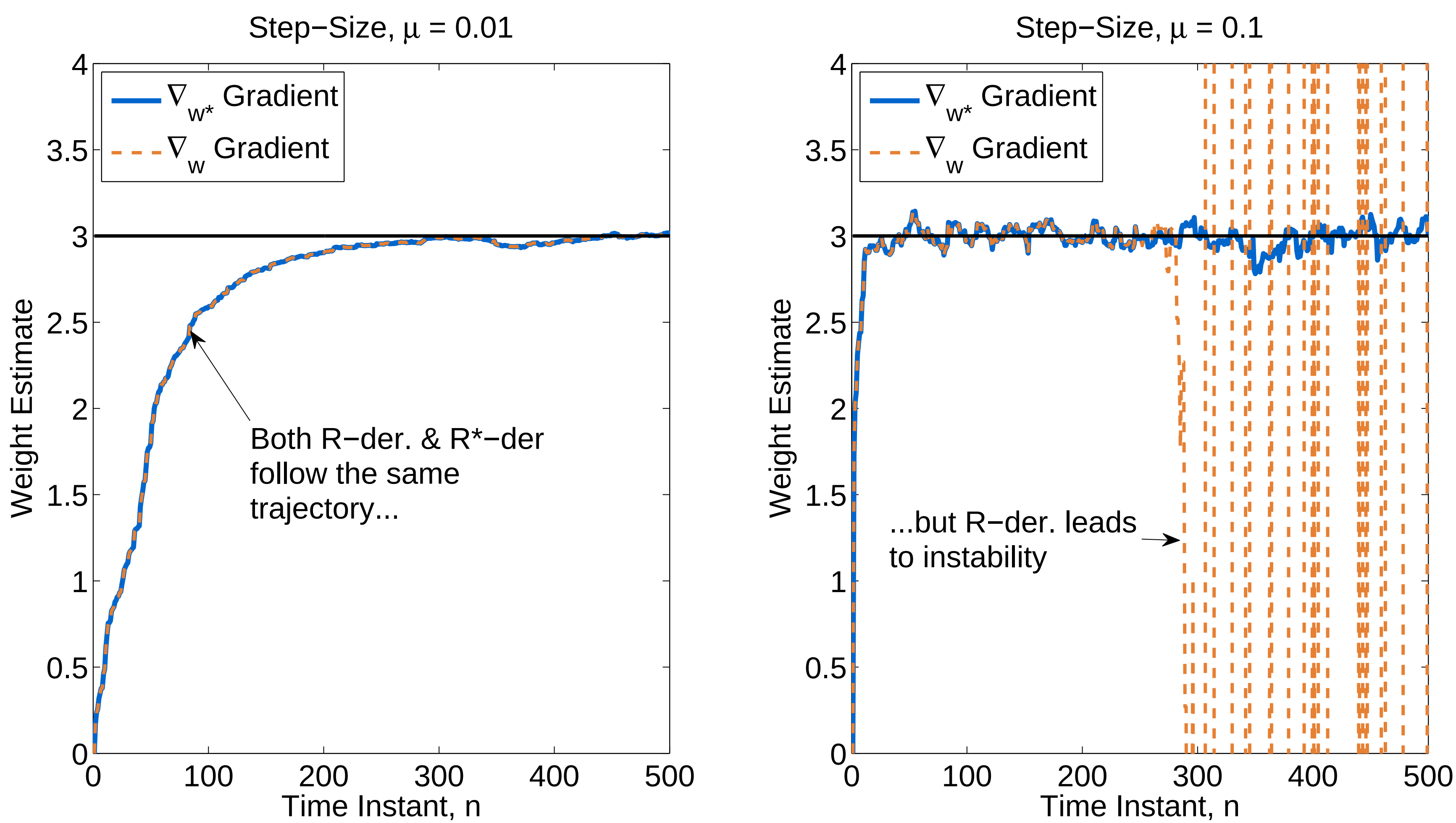
$$\mathbf{w}(k+1) \quad = \quad \mathbf{w}(k) - \mu\frac{\partial e(k)}{\partial \mathbf{w}^*(k)}e^*(k) = \mathbf{w}(k) + \mu e^*(k)\mathbf{x}(k)$$

Thus, no need for tedious computations $\looparrowright$ the CLMS is derived in one line

Which derivative to we choose to compute the gradient?

$\mathbb{R}$-derivative vs. $\mathbb{R}^*$-derivative?

Simulation for the CLMS derived using $\mathbb{R}$-der. and $\mathbb{R}^*$-der. $(\mathbf{w}_o = 3)$



Orthogonality principle & alternative forms for the CLMS

Consider the expected value for the CLMS update

$$E\big[\mathbf{w}(k+1)\big] = E\big[\mathbf{w}(k)\big] + \mu E\big[e(k)\mathbf{x}^*(k)\big]$$

It has converged when $\mathbf{w}(k+1) = \mathbf{w}(k) = \mathbf{w}(\infty)$ and thus the weight update $\Delta\mathbf{w}(k) = \mu E[e(k)\mathbf{x}^*(k)] = \mathbf{0}$.

This is achieved for $E\big[d^*(k)\mathbf{x}(k)\big] - E\big[\mathbf{x}(k)\mathbf{x}^H(k)\big]\mathbf{w}(k) = \mathbf{0} \Leftrightarrow \mathbf{R}\mathbf{w}_o = \mathbf{p}$ that is, for the Wiener solution, $\mathbf{w}_o = \mathbf{R}^{-1}\mathbf{p}$.

The condition $E[e(k)\mathbf{x}^*(k)] = \mathbf{0}$ is called the "orthogonality condition" and states that the output error of the filter and the tap input vector are orthogonal ($e \perp \mathbf{x}$) when the filter has converged to the optimal solution.

The following formulations for the CLMS produce identical results:

$$y(k) = \mathbf{x}^T(k)\mathbf{w}(k) = \mathbf{w}^T(k)\mathbf{x}(k) \qquad \rightarrow \qquad \mathbf{w}(k+1) = \mathbf{w}(k) + \mu e(k)\mathbf{x}^*(k)$$

$$y(k) = \mathbf{w}^H(k)\mathbf{x}(k) = \mathbf{x}^T(k)\mathbf{w}^*(k) \qquad \rightarrow \qquad \mathbf{w}(k+1) = \mathbf{w}(k) + \mu e^*(k)\mathbf{x}(k)$$

$$y(k) = \mathbf{x}^H(k)\mathbf{w}(k) = \mathbf{w}^T(k)\mathbf{x}^*(k) \qquad \rightarrow \qquad \mathbf{w}(k+1) = \mathbf{w}(k) + \mu e(k)\mathbf{x}(k)$$

Types of convergence of CLMS

It is of greater interest, however, to analyse the evolution of the weights in time. As with any other estimation problem, we need to analyse the "bias" and "variance" of the estimator, that is:

- **Convergence in the mean**, to ascertain whether $\mathbf{w}(k) \rightarrow \mathbf{w}_o$ when $k \rightarrow \infty$;
- **Convergence in the mean square**, in order to establish whether the variance of the weight error vector $\mathbf{v}(k) = \mathbf{w}(k) - \mathbf{w}_o(k)$ approaches J_{min} as $k \rightarrow \infty$.

The analysis of convergence of linear adaptive filters is made mathematically tractable if we use so called **independence assumptions**, such that the filter coefficients are

⊛ statistically independent of the data currently in filter memory, and

⊛ $\{d(l), x(l)\}$ is independent of $\{d(k), x(k)\}$ for $k \neq l$.

Convergence of CLMS in the Mean

Assume $d(k) = \mathbf{x}^H(k)\mathbf{w}_o + q(k),\; q(k) \sim \mathcal{N}(0, \sigma_q^2)$. Then

$$\begin{aligned} e(k) &= \mathbf{x}^H(k)\mathbf{w}_o + q(k) - \mathbf{x}^H(k)\mathbf{w}(k) \\ \mathbf{w}(k+1) &= \mathbf{w}(k) + \mu\mathbf{x}(k)\mathbf{x}^H\mathbf{w}_o - \mu\mathbf{x}(k)\mathbf{x}^H(k)\mathbf{w}(k) + \mu q(k)\mathbf{x}(k) \end{aligned}$$

Subtract $\mathbf{w}_o$ from both sides, and define the weight error vector $\mathbf{v}(k) \stackrel{\text{def}}{=} \mathbf{w}(k) - \mathbf{w}_o$

$$\mathbf{v}(k+1) = \mathbf{v}(k) - \mu\mathbf{x}(k)\mathbf{x}^H(k)\mathbf{v}(k) + \mu q(k)\mathbf{x}(k)$$

Applying the statistical expectation operator $E\{\cdot\}$ gives the mean weight error recursion

$$E[\mathbf{v}(k+1)] = \big(\mathbf{I} - \mu E[\mathbf{x}(k)\mathbf{x}^H(k)]\big)E[\mathbf{v}(k)] + \mu\,\underbrace{E[q(k)\mathbf{x}(k)]}_{=\mathbf{0}}$$

$$\Longrightarrow\; E[\mathbf{v}(k+1)] = \big(\mathbf{I} - \mu\mathbf{R}\big)E[\mathbf{v}(k)]$$

Challenge: Unless the correlation matrix $\mathbf{R}$ is diagonal, there will be cross–coupling between the coefficients of the weight error vector.

Convergence of CLMS in the Mean (contd.)

Solution: Since the eigenvalue decomposition of $\mathbf{R} = \mathbf{Q}\boldsymbol{\Lambda}\mathbf{Q}^H$, the rotation of the weight error vector $\mathbf{v}(k)$ by the eigenvector matrix $\mathbf{Q}$, that is, $\mathbf{v}'(k) \stackrel{\text{def}}{=} \mathbf{Q}^H E\{\mathbf{v}(k)\}$, decouples the evolution of its coefficients.

Proof: Pre-multiply both sides of mean weight vector error recursion by $\mathbf{Q}^H$

$$\underbrace{\mathbf{Q}^H E[\mathbf{v}(k+1)]}_{\mathbf{v}'(k+1)} = \mathbf{Q}^H \big(\mathbf{I} - \mu\mathbf{Q}\boldsymbol{\Lambda}\mathbf{Q}^H\big) E[\mathbf{v}(k)] = \big(\mathbf{I} - \mu\boldsymbol{\Lambda}\big) \underbrace{\mathbf{Q}^H E[\mathbf{v}(k)]}_{\mathbf{v}'(k)}$$

Since the eigenvector matrix is unitary, i.e. $\mathbf{Q}^H\mathbf{Q} = \mathbf{I}$, the **modes of convergence** are

$$\mathbf{v}'(k+1) = (\mathbf{I} - \mu\boldsymbol{\Lambda})\,\mathbf{v}'(k)$$

This recursion will be stable if the diagonal elements of $(\mathbf{I} - \mu\boldsymbol{\Lambda})$ satisfy

$$|1 - \mu\lambda_i| < 1, \quad \text{for } i = 1, \ldots, N \qquad \Longrightarrow \; 0 < \mu < \frac{2}{\lambda_{\max}} < \frac{2}{\text{tr}[\mathbf{R}]}$$

Since $\mathrm{tr}[\mathbf{R}] = NE\{|x(k)|^2\}$, an easier to estimate bound $0 < \mu < \frac{2}{NE[|x(k)|^2|]}$.

Convergence of CLMS in the Mean Square

As the filter coefficients converge in the mean, they fluctuate around their optimum values $\mathbf{w}_o$. As a result, the mean square error $\xi(k) = E[|e(k)|^2]$ exceeds the minimum mean square error J_{min} by an amount referred to as the **excess mean square error**, denoted by $\xi_{EMSE}(k)$, that is

$$\xi(k) = J_{min} + \xi_{EMSE}(k) \quad \overset{J_{min}=\sigma_d^2}{\Longrightarrow} \quad \xi(k) = \sigma_q^2 + E\left[\mathbf{v}^H(k)\mathbf{R}\mathbf{v}(k)\right] = \sigma_q^2 + \text{tr}[\mathbf{R}\mathbf{K}(k)]$$

We have used the identity $E[\mathbf{v}^H(k)\mathbf{R}\mathbf{v}(k)] = \text{tr}[\mathbf{R}\mathbf{K}(k)] = \text{tr}[\mathbf{K}(k)\mathbf{R}]$, where $\mathbf{K}(k) = E[\mathbf{v}(k)\mathbf{v}^H(k)]$.

The excess mean square error depends on second order statistical properties of $d(k), \mathbf{x}(k), \mathbf{w}(k), e(k)$. **The plot showing time evolution of the mean square error is called the *learning curve*.**

For convergence in the **mean square** the misadjustment

$$\mathcal{M} = \frac{\xi_{EMSE}(\infty)}{\xi_{min}} = \frac{\xi_{EMSE}(\infty)}{\sigma_q^2} \approx \frac{1}{2}\mu\sigma_x^2 N$$

must be bounded and positive, that is, $(1 - \frac{1}{2}\mu\text{tr}[\mathbf{R}] > 0)$, and therefore

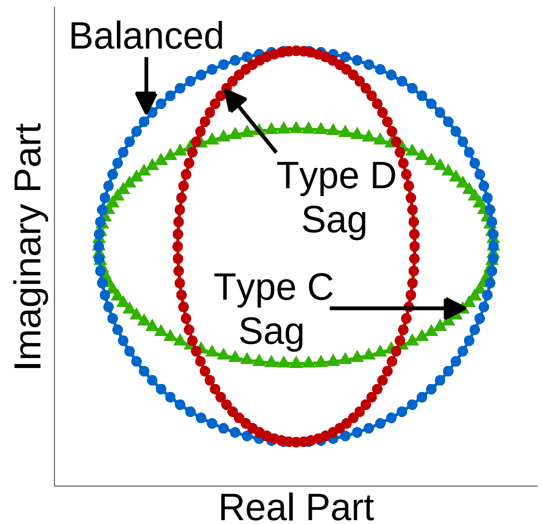
$$0 < \mu < 2/\text{tr}[\mathbf{R}] \quad \leftrightarrow \quad 0 < \mu < 2/(\sigma_\text{x}^2\text{N}).$$

Does Circularity Influence Estimation in $\mathbb{C}$?

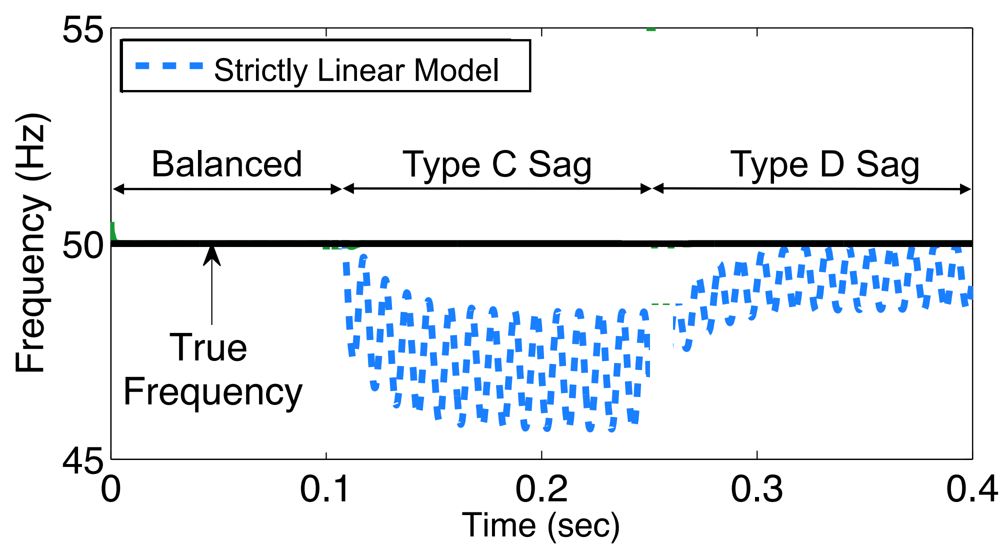
Voltage Sag: A magnitude and/or phase imbalance

- For balanced systems, $v(k) = A(k)e^{\jmath\omega k\Delta T} \rightarrow$ circular trajectory.
- Unbalanced systems, $v(k) = A(k)e^{\jmath\omega k\Delta T} + B(k)\mathbf{e}^{-\jmath\omega\mathbf{k}\boldsymbol{\Delta}\mathbf{T}}$ are influenced by the "conjugate" component.
- We need the complex conjugate when the modelling the signal.

Circularity Diagram



Strictly linear model yields biased estimates when system is unbalanced



What are we doing wrong $\looparrowright$ the Widely Linear Model

Consider the MSE estimator of a signal y in terms of another observation x

$$\hat{y} = E[y|x]$$

For zero mean, jointly normal y and x, the solution is

$$\hat{y} = \mathbf{h}^T\mathbf{x}$$

In standard MSE in the complex domain $\hat{y} = \mathbf{h}^H\mathbf{x}$, however

$$\hat{y}_r = E[y_r|x_r, x_i] \quad \& \quad \hat{y}_i = E[y_i|x_r, x_i]$$

$$thus \qquad \hat{y} = E[y_r|x_r, x_i] + \jmath E[y_i|x_r, x_i]$$

Upon employing the identities $x_r = (x + x^*)/2$ and $x_i = (x - x^*)/2\jmath$

$$\hat{y} = E[y_r|x, x^*] + \jmath E[y_i|x, x^*]$$

and thus arrive at the **widely linear** estimator for general complex signals

$$y = \mathbf{h}^T\mathbf{x} + \mathbf{g}^T\mathbf{x}^*$$

We can now process general (noncircular) complex signals!

Widely linear autoregressive modelling in $\mathbb{C}$

Standard AR model of order n is given by

$$z(k) = a_1 z(k-1) + \cdots + a_n z(k-n) + q(k) = \mathbf{a}^T \mathbf{z}(k) + q(k),$$

Using the Yule-Walker equations the AR coefficients are found from

$$\begin{aligned} \mathbf{a}^* &= \mathcal{C}^{-1}\mathbf{c} \\ \begin{bmatrix} a_1^* \\ a_2^* \\ \vdots \\ a_n^* \end{bmatrix} &= \begin{bmatrix} c(0) & c^*(1) & \dots & c^*(n-1) \\ c(1) & c(0) & \dots & c^*(n-2) \\ \vdots & \vdots & \ddots & \vdots \\ c(n-1) & c(n-2) & \dots & c(0) \end{bmatrix}^{-1} \begin{bmatrix} c(1) \\ c(2) \\ \vdots \\ c(n) \end{bmatrix} \end{aligned}$$

where $\mathbf{c} = [c(1), c(2), \dots, c(n)]^T$ is the time shifted correlation vector.

Widely linear model

$$y(k) = \mathbf{h}^T(k)\mathbf{x}(k) + \mathbf{g}^T(k)\mathbf{x}^*(k) + q(k)$$

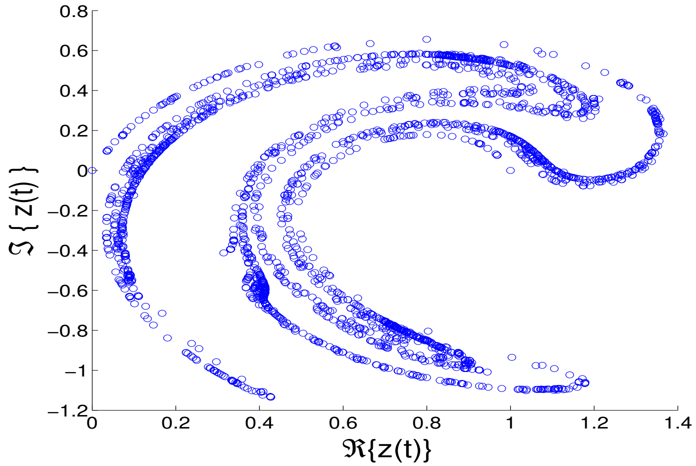
Widely linear normal equations

$$\begin{bmatrix} \mathbf{h}^* \\ \mathbf{g}^* \end{bmatrix} = \begin{bmatrix} \mathcal{C} & \mathcal{P} \\ \mathcal{P}^* & \mathcal{C}^* \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{c} \\ \mathbf{p}^* \end{bmatrix}$$

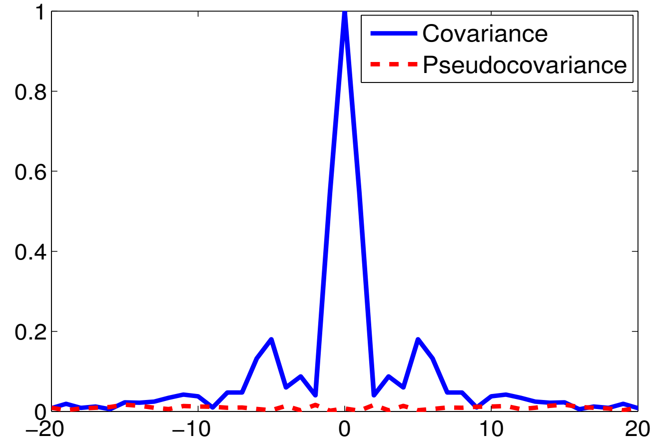
where $\mathbf{h}$ and $\mathbf{g}$ are coefficient vectors and $\mathbf{x}$ the regressor vector.

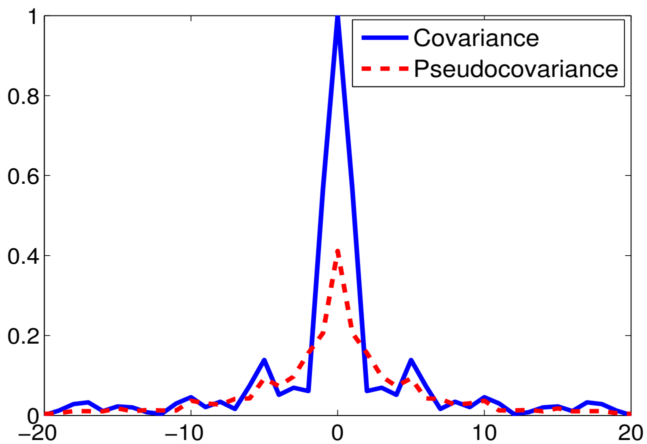
This is a rigorous way to model general complex signals!

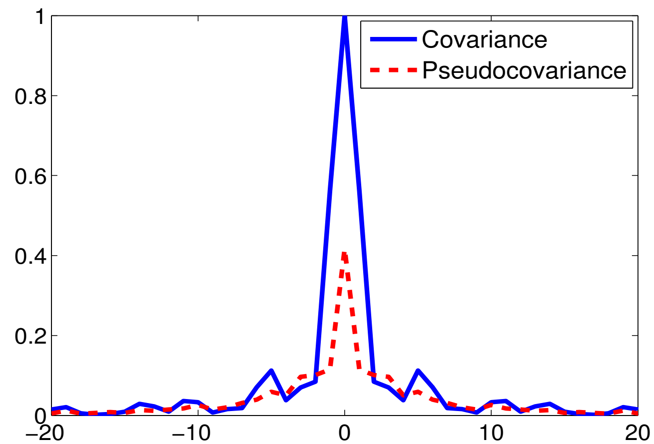
Circularity for Ikeda map



AR model of Ikeda signal







Covariances: Original Ikeda Widely linear AR of Ikeda

The Augmented (widely linear) CLMS (ACLMS)

Widely linear model $y(k) = \mathbf{h}^{\mathsf{T}}(k)\mathbf{z}(k) + \mathbf{g}^{\mathsf{T}}(k)\mathbf{z}^{*}(k)$

$$\mathbf{h}(k+1) = \mathbf{h}(k) - \mu\nabla_{\mathbf{h}^*}J \qquad \Rightarrow \qquad \nabla_{\mathbf{h}^*}J = -e(k)\mathbf{x}^{*}(k)$$

$$\mathbf{g}(k+1) = \mathbf{g}(k) - \mu\nabla_{\mathbf{g}^*}J \qquad \Rightarrow \qquad \nabla_{\mathbf{g}^*}J = -e(k)\mathbf{x}(k)$$

Therefore, the ACLMS update

$$\begin{aligned}\mathbf{h}(k+1) &= \mathbf{h}(k) + \mu e(k)\mathbf{x}^{*}(k)\\ \mathbf{g}(k+1) &= \mathbf{g}(k) + \mu e(k)\mathbf{x}(k)\end{aligned}$$

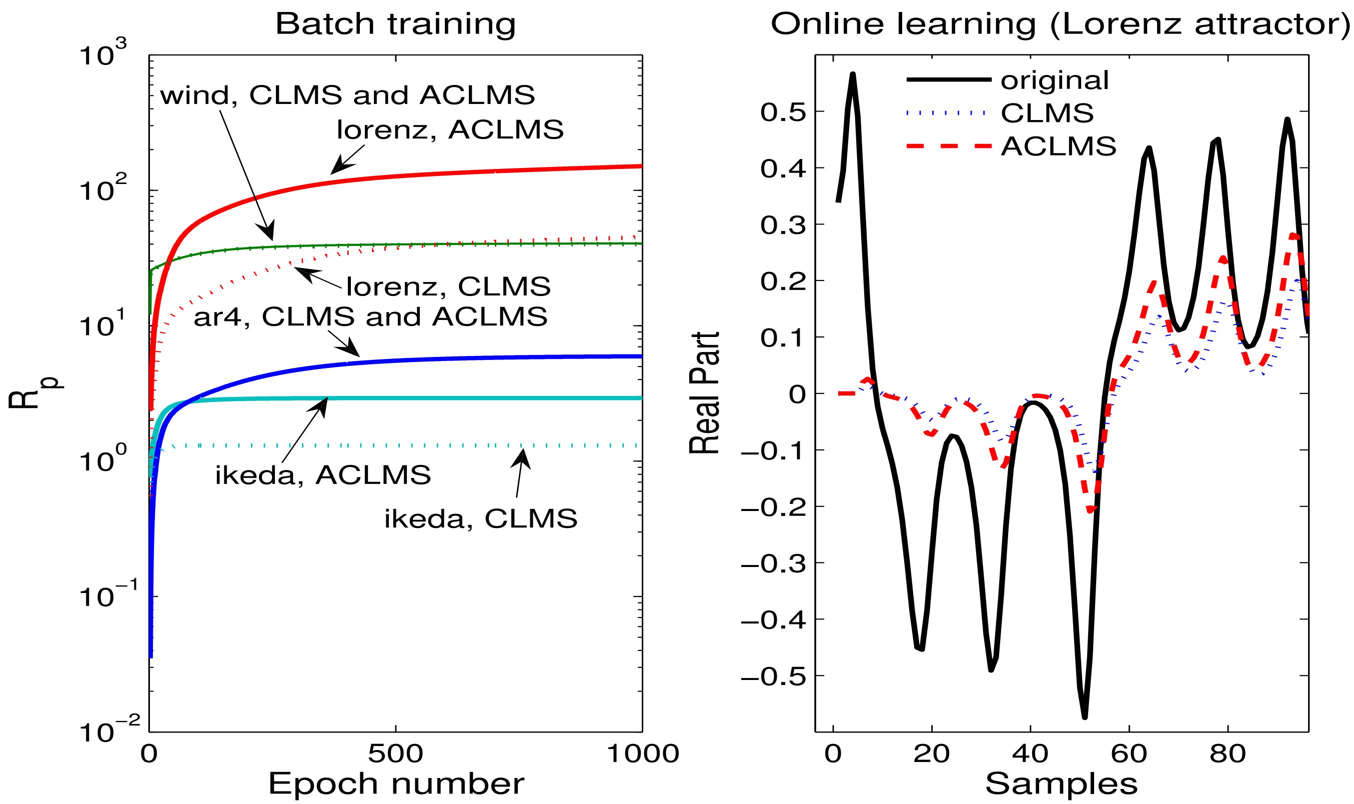
or in a more compact form (using augmented input and weight vectors)

$$\mathbf{w}^{a}(k+1) = \mathbf{w}^{a}(k) + \eta e^{a}(k)\mathbf{x}^{a^*}(k)$$

where $\eta = \mu_h = \mu_g$, $\mathbf{w}^{a}(k) = [\mathbf{h}^{T}(k),\ \mathbf{g}^{T}(k)]^{T}$, $\mathbf{x}^{a}(k) = [\mathbf{x}^{T}(k),\ \mathbf{x}^{H}(k)]^{T}$, $e^{a}(k) = d(k) - \mathbf{x}^{a^T}(k)\mathbf{w}^{a}(k)$

Performance of the ACLMS

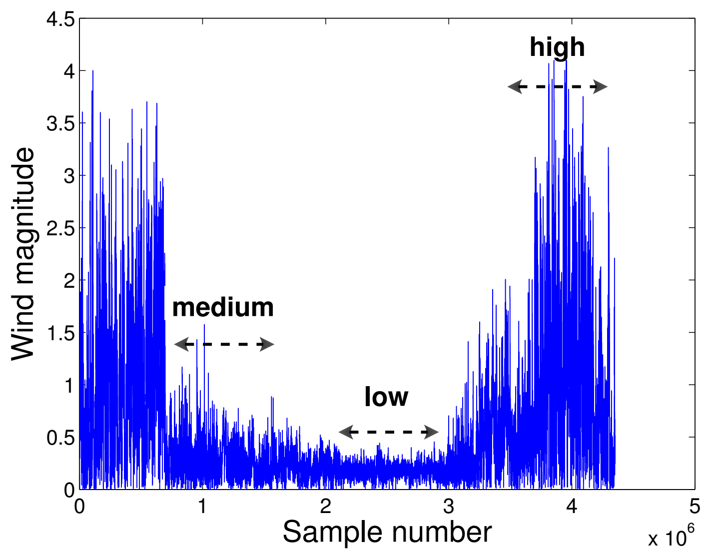
Evaluated for both second order circular (proper) and improper signals.



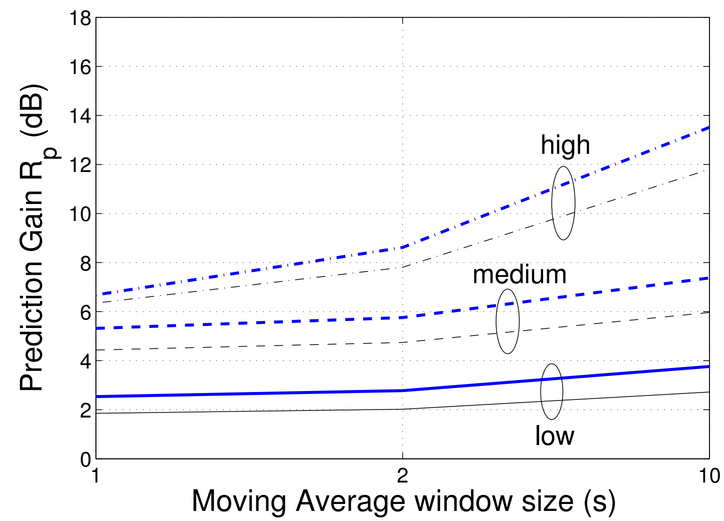
The ACLMS outperforms CLMS for second order noncircular signals.

Wind modelling: Performance vs dynamics vs circularity

Data recorded in an urban environment over one day



(a) Modulus of complex windover one day



(b) CLMS vs ACLMS for different wind regimes. CLMS - black, ACLMS - blue

Different wind regimes $\rightsquigarrow$ different dynamics,

$$v(k) = |v(k)|e^{\jmath\Phi(k)},\ |v| \text{ - speed, } \Phi \text{ - direction}$$

Different dynamics $\rightsquigarrow$ different circularity properties $\rightsquigarrow$ impact of ACLMS

Real bivariate or complex: Isomorphism between $\mathbb{C}$ and $\mathbb{R}^2$

(also serves as a basis for the $\mathbb{CR}$ calculus)

$$z \rightarrow z^a \quad \leftrightarrow \quad \begin{bmatrix} z \\ z^* \end{bmatrix} = \begin{bmatrix} 1 & \jmath \\ 1 & -\jmath \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

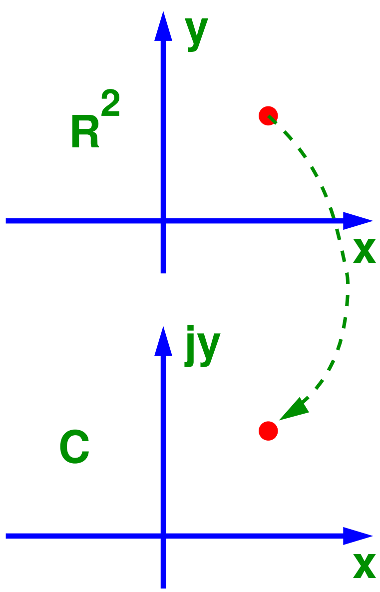
whereas in the case of complex–valued signals, we have

$$\mathbf{z} \rightarrow \mathbf{z}^a \quad \leftrightarrow \quad \begin{bmatrix} \mathbf{z} \\ \mathbf{z}^* \end{bmatrix} = \begin{bmatrix} \mathbf{I} & \jmath\mathbf{I} \\ \mathbf{I} & -\jmath\mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix}$$

For convenience, the "augmented" complex vector $\mathbf{v} \in \mathbb{C}^{2N\times 1}$ can be introduced as

$$\begin{aligned} \mathbf{v} &= [z_1, z_1^*, \ldots, z_N, z_N^*]^T \\ \mathbf{v} &= \mathbf{A}\mathbf{w}, \qquad \mathbf{w} = [x_1, y_1, \ldots, x_N, y_N]^T \end{aligned}$$

where matrix $\mathbf{A} = diag(\mathbf{J}, \ldots, \mathbf{J}) \in \mathbb{C}^{2N\times 2N}$ is block diagonal and transforms the **composite** real vector $\mathbf{w}$ into the augmented complex vector $\mathbf{v}$.



Duality between bivariate real and complex filters

Desired signal: $\hat{d}(k) = \hat{d}_r(k) + \jmath \hat{d}_i(k)$, **Input:** $\mathbf{x}(k) = \mathbf{x}_r(k) + \jmath \mathbf{x}_i(k)$,

The bivariate (dual channel) real filter:

$$\begin{bmatrix} \hat{d}_r(k) \\ \hat{d}_i(k) \end{bmatrix} = \begin{bmatrix} \mathbf{a}^T(k) & \mathbf{b}^T(k) \\ \mathbf{c}^T(k) & \mathbf{d}^T(k) \end{bmatrix} \begin{bmatrix} \mathbf{x}_r(k) \\ \mathbf{x}_i(k) \end{bmatrix}$$

The strictly linear model $\hat{d}(k) = \mathbf{w}^H(k)\mathbf{x}(k)$ $(\mathbf{w}(k) = \mathbf{w}_r(k) + \jmath \mathbf{w}_i(k))$ in the CLMS is highly restrictive since it imposes the condition

$$\mathbf{a}(k) = \mathbf{d}(k) = \mathbf{w}_r(k) \quad \text{and} \quad \mathbf{b}(k) = -\mathbf{c}(k) = \mathbf{w}_i(k)$$

Augmented CLMS: $\hat{d}(k) = \mathbf{h}^H(k)\mathbf{x}(k) + \mathbf{g}^H(k)\mathbf{x}^*(k)$

$$\begin{bmatrix} \hat{d}_r(k) \\ \hat{d}_i(k) \end{bmatrix} = \begin{bmatrix} (\mathbf{h}_r(k) + \mathbf{g}_r(k))^T & (\mathbf{h}_i(k) - \mathbf{g}_i(k))^T \\ -(\mathbf{h}_i(k) + \mathbf{g}_i(k))^T & (\mathbf{h}_r(k) - \mathbf{g}_r(k))^T \end{bmatrix} \begin{bmatrix} \mathbf{x}_r(k) \\ \mathbf{x}_i(k) \end{bmatrix}$$

Sufficient degrees of freedom to model general complex signals!

The bound on the step-size which preserves convergence: $0 < \mu < \frac{2}{\lambda_{max}}$

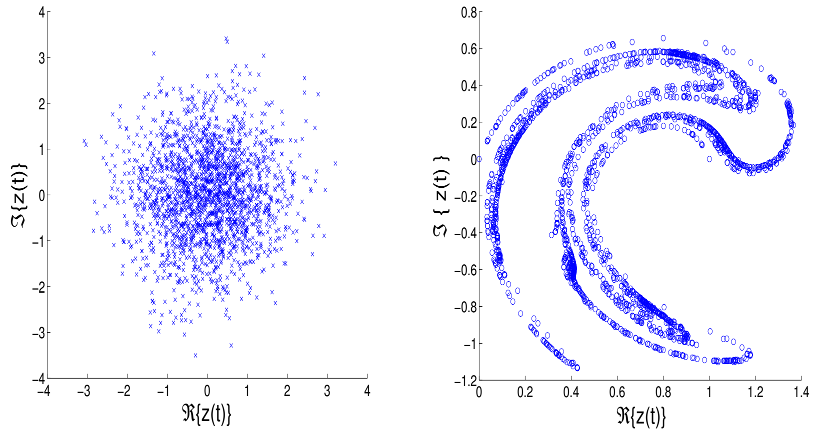
The bivariate and augmented complex correlation matrices related as

$$\mathcal{C}^a = E[\mathbf{z}^a \mathbf{z}^{aH}] = E\big[\mathbf{A}\boldsymbol{\omega}\boldsymbol{\omega}^T\mathbf{A}^H\big] = \mathbf{AWA}^H$$

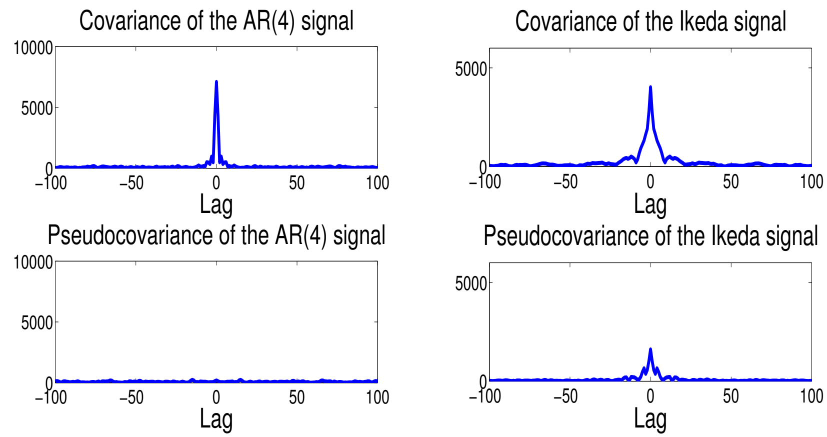
It then follows that $\lambda^a = 2\lambda^\omega$.

☞ ACLMS and DCRLMS converge at the same speed when $\mu_r = 2\mu_{aclms}$.

Duality: Simulations



AR(4) and Ikeda signal



Covariances and pseudocovariances

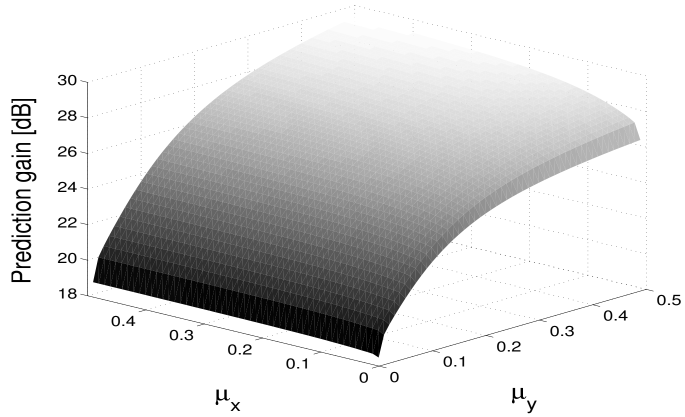
$$R_p = 10 \log \frac{\sigma_z^2}{\sigma_e^2}$$

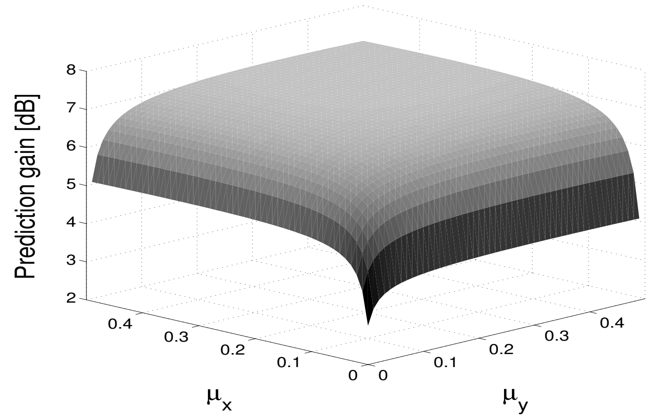
Algorithm	AR4	Ikeda	Wind
R_p for DCRLMS	5.8423	3.9733	13.2604
R_p for CLMS	6.6380	2.4278	14.2941
R_p for ACLMS	6.6096	4.0330	14.8926
R_p for DCRLMS (double μ)	6.6096	4.0330	14.8926

Duality simulations: Dependence on the parameters

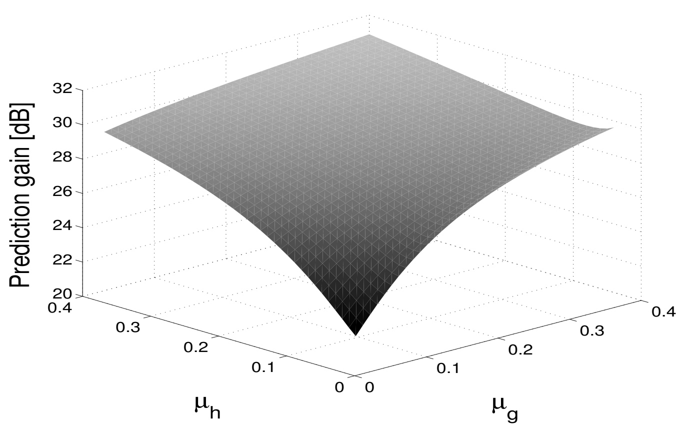
(observe the possibility for better tuning using the complex ACLMS)

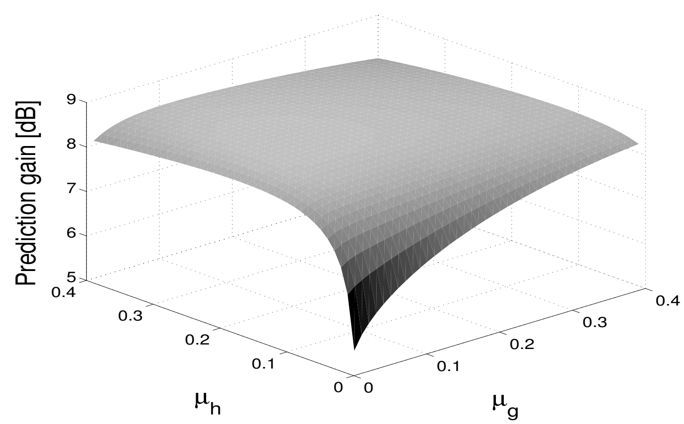
DCRLMS Lorenz (left) and wind signal (right)





ACLMS Lorenz (left) and wind signal (right)





Part 2: Advanced learning algorithms and applications of Complex-Valued Adaptive Filters

Relationship between a complex variable and its conjugate

Consider the problem of using a zero-mean r.v. $z \in \mathbb{C}$ to estimate its complex conjugate, that is

$$\hat{z}^* = w^* z$$

find an estimate of w that minimizes

$$J_{\mathsf{MSE}} = E\{|e|^2\} = E\{|z^* - \hat{z}^*|^2\}$$

The Wiener solution

$$w_{\mathsf{opt}} = c^{-1} r = \frac{E\{z^2\}}{E\{|z|^2\}} = \frac{p}{c}$$

$\rightarrow$ Circularity Quotient!!!

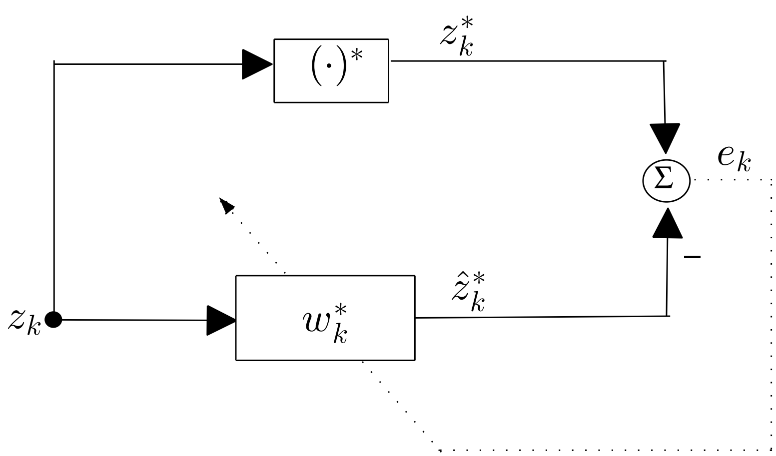
Real time tracker of complex circularity

Idea: Use an **adaptive filter** to estimate the complex conjugate of a signal using the original signal as the input.

Since the complex least mean square (CLMS) [Widrow 1975] estimates the Wiener solution, we can configure it with input signal z_k and the desired signal z_k^*.

The one-tap filter weight is the estimate of the circularity quotient ρ.

Adaptive filtering configuration



CLMS algorithm

$$\hat{z}_k^* = w_k^* z_k$$

$$e_k = z_k^* - \hat{z}_k^*$$

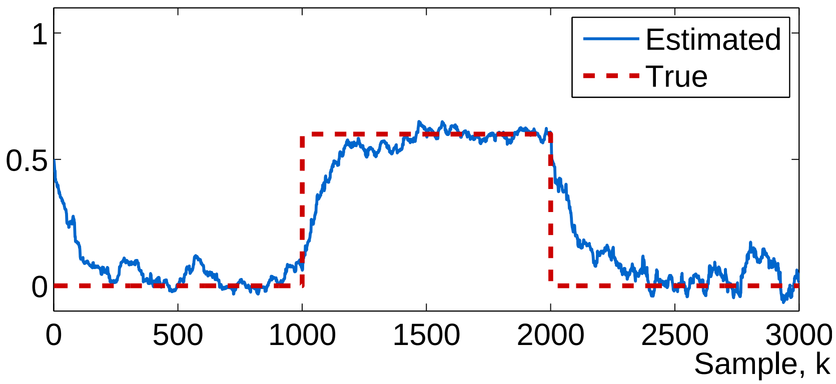
$$w_{k+1} = w_k + \mu e_k^* z_k$$

Real time tracker of complex circularity

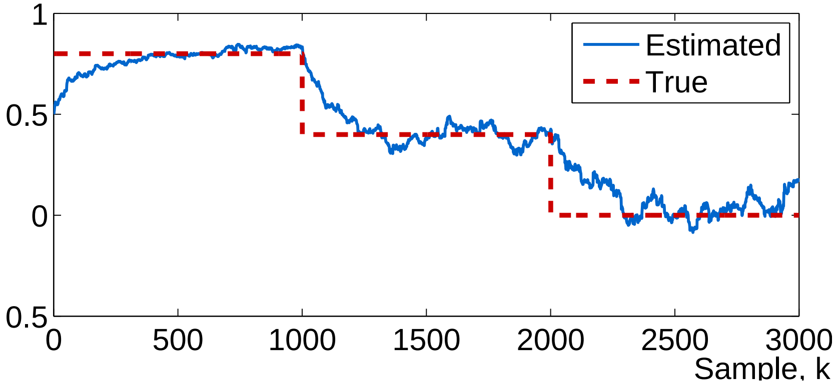
Simulations on synthetic white Gaussian data

The real and imaginary parts of the evolution of the CLMS weights when tracking circularity.

Real Part of the Circularity Quotient



Imaginary Part of the Circularity Quotient



Synthetic signal was generated by concatenating three segments of zero-mean white Gaussian signals, $z_{i,k}$, with different properties, where

$$z_{i,k} = x_{i,k} + jy_{i,k} \quad z_i \sim \mathcal{N}(0, c, p_i)$$

$$i = \{1, 2, 3\}$$

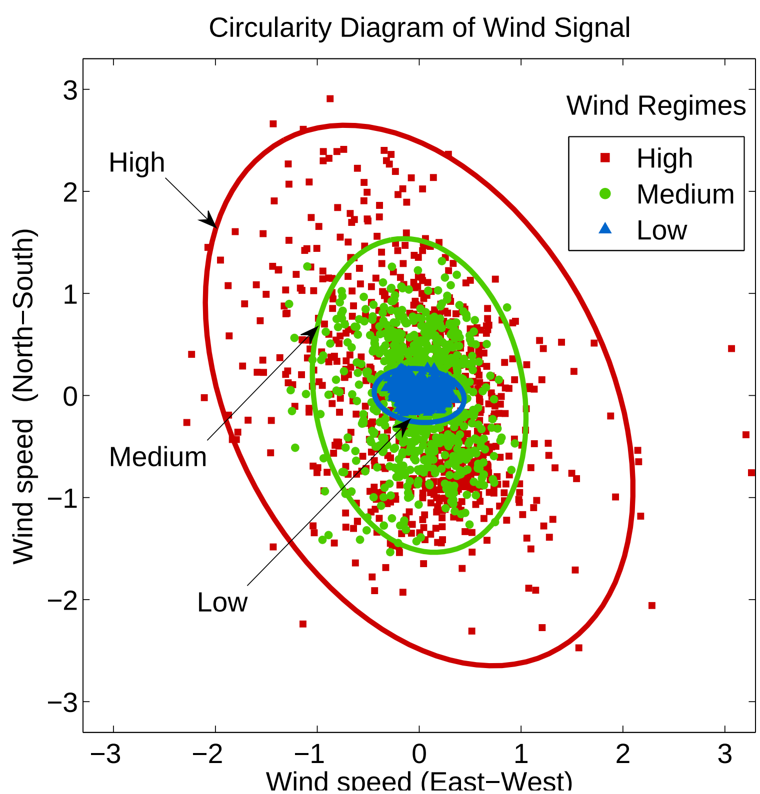
Each segment:

Sample, k	c	p_i	$\lvert\rho_i\rvert$
$1 - 1000$	1	$0.8j$	0.8
$1001 - 2000$	1	$0.6 + 0.4j$	0.72
$2001 - 3000$	1	0	0

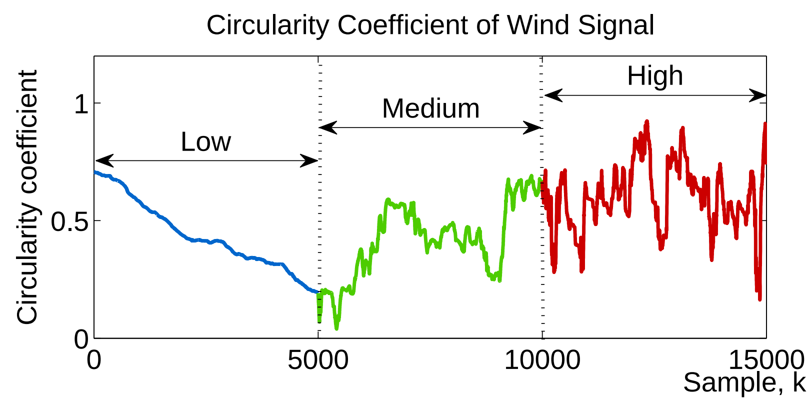
Real time tracker of complex circularity

Simulations on wind sensor data

The circularity diagram of wind speeds in the low, medium and high dynamic regimes.



The estimate of circularity coefficient for wind signals in the low, medium and high dynamic regimes using the proposed algorithm.

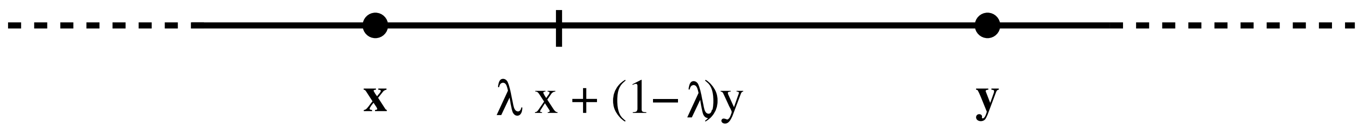


Wind signal modelled as wind a complex number

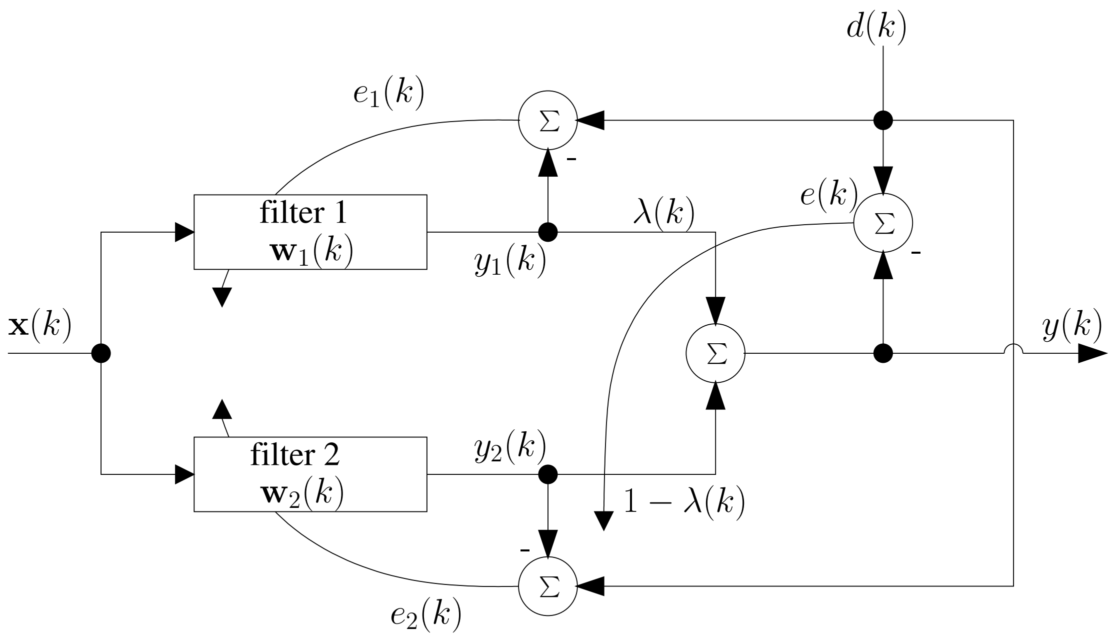
$$s = s_{\mathrm{E}} + j s_{\mathrm{N}}$$

Circular vs. Noncircular: Fast convergence vs. accuracy
Best of both worlds $\looparrowright$ a Hybrid Filter

Virtues of Convex Combination ($\lambda \in [0, 1]$)



Convexity $\Rightarrow$ existence and uniqueness of solution ☺



Let Filter1 be trained by CLMS and Filter2 by ACLMS

Adaptation of the mixing parameter λ

To preserve their inherent characteristics, subfilters, $Filter1$ and $Filter2$ updated based on their own errors:

- Linear $e_{clms}(k)$

- Widely linear $e_{aclms}(k)$

The convex mixing parameter λ is updated based on based on

$$J(k) = \frac{1}{2}e(k)e^*(k) = \frac{1}{2}|e(k)|^2 \quad \rightsquigarrow \quad \nabla_\lambda J(k)_{\lambda=\lambda(k)} = e(k)\frac{\partial e^*(k)}{\partial \lambda(k)} + e^*(k)\frac{\partial e(k)}{\partial \lambda(k)}$$

and the stochastic gradient based update of λ becomes

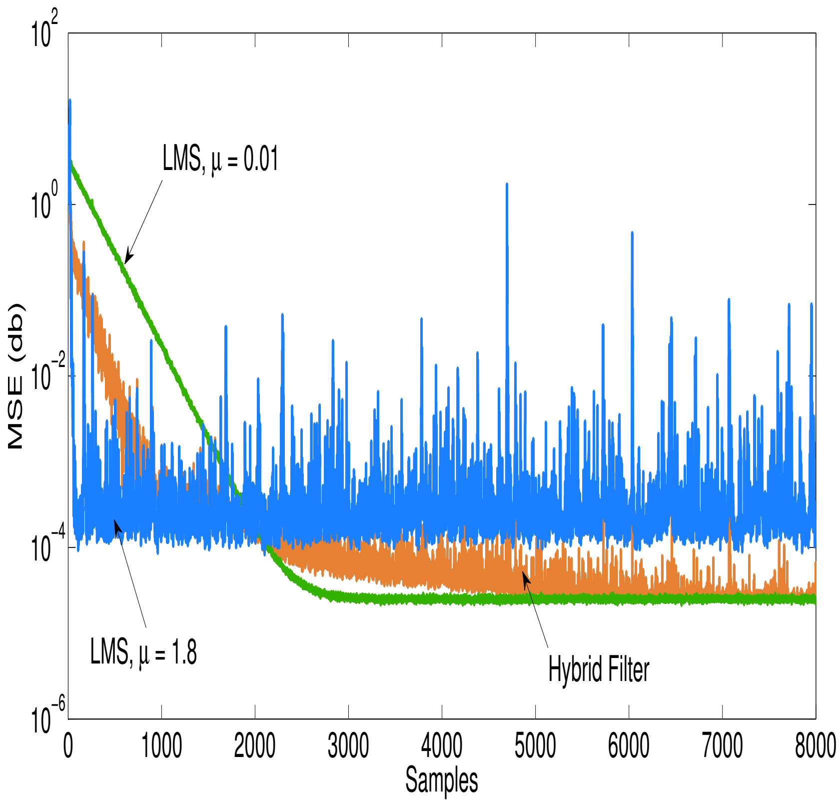
$$\begin{aligned} \lambda(k+1) &= \lambda(k) + \mu_\lambda \Big[e(k)\big(y_{aclms}(k) - y_{clms}(k)\big)^* \\ &+ e^*(k)\big(y_{clms}(k) - y_{aclms}(k)\big)\Big] \end{aligned}$$

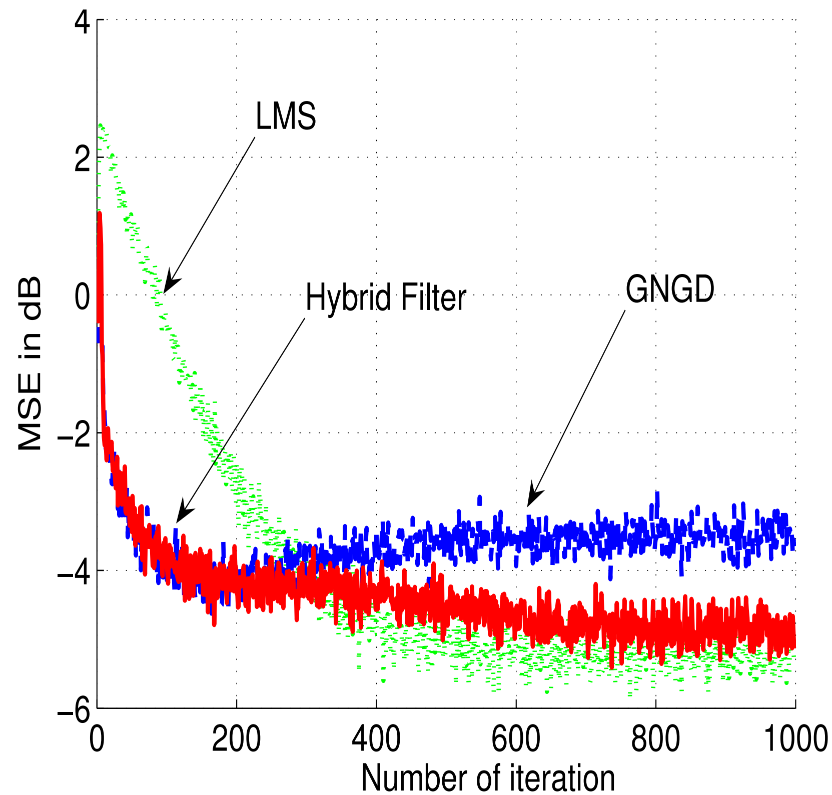
We must ensure that the value of $\lambda(k)$ belongs to $0 \leq \lambda(k) \leq 1$.

Performance of hybrid filters – prediction setting

consider an LMS/GNGD hybrid – GNGD is fast, LMS with small μ has good $\mathcal{M}$

Hybrid attempts to follow the subfilter with better performance.
If one of the subfilters diverges, hybrid filters still converges.

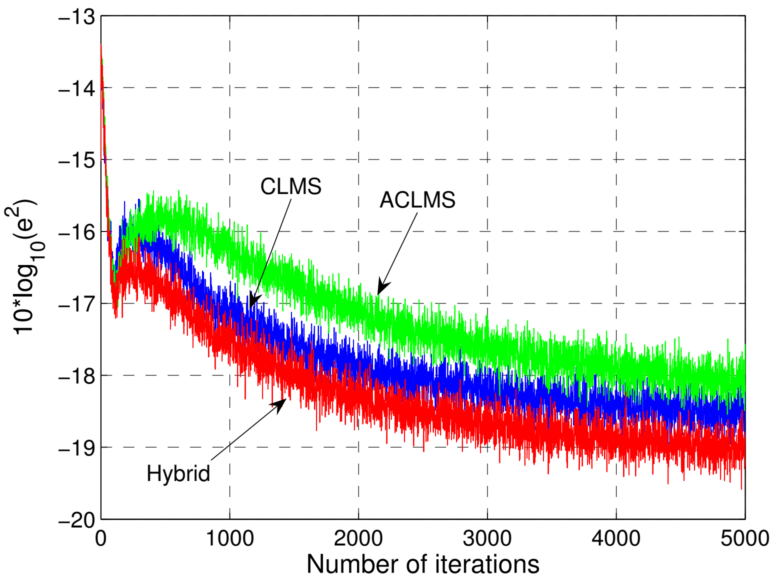




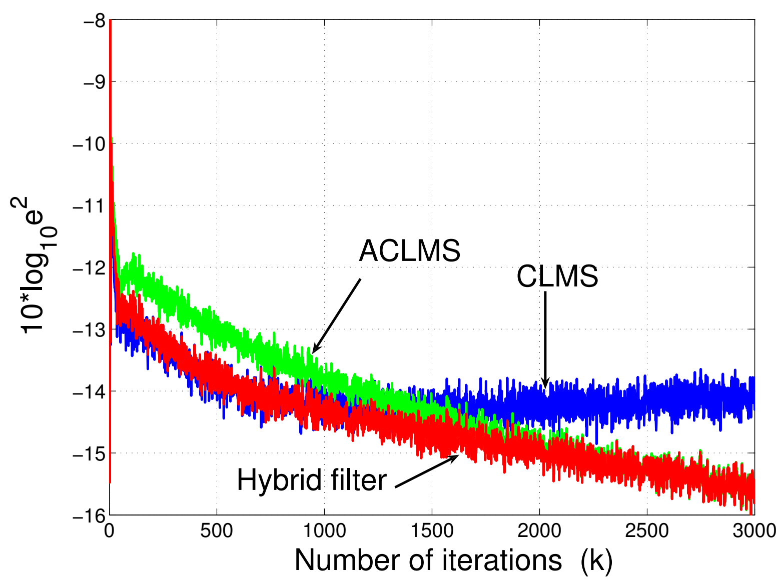
Learn. curves for pred.: Left $\looparrowright$ linear signal Right $\looparrowright$ nonlinear signal

The hybrid CLMS ↔ ACLMS filter (prediction setting)

Left: circular AR(4) process



Right: Noncircular Ikeda map



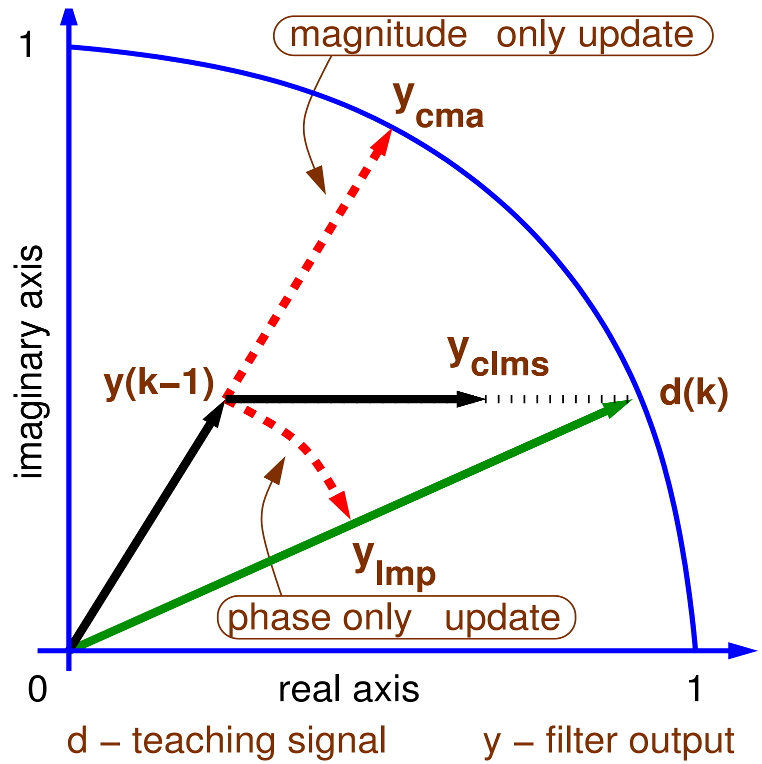
- The CLMS has half the number of parameters of ALMS
- Hence, it initially converges faster for all test signals

⊛ Both filters perform similarly for proper data in terms of the steady state

⊛ ACLMS has superior steady state properties for the improper Ikeda map

⊛ Hybrid filter: both fast covergence and excellent steady state properties!

A continuum between phase-only and magnitude-only cost functions (when does phase error matter?)



So, when does the (complex) phase error matter to the overall performance?

- **Answer #1:** Constant Modulus Channel Estimator (CMCE) [Rupp 1998] can estimate time-varying channels better *if phase error is ignored*.
- **Answer #2:** Least Mean Phase (LMP) [Tarighat/Sayed 2004] can estimate complex symbols better if *phase error term is added to update*.
- **Answer #3:** Least Mean Magnitude Phase (LMMP) [Douglas/Mandic ICASSP 2011] employs a combined magnitude- and phase-based criterion *best performance with stable behavior*.

On phase-only cost functions

The least mean phase (LMP) learning is based on the cost function

$$J_p(k) = \big(\angle d_k - \angle y_k\big)^2$$

where $e(k) = \angle d_k - \angle y(k)$, and $y(k) = \mathbf{w}^H(k)\mathbf{x}(k)$.

Then,

$$J_p(k) = \big(\angle d_k - \angle y_k\big)^2 = \left[\angle d_k - \arctan\frac{\Im(y_k)}{\Re(y_k)}\right]^2 = \left[\angle d_k - \arctan\frac{\Im(\mathbf{w}_k^H\mathbf{x}_k)}{\Re(\mathbf{w}_k^H\mathbf{x}_k)}\right]^2$$

and $J_{p,min}$ is achieved for $\mathbf{w}_k = \mathbf{w}_{opt}$, to give

$$J_{p,min} = J_p(\mathbf{w}_{opt}) = \left[\angle d_k - \arctan\frac{\Im(\mathbf{w}_{opt}^H\mathbf{x}_k)}{\Re(\mathbf{w}_{opt}^H\mathbf{x}_k)}\right]^2 = \left[\angle d_k - \arctan\frac{\Im(\alpha\mathbf{w}_{opt}^H\mathbf{x}_k)}{\Re(\alpha\mathbf{w}_{opt}^H\mathbf{x}_k)}\right]^2$$

Therefore $J_p(\mathbf{w}_{opt}) = J_p(\alpha\mathbf{w}_{opt})$, for any α and there is no unique minimum.

Decomposing the squared error cost function

Decompose the well-known squared error cost function:

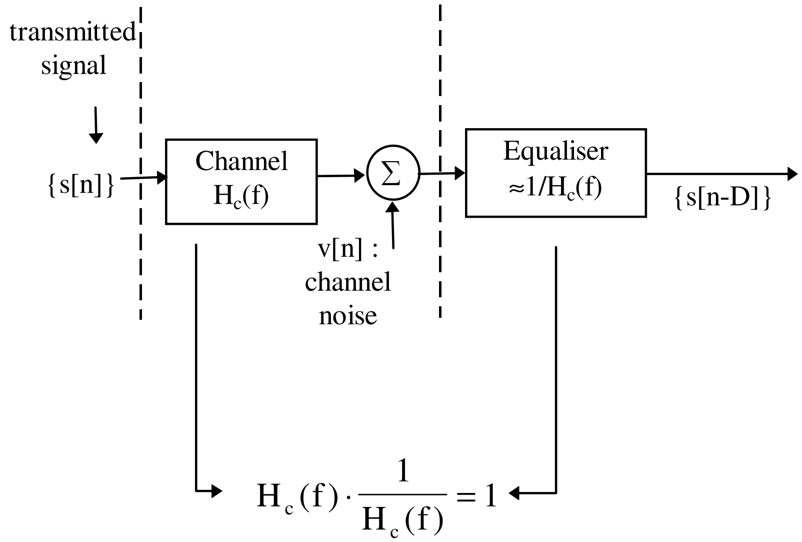
$$
\begin{aligned}
J_{\text{LMS}} &= |e(k)|^2 = |d(k) - y(k)|^2 \\
&= |d(k)|^2 + |y(k)|^2 - [d(k)y^*(k) + d(k)y^*(k)] \\
&= (|d(k)| - |y(k)|)^2 + 2|d(k)||y(k)| - [d(k)y^*(k) + d(k)y^*(k)] \\
&= \underbrace{(|d(k)| - |y(k)|)^2}_{\text{Magnitude Error}} + \underbrace{2|d(k)||y(k)|\,[1 - \cos(\theta_d - \theta_y)]}_{\text{Phase Error}}
\end{aligned}
$$

The Least Mean-Magnitude Phase (LMMP) algorithm can be derived as

$$
\begin{aligned}
\mathbf{w}(k+1) &= \mathbf{w}(k) - \mu_m \nabla_{\mathbf{w}^*} J_{\text{Mag}} - \mu_p \nabla_{\mathbf{w}^*} J_{\text{Phase}} \\
&= \mathbf{w}(k) + \mu_m \left(|d(k)| - |y(k)|\right) \frac{y_k}{|y_k|} \mathbf{x}_k^* + \mu_p \left(d(k) - |d(k)| \frac{y_k}{|y_k|} \right) \mathbf{x}_k^*
\end{aligned}
$$

If $\mu_p = \mu_m$, the LMMP simplifies into the CLMS!

Example: Channel equalisation in digital communications



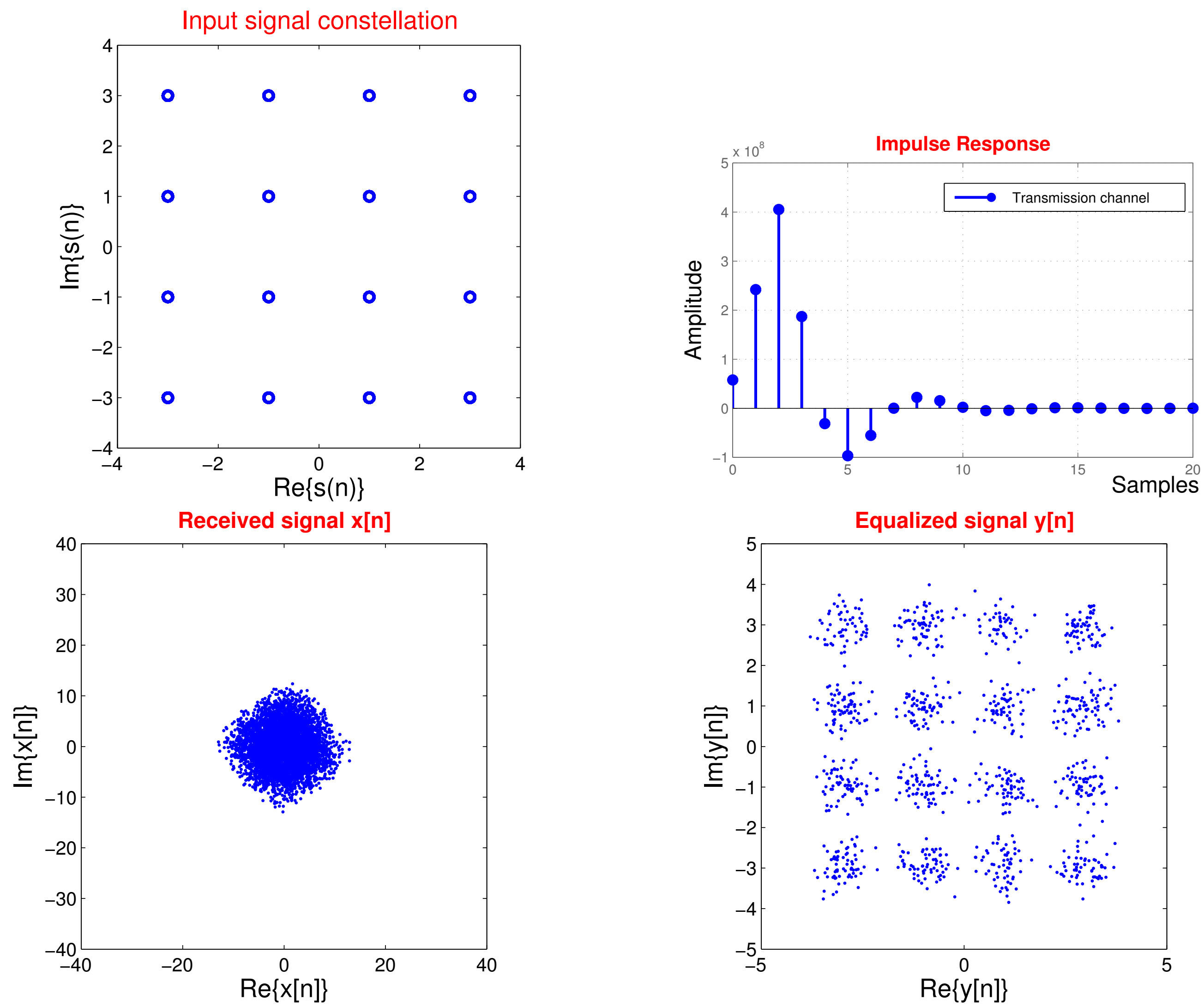
□ Channel equalization is a simple way of mitigating the detrimental effects caused by a frequency-selective and/or dispersive communication link between sender and receiver.

□ During the training phase of channel equalization, a digital signal $s[n]$ that is known to both the transmitter and receiver is sent by the transmitter to the receiver

□ The received signal $x[n]$ contains two signals: the signal $s[n]$ filtered by the channel impulse response, and an unknown broadband noise signal $v[n]$

□ The goal is to filter $x[n]$ to remove the inter-symbol interference (ISI) caused by the dispersive channel and to minimize the effect of the additive noise $v[n]$

Example: Digital communications $\hookrightarrow$ continued



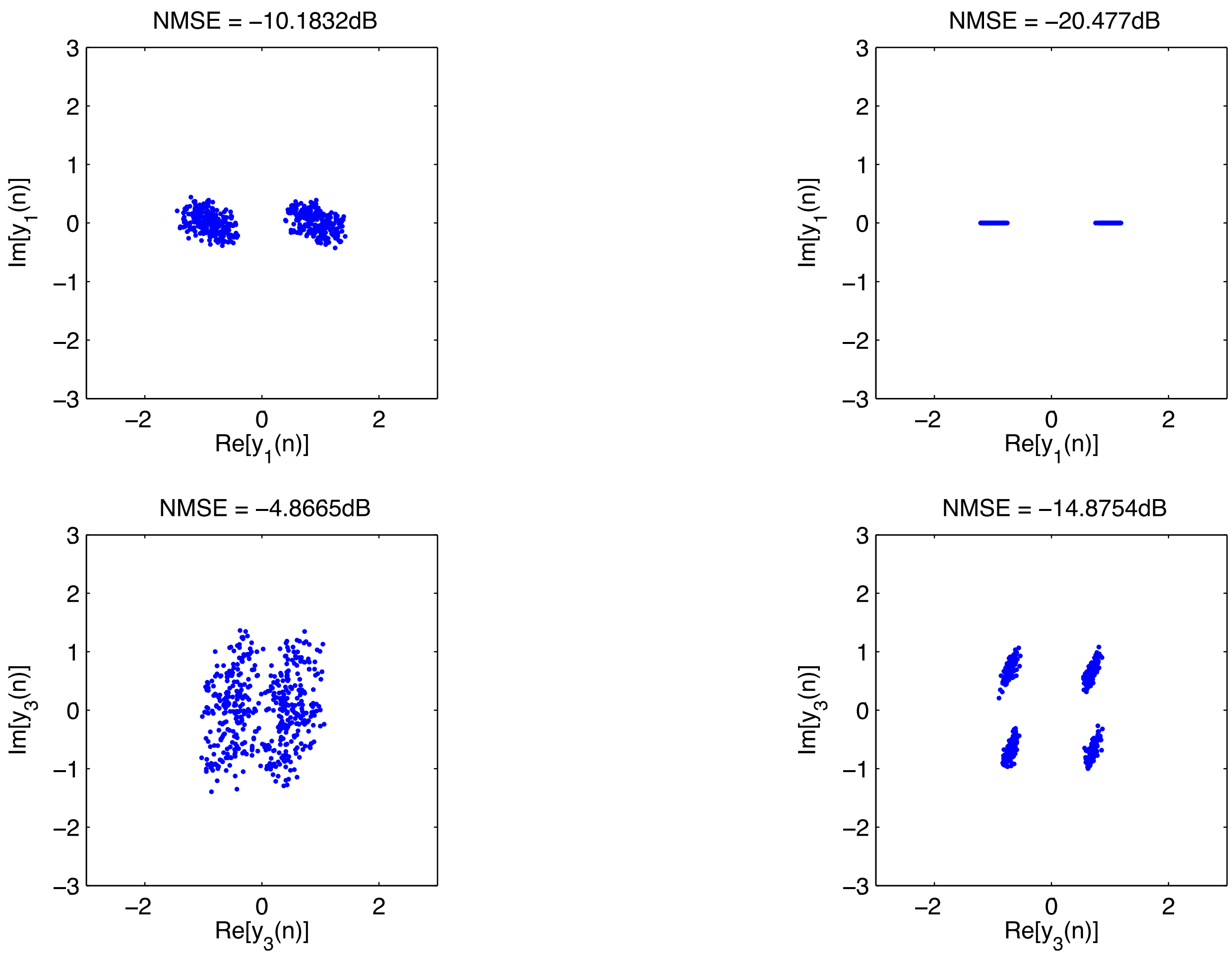
Beamforming example

- Beamforming Model: ULA, $\lambda/2$ spacing
- Sources: BPSK, BPSK, QPSK, QPSK
- Angle of Arrival: $-45°, 8°, -13°, 30°$
- Number of Antenna Elements: 3
- Algorthims: ACLMS and CCLMS
- Desired Signal: $d(k) = s_p^*(k)$, $1 \leq p \leq 4$
- Step Size: $\mu = 0.0001$.

Compare: Convergence Rates, Steady State MSE

Output signal constellations

CCLMS **ACLMS**

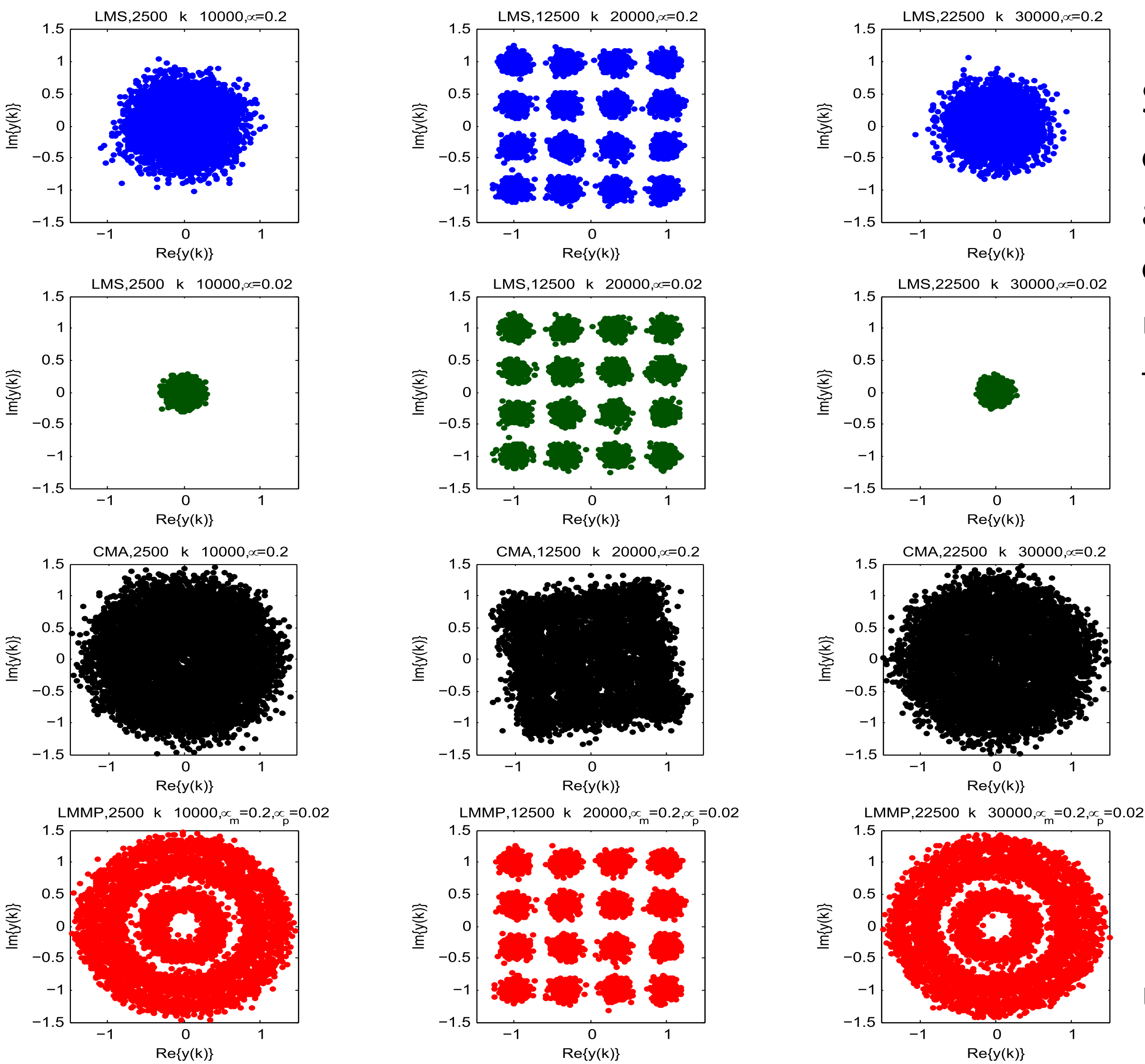


ACLMS has lower MSE because of non-circular binary sources

Signal constellations: channel equalization

Channel: 3-tap with frequency offset **Source phases: time-varying**

Performance meas.: average inter-symbol interfer., ISI (not dependent on phase)

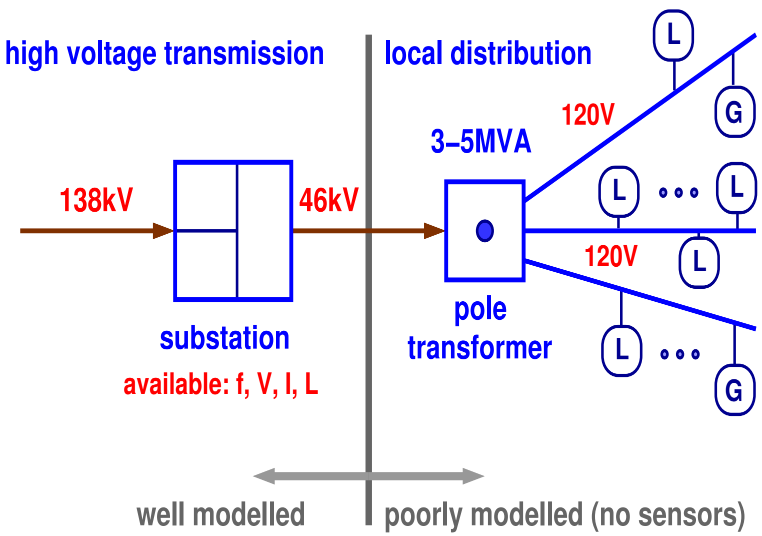


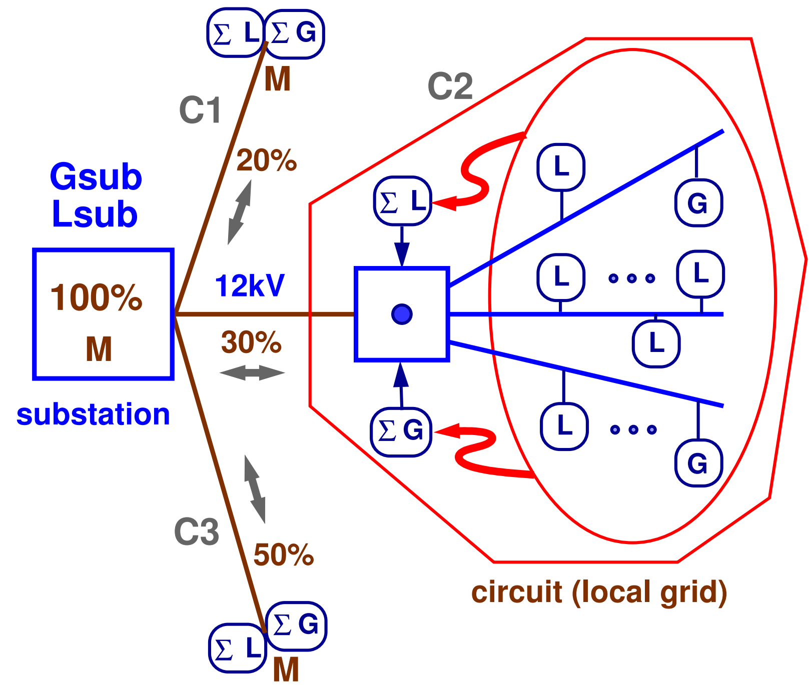
Moral:
Signal phase uncertainty can harm channel amplitude and phase estimation performance, unless it is mitigated within the algorithm itself.

LMMP in red addresses this uncertainty explicitly.

Case Study: Frequency estimation in smart grid
Sources of frequency deviation

Transmission: well-modelled distribution side is not





Block diagram of power grid

Nodal estimation

- Dual character of load/supply $\rightsquigarrow$ $f+$ for $G > L$ and $f-$ for $G < L$
- Harmonics and freq. drifts from loads with nonlinear $V - I$ properties
- Transient stability issues cause inaccurate frequency estimates, also switching on/off the shunt capacitors in reactive power compensation

The three-phase power system and $\alpha\beta$ transformation

Three-phase system where $V_a(k), V_b(k), V_c(k)$ are the peak values.

$$v_{\mathsf{a}}(k) = V_a(k)\cos(\omega k \triangle \mathsf{T} + \phi)$$

$$v_{\mathsf{b}}(k) = V_b(k)\cos(\omega k \triangle \mathsf{T} + \phi - \frac{2\pi}{3})$$

$$v_{\mathsf{c}}(k) = V_c(k)\cos(\omega k \triangle \mathsf{T} + \phi + \frac{2\pi}{3})$$

The $\alpha\beta$ transform - a complex signal which carries the same information

$$\begin{bmatrix} v_0 \\ v_\alpha(k) \\ v_\beta(k) \end{bmatrix} = \sqrt{\frac{2}{3}} \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ 1 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & \frac{\sqrt{3}}{2} & -\frac{\sqrt{3}}{2} \end{bmatrix} \begin{bmatrix} v_{\mathsf{a}}(k) \\ v_{\mathsf{b}}(k) \\ v_{\mathsf{c}}(k) \end{bmatrix}$$

For **balanced systems** $v_0 = 0$, and thus the complex Clarke's voltage

$$v(k) = v_\alpha(k) + \jmath v_\beta(k) = A(k)e^{\jmath(\omega k \Delta T + \phi)} + B(k)e^{-\jmath(\omega k \Delta T + \phi)}$$

$A(k) = \frac{\sqrt{6}(V_a(k)+V_b(k)+V_c(k))}{6}$ and $B(k) = \frac{\sqrt{6}(2V_a(k)-V_b(k)-V_c(k))}{12} - \frac{\sqrt{2}(V_b(k)-V_c(k))}{4}\jmath$

Clearly, unbalanced systems are noncircular!

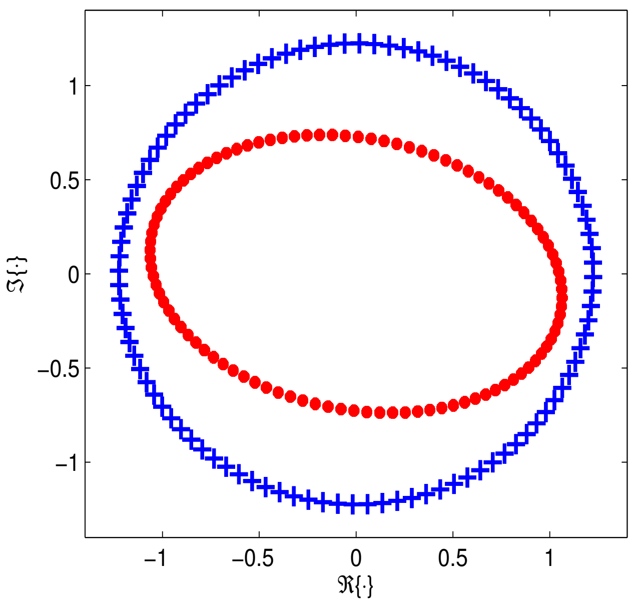
Noncircularity in unbalanced voltage conditions

◦ **Balanced system:** $V_a(k) = V_b(k) = V_c(k)$, $A(k) = \text{const}$, $B(k) = 0$, and $v(k)$ is on a circle

◦ **Unbalanced system:** $V_a(k)$, $V_b(k)$, $V_c(k)$ are not identical

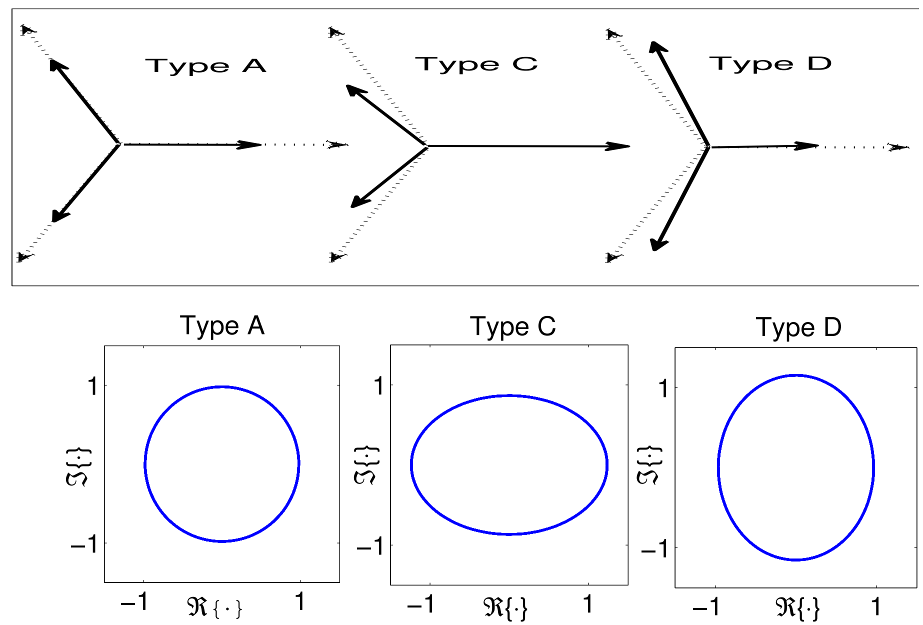
⊛ $A(k)$ is no longer constant, $B(k) \neq 0$

⊛ $v(k)$ is not on a circle $\rightarrow$ **a degree of noncircularity**



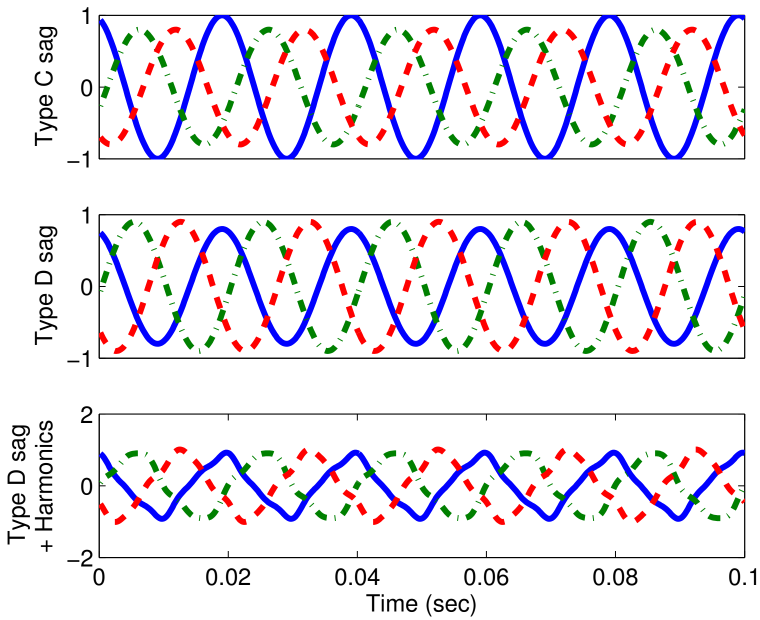
balanced and unbalanced system

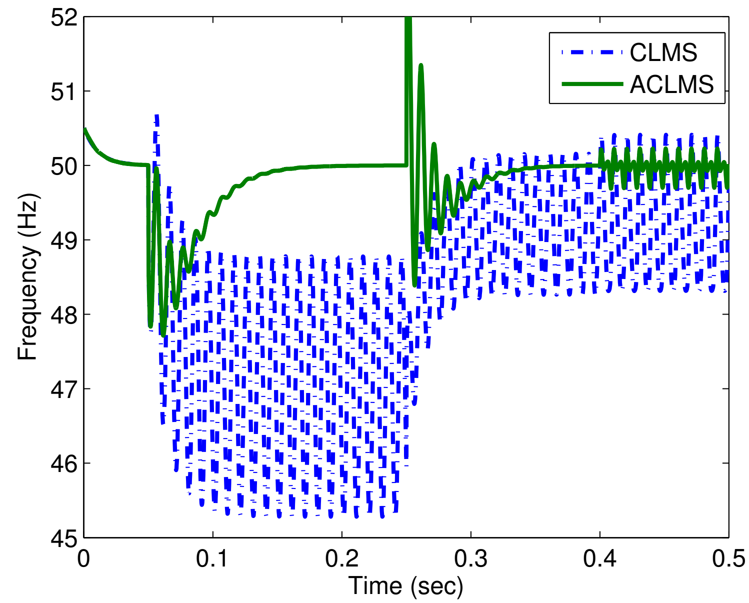
Some types of voltage sags



Their circularity diagrams

Simulations: Several successive sags





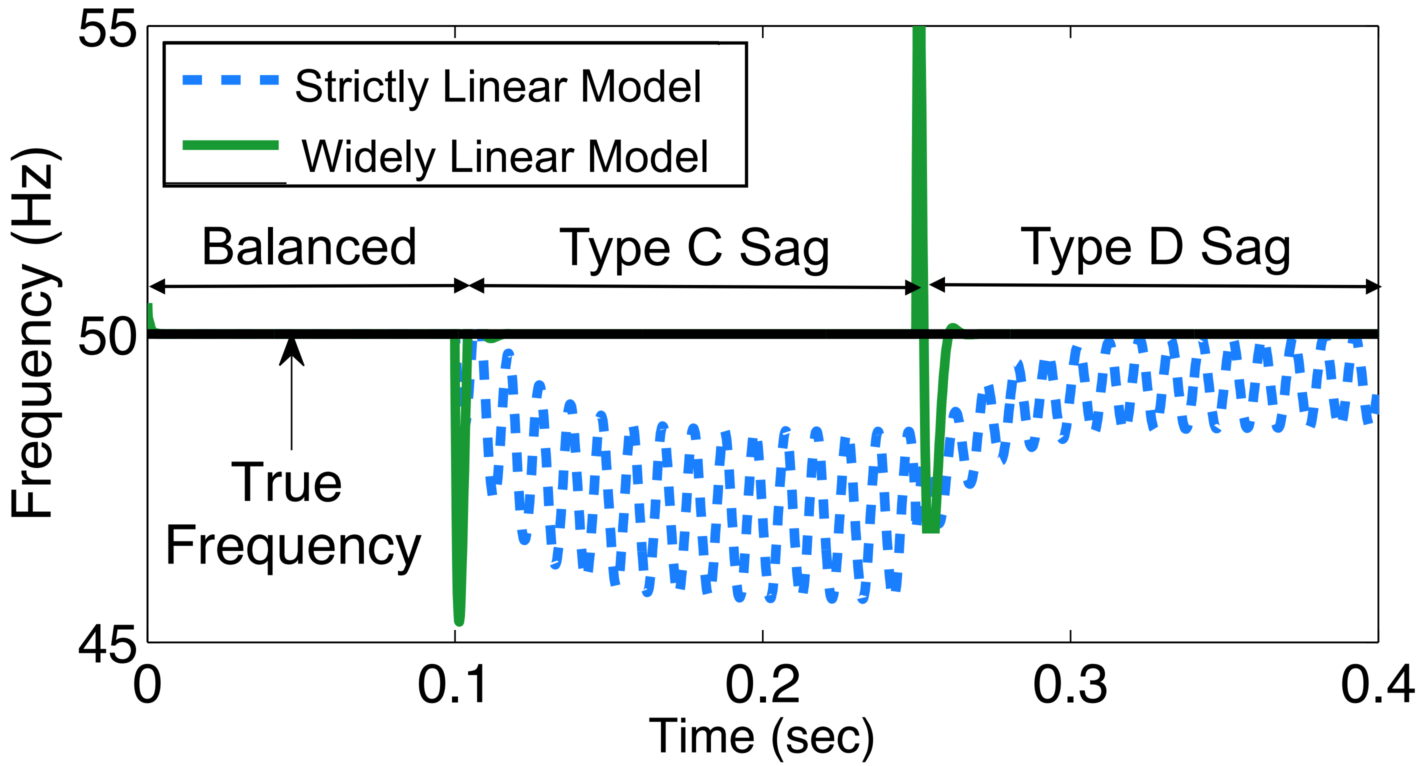
Phase voltages for different sags

Linear and widely linear frequency estimation

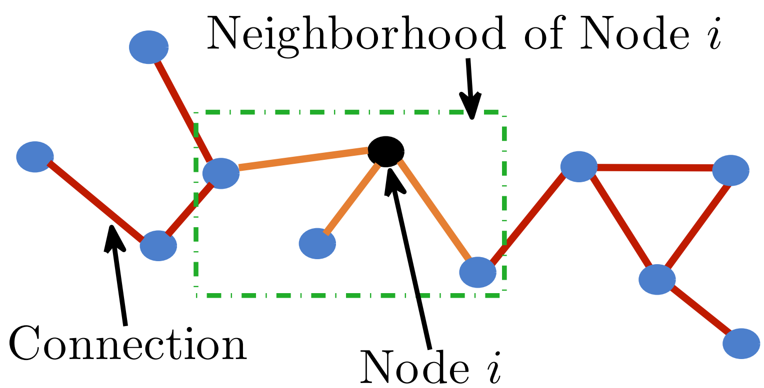
- Initially, the power system (50Hz) was operating normally and both CLMS and ACLMS converged to 50Hz
- The widely linear ACLMS had advantage in the subsequent type C and D sags and under harmonics

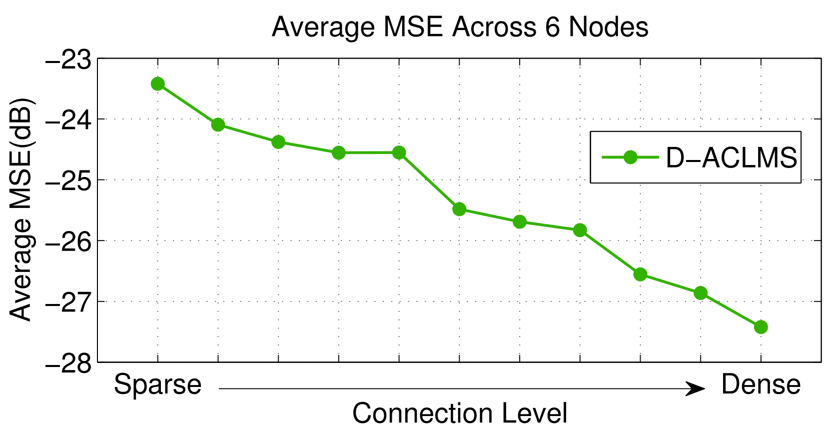
Using the widely linear model for frequency estimation

The widely linear model is able to estimate the frequency for both **circular** (balanced) and **noncircular** (unbalanced) voltages.



Multiple adaptive filters collaborating in a network





The *Diffusion* ACLMS at node i

Input: $\mathbf{z}_i(k) = [\mathbf{x}_i^T(k), \mathbf{x}_i^H(k)]^T$,

Weights: $\mathbf{w}_i(k) = [\mathbf{h}_i^T(k), \mathbf{g}_i^T(k)]^T$

$$y_i(k) = \mathbf{w}_i^H(k)\mathbf{z}_i(k)$$

$$e_i(k) = d_i(k) - y_i(k)$$

$$\boldsymbol{\psi}_i(k+1) = \mathbf{w}_i(k) + \mu_i e_i^*(k)\mathbf{z}_i(k)$$

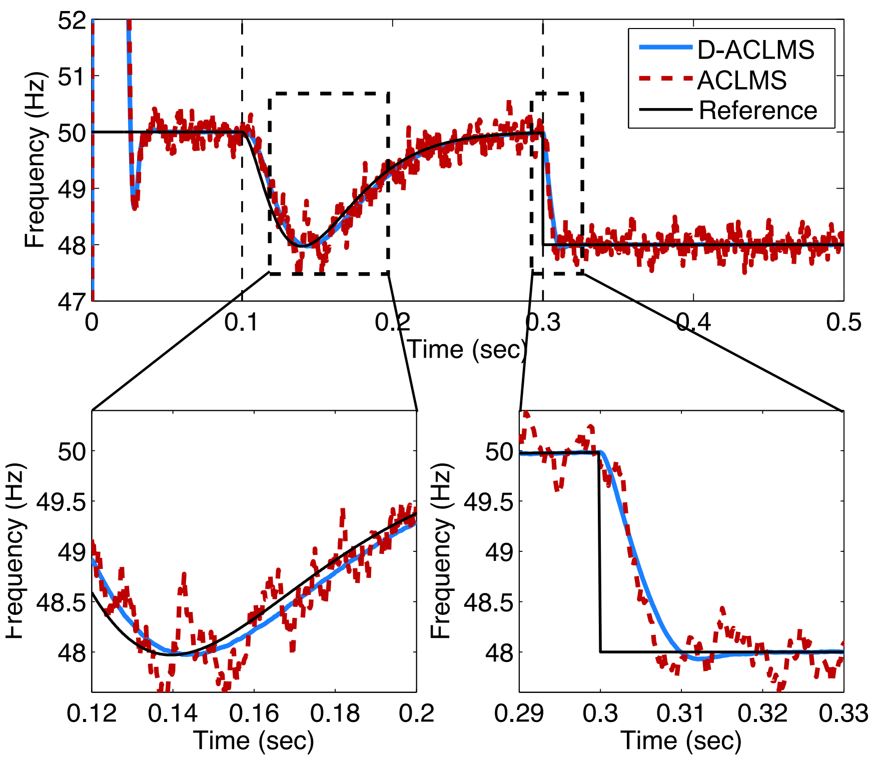
$$\mathbf{w}_i(k+1) = \sum_{\ell=1}^{N} a_{\ell i}\boldsymbol{\psi}_\ell(k+1)$$

The weighting coefficients satisfy

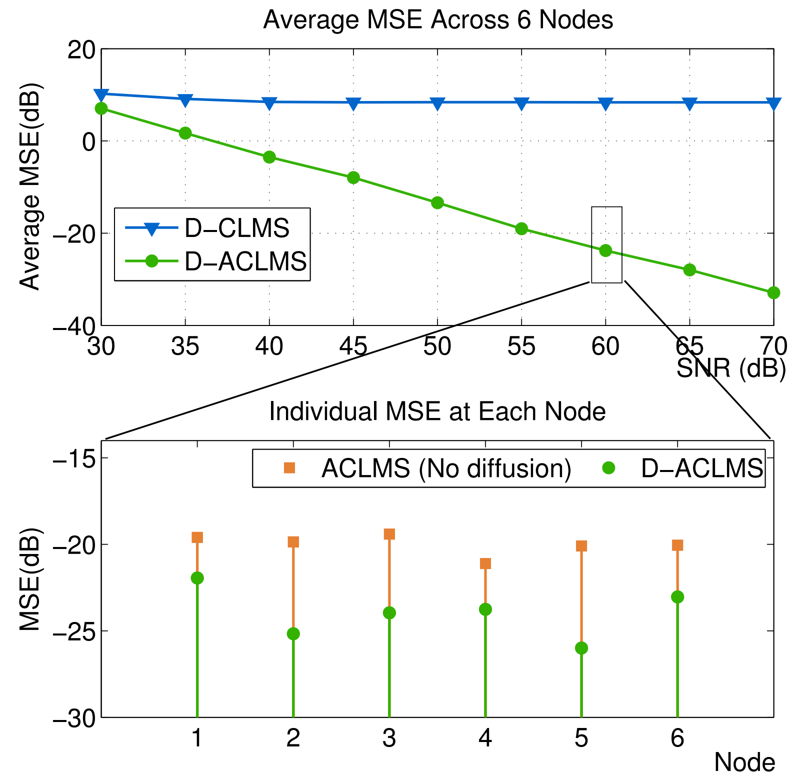
$$\sum_{\ell=1}^{N} a_{\ell i} = 1 \implies a_{ii} = 1 - \sum_{\ell\neq i} a_{\ell i}$$

Performance of the Diffusion-CLMS and Diffusion-ACLMS in the smart grid

Frequency estimate at a randomly selected node with multiple frequency events between 0.1 s and 0.5 s



The average MSE of the frequency estimate of D-ACLMS and D-CLMS under a Type D sag.

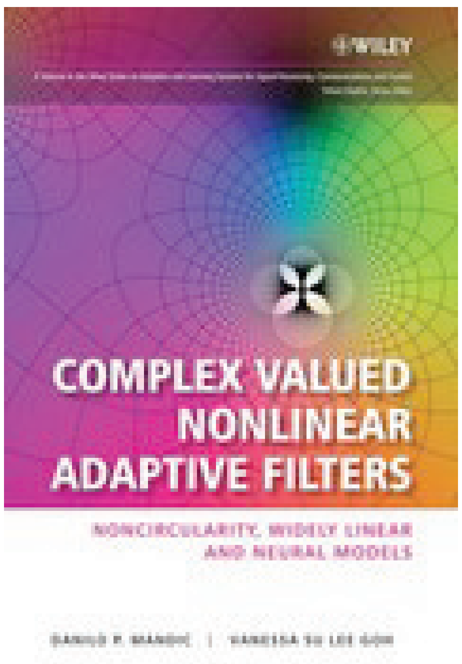


Conclusions

- Adaptive processing of noncircular complex signals
- These arise e.g. due to the nonlinearity of transievers (I/Q imbalance mitigation), multipath, in some modulation schemes (BPSK, GSMK, PAM, offset-QPSK) widely used in practical communications systems
- Standard solutions assume second order circularity of signal distributions and are inadequate when the signals are observed through nonlinear sensors, mixtures of sources, or noise model which is not doubly white
- This is achieved based on *augmented complex statistics* and *widely linear modelling*
- The complex LMS (CLMS) and augmented CLMS (ACLMS) introduced
- Convergence of CLMS and ACLMS – from the booklet provided (Chapter 6)
- This promises enhanced practical solutions in a variety of applications (interference supression, DoA estimation, blind estimation)

A comprehensive account of widely linear modelling

D.Mandic and V. Goh, "Complex Valued Nonlinear Adaptive FIlters: Noncircularity, Widely Linear and Neural Models", Wiley 2009.



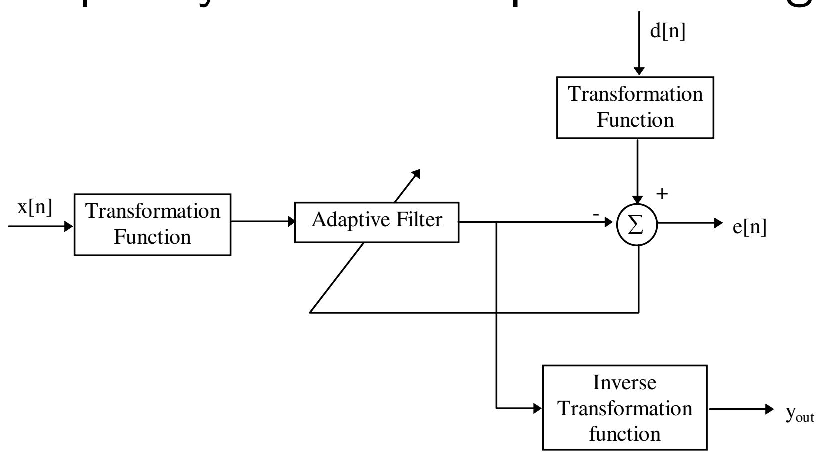
- Unified approach to the design of complex valued adaptive filters and neural networks
- Augmented learning algorithms based on widely linear models
- Suitable for processing both second order circular (proper) and noncircular (improper) complex signals
- ACLMS, augmented Kalman filters, augmented CRTRL, linear and nonlinar IIR filters
- Adaptive stepsizes, dynamical range reduction, collaborative adaptive filters, statitical tests for the validity of complex representations

Some back–up material

Appendix: Transform domain signal processing

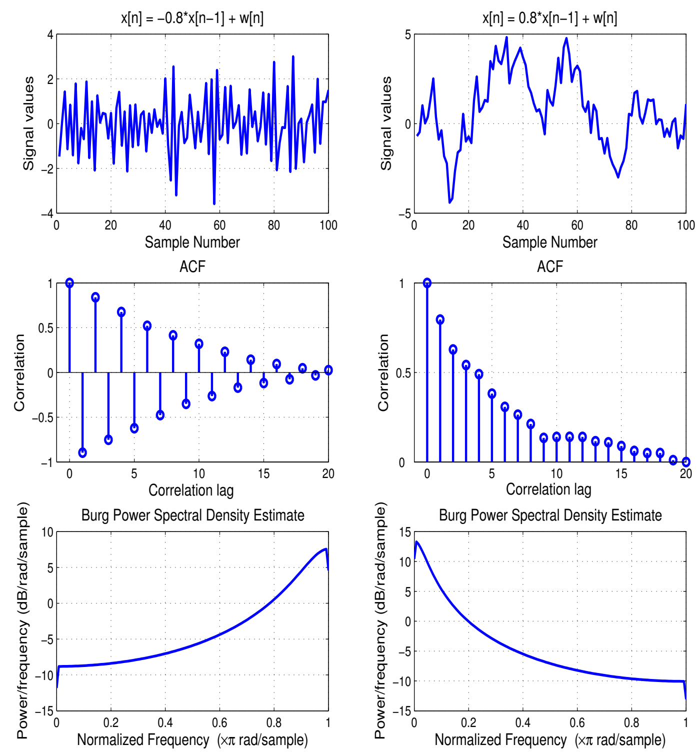
Usually complex valued (e.g. Fourier Domain)

Frequency domain adaptive filtering



A transform can be applied to the inputs of an adaptive filter in order to maximise the performance of the LMS algorithm, i.e. a white input with equal eigenvalues.

Various techniques can be used such as Lattice Filters, the FFT which requires the Complex LMS algorithm, the Discrete Cosine or Wavelet transforms, or sub-band filters.



$a < 0 \rightarrow$ highpass, $a > 0 \rightarrow$ lowpass

$$x(k) = a_1 x(k-1) + w(k)$$

Highpass signal: fast changing in time $\looparrowright$ however, smooth spectrum

Appendix: The CLMS algorithm

Step-by-step derivation

Consider a complex valued FIR filter.

The weight update equation for a real valued filter is

$$\mathbf{w}(k+1) = \mathbf{w}(k) + \eta e(k)\mathbf{x}(k)$$

Now, the weights and errors and teaching signal and input are complex valued.

Hence

$$\begin{aligned}
e(k) &= e_r(k) + \jmath e_i(k) \\
d(k) &= d_r(k) + \jmath d_i(k) \\
\mathbf{x}(k) &= \mathbf{x}_r(k) + \jmath \mathbf{x}_i(k) \\
\mathbf{w}(k) &= \mathbf{w}_r(k) + \jmath\, \mathbf{w}_i(k) \\
y(k) &= \mathbf{x}^T(k)\mathbf{w}(k)
\end{aligned}$$

Appendix: The CLMS cost function

The complex LMS should simultaneously adapt the real and imaginary part, minimising in some sense both $e_r(k)$ and $e_i(k)$, with respect to the average total error power, given by

$$E\left[e(k)e^*(k)\right] = \frac{1}{2}E\left[e_r^2(k) + e_i^2(k)\right] = \frac{1}{2}E\left[e_r^2(k)\right] + \frac{1}{2}E\left[e_i^2(k)\right]$$

Since the two components of the error are in quadrature relative to each other, they cannot be minimised independently.

The derivation of the complex LMS is similar to the derivation fo the original LMS, except that the rules of complex algebra must be observed.

Notice that $\left(\mathbf{x}^T(k)\mathbf{w}(k)\right)^* = \left(\mathbf{x}^*(k)\right)^T \mathbf{w}^*(k)$, e.g.

$x \times w = [(x_r + \jmath x_i)(w_r + \jmath w_i)] = [x_r w_r - x_i w_i + \jmath(x_r w_i + x_i w_r)]$ After conjugation we have $[x_r w_r - x_i w_i - \jmath(x_r w_i + x_i w_r)]$. On the other hand $x^* w^* = [(x_r - \jmath x_i)(w_r - \jmath w_i] = [x_r w_r - x_i w_i - \jmath(x_r w_i + x_i w_r)]$,

which is the same as when we conjugate the whole expression.

Appendix: The CLMS Derivation

$$e^*(k) = d^*(k) - \left(\mathbf{x}^T(k)\right)^* \mathbf{w}^*(k)$$

Therefore, for a GD adaptation we have

$$\mathbf{w}(k+1) = \mathbf{w}(k) - \frac{1}{2}\eta \nabla_{\mathbf{w}} \left(e(k)e^*(k)\right)$$

where $$\nabla(e(k)e^*(k)) = \nabla_r(e(k)e^*(k)) + \jmath \nabla_i(e(k)e^*(k)).$$

Now,

$$\nabla_r(e(k)e^*(k)) = \begin{bmatrix} \frac{\partial(e(k)e^*(k))}{\partial w_{1r}} \\ \frac{\partial(e(k)e^*(k))}{\partial w_{2r}} \\ \vdots \\ \frac{\partial(e(k)e^*(k))}{\partial w_{Nr}} \end{bmatrix} = e(k)\nabla_r(e^*(k)) + e^*(k)\nabla_r(e(k))$$

Appendix: The CLMS derivation – contd.

Notice that (in a simplified way)

$$\begin{aligned} e(k) &= d(k) - x(k)w(k) = d(k) - \\ &- \left[x_r(k)w_r(k) - x_i(k)w_i(k) + \jmath(x_r(k)w_i(k) + x_i(k)w_r(k))\right] \\ e^*(k) &= d(k) - \left[x_r(k)w_r(k) + x_i(k)w_i(k) - \jmath(x_r(k)w_i(k) + x_i(k)w_r(k))\right] \end{aligned}$$

The partial derivatives wrt to w_r and w_i are

$$\begin{aligned} \frac{\partial e(k)}{\partial w_r(k)} &= -[x_r(k) + \jmath x_i(k)] = -\mathbf{x}(k) \\ \frac{\partial e(k)}{\partial w_i(k)} &= -[-x_i(k) + \jmath x_r(k)] = -\jmath \mathbf{x}(k) \\ \frac{\partial e^*(k)}{\partial w_r(k)} &= -[x_r(k) - \jmath x_i(k)] = -\mathbf{x}^*(k) \\ \frac{\partial e^*(k)}{\partial w_i(k)} &= -[x_i(k) - \jmath x_r(k)] = -[-\jmath \mathbf{x}^*(k)] \end{aligned}$$

Appendix: The CLMS Derivation – Complex Gradients

The instantaneous gradient with respect to its real and imaginary component becomes

$$
\begin{aligned}
\nabla_r(e(k)e^*(k)) &= e(k)\nabla_r(e^*(k)) + e^*(k)\nabla_r(e(k)) \\
&= e(k)(-\mathbf{x}^*(k)) + e^*(k)(-\mathbf{x}(k)) \\
\nabla_i(e(k)e^*(k)) &= e(k)\nabla_i(e^*(k)) + e^*(k)\nabla_i(e(k)) \\
&= e(k)\left(\jmath\mathbf{x}^*(k)\right) + e^*(k)\left(-\jmath\mathbf{x}(k)\right)
\end{aligned}
$$

Now, applying the method of steepest descent, to the real and imaginary part of the weights we have

$$
\begin{aligned}
\mathbf{w}_r(k+1) &= \mathbf{w}_r(k) - \frac{1}{2}\eta\nabla_r\left(e(k)e^*(k)\right) \\
\mathbf{w}_i(k+1) &= \mathbf{w}_i(k) - \frac{1}{2}\eta\nabla_i\left(e(k)e^*(k)\right)
\end{aligned}
$$

Appendix: The CLMS Derivation – Filter Update

Having in mind that $\mathbf{w}(k+1) = \mathbf{w}_r(k+1) + \jmath\mathbf{w}_i(k+1)$, we have

$$\mathbf{w}(k+1) = \mathbf{w}(k) - \frac{1}{2}\eta\left[\nabla_r\left(e(k)e^*(k)\right) + \jmath\nabla_i\left(e(k)e^*(k)\right)\right]$$

If the gradients are now substituted in the above equation, we have

$$\begin{aligned}
\nabla_r\left(e(k)e^*(k)\right) + \jmath\nabla_i\left(e(k)e^*(k)\right) &= \\
e(k)(-\mathbf{x}^*(k)) + e^*(k)(-\mathbf{x}(k)) + \jmath\left[e(k)\left(\jmath\mathbf{x}^*(k)\right) + e^*(k)\left(-\jmath\mathbf{x}(k)\right)\right] &= \\
-e_r\mathbf{x}^* - \jmath e_i\mathbf{x}^* - e_r\mathbf{x} + \jmath e_i\mathbf{x} - e_r\mathbf{x}^* - \jmath e_i\mathbf{x}^* + e_r\mathbf{x} - \jmath e_i\mathbf{x} &= \\
-2e_r\mathbf{x}^* - 2\jmath e_i\mathbf{x}^* = -2e(k)\mathbf{x}^*(k)
\end{aligned}$$

Therefore, the complex form of the LMS algorithm is given by

$$\mathbf{w}(k+1) = \mathbf{w}(k) + \eta e(k)\mathbf{x}^*(k)$$

Notes

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