Multi-Rate DSP

Applications:

Oversampling, Undersampling,

Quadrature Mirror Filters

Multi-Rate DSP

Oversampling

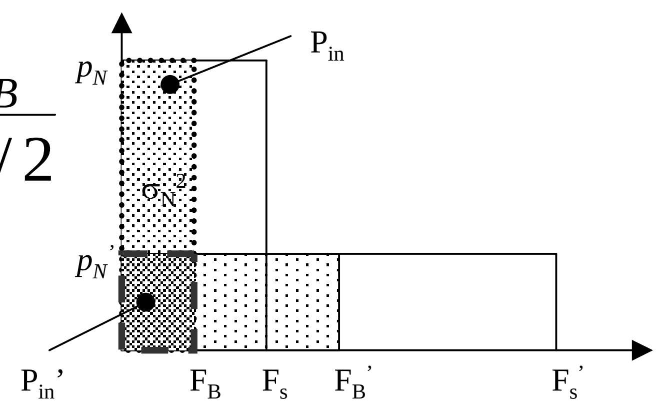
Optimal Sampling vs. Oversampling

- Sampling at Nyquist rate $F_s = 2F_B$
  - Allows perfect reconstruction in principle, but...
  - Pre-sampling anti-aliasing filter must have very steep roll-off:
    - High-order analogue filter: expensive, difficult, imprecise, large phase distortion, ...
- Sampling at $F_s >> 2F_B$ & decimation to $2F_B$
  - Larger separation between images $\Rightarrow$ easier filtering of aliases (lower-order filter)
    - Cheaper analogue component; easier digital than analogue VLSI filters; greater digital complexity

Oversampling Noise Reduction

- Quantisation step (*b*-bit ADC, range *R*): $Q = \frac{R}{2^b}$
- Noise power density (per unit sampling bandwidth): $$p_N = \frac{\sigma_N^2}{F_s/2} = \frac{Q^2/12}{F_s/2} = \frac{Q^2}{6F_s} \quad \text{[W/Hz]}$$
- Total in-band noise power:

$$P_{in} = \int_0^{F_B} p_N(f)\,\mathrm{d}f = \frac{Q^2 F_B}{6F_s} = \frac{(R/2^b)^2}{12} \times \frac{F_B}{F_s/2}$$



- Low P_{in} for high F_s:

$$F_s' \gg 2F_B \Rightarrow P_{in}' \ll \sigma_N^2$$

Oversampling: Effective Resolution

- Equivalent β-bit ADC operating at F_s=2F_B giving same noise power as b-bit ADC operating at F_s>>2F_B over same range R:

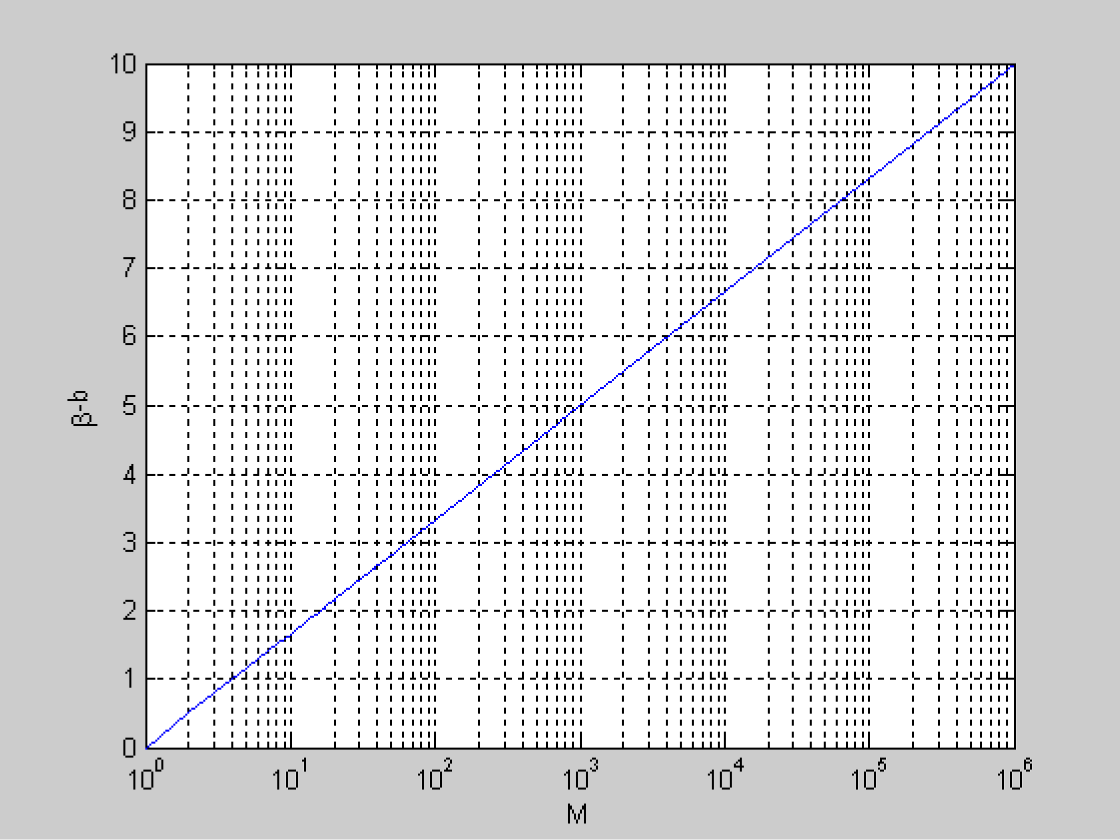
$$\frac{(R/2^b)^2}{12} \times \frac{F_B}{F_s/2} = \frac{(R/2^\beta)^2}{12} \times \frac{F_B}{2F_B/2}$$

i.e.,

$$\boxed{\beta = b + \frac{\log_2\left(F_s/(2F_B)\right)}{2} = b + \frac{\log_2(M)}{2}}$$

  - M = oversampling ratio, β-b = resolution increase

Oversampling Ratio

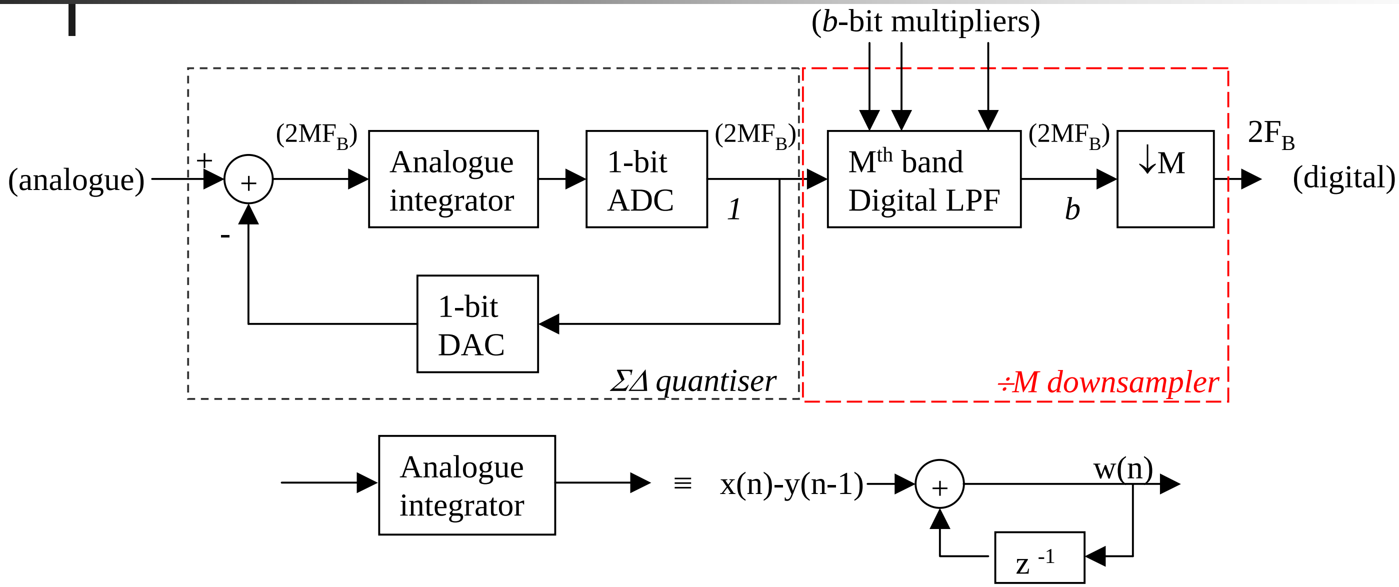


Example:

M=10^6: β=b+10, i.e., standard 16-bit ADC @ F_s=2F_B has equivalent resolution w.r.t. noise power as an oversampling 6-bit ADC @ F_s=2×$10^6$$F_B$ or as oversampling 11-bit ADC @ F_s=2000F_B

Thus, 0.5 bit length reduction of ADC per doubling of M

ΣΔ (Sigma-Delta) Converters



-High-rate oversampling allows for differential encoding: only 1 bit needed to quantify change Δ between input and delayed output of 1-bit ADC, for closely spaced consecutive samples in output of ΣΔ quantiser (bitmap of sample increments)

-*M*th order alias digital LPF eliminates out-of-band quantisation noise (sharp cut-off); heavy comp.; remedy: perform multiplications at later stage (down-rate $2F_B$) instead

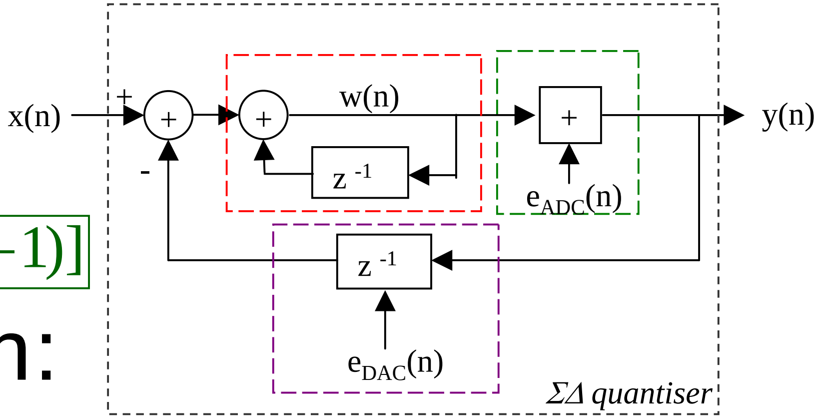
-Long word length of digital o/p determines overall oversampling rate

ΣΔ Converters

- Accumulator (integrator) & quantiser outputs:

$$w(n) = x(n) - y(n-1) + w(n-1)$$

$$y(n-1) = w(n-1) + e(n-1)$$



$$\Rightarrow \boxed{y(n) = w(n) + e(n) = x(n) + [e(n) - e(n-1)]}$$

$\Rightarrow$ noise transfer function:

$$H(z) = 1 - z^{-1}$$

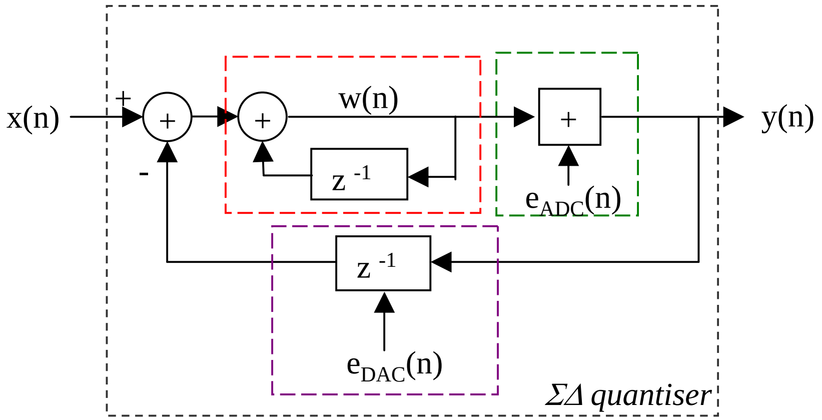
- Output psdf (one-sided spectrum):

$$p_y(f) = \left|H\left(e^{j\omega T_s}\right)\right|^2 p_N(f) = 4\sin^2\left(\frac{\omega T_s}{2}\right) p_N(f)$$

ΣΔ Oversampling

- Precise psdf of output noise depends on pdf & spectral characteristics of {x(n)}
  - Assume: {e(n)} is random, uncorrelated, white:

$$p_N = \frac{\sigma_N^2}{F_s/2} = \frac{Q^2/12}{F_s/2}$$



$\Rightarrow$ psdf of output noise:

$$p_y(f) = \frac{2\sin^2(\pi f T_s)}{3} \frac{Q^2}{F_s}$$

Oversampling: Performance Comparison

- For efficient oversampling $(M >> 1,\ f << F_s)$:

$$p_y(f) \approx \frac{2(\pi f / F_s)^2 Q^2}{3F_s} = \frac{2\pi^2 Q^2 f^2}{3F_s^3}$$

- Output noise power for ΣΔ modulator:

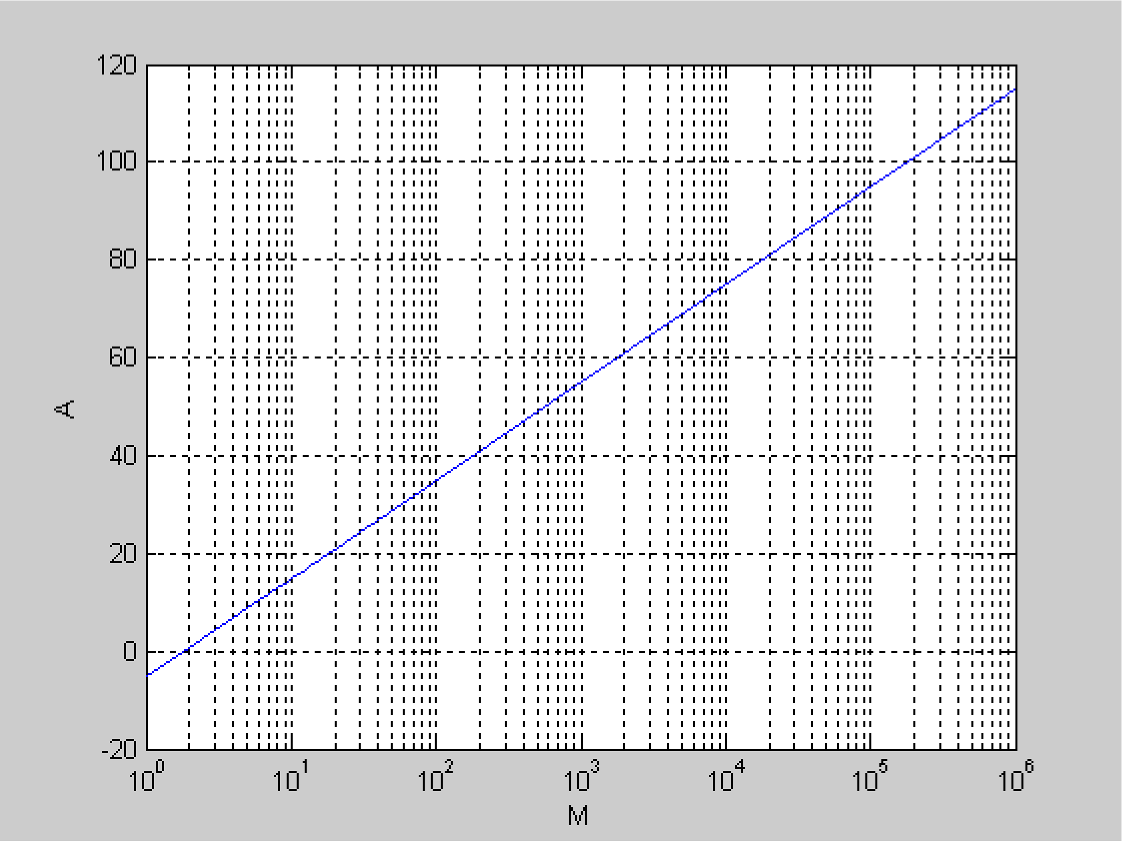
$$P_y = \int_0^{F_B} p_y(f)\,\mathrm{d}f = \frac{2\pi^2 Q^2 F_B^3}{9F_s^3}$$

⇒ Improvement over standard oversampling:

$$10\log_{10}\left(\frac{P_{in}}{P_y}\right) = 10\log_{10}\left(\frac{3M^2}{\pi^2}\right) = [-5.17 + 20\log_{10}(M)]\ \ \mathrm{dB}$$

Oversampling: Performance Comparison

$$A = 10\log_{10}\left(\frac{P_{in}}{P_y}\right) = [-5.17 + 20\log_{10}(M)] \text{ dB}$$



M=10: A=14.8 dB

M=1000: A=54.8 dB

M=10^6: A=114.8 dB

1.5 bit length reduction of ADC per doubling of *M* (relative to standard oversampling):

is due to noise shaping by $|H(f)|$: noise reduction if $f < F_B$

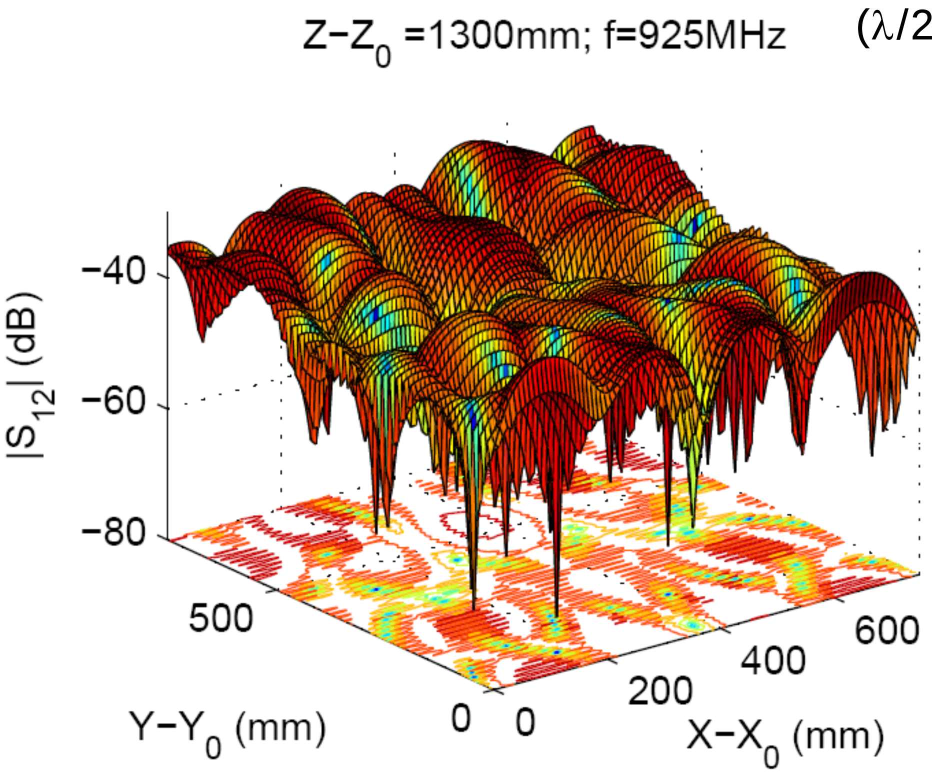
Oversampling: Application

Example: measurement of indoor EM wave propagation: GSM

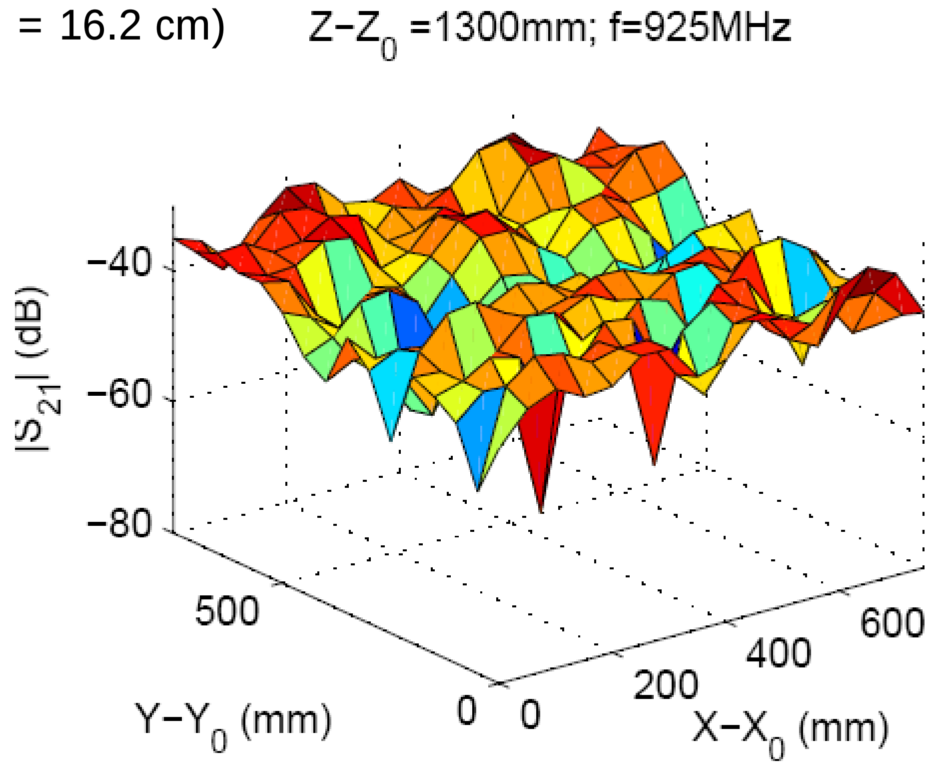
Oversampling: Application

GSM band (925 MHz)

($\lambda/2$ = 16.2 cm)



1 cm × 1 cm cell size

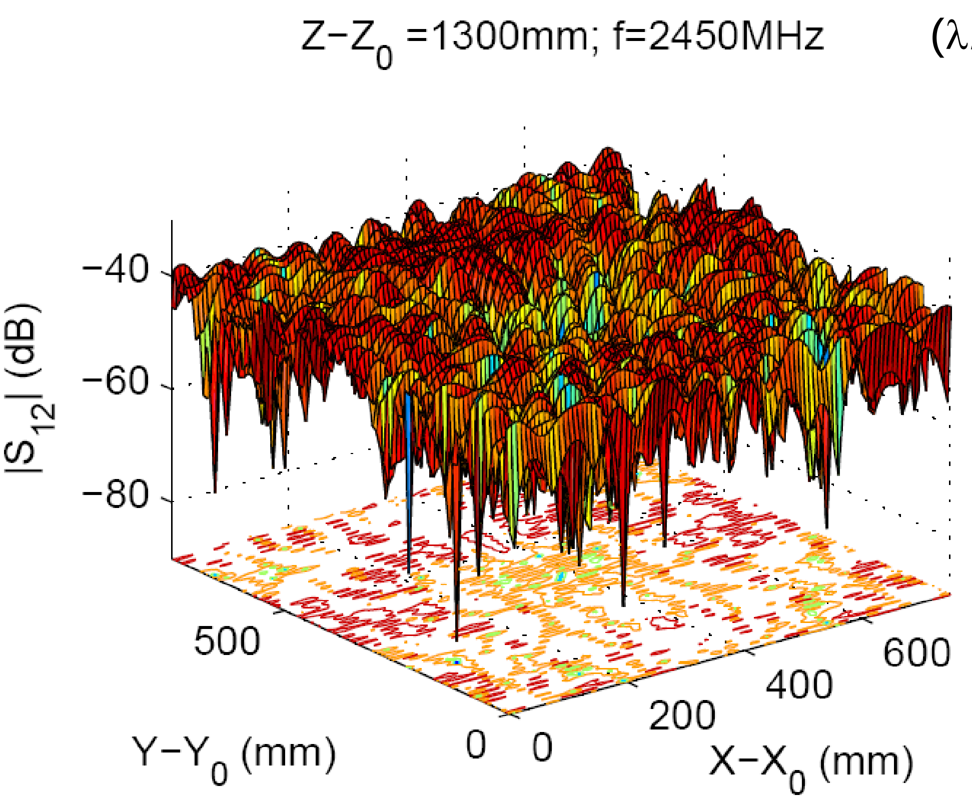


5 cm × 5 cm cell size

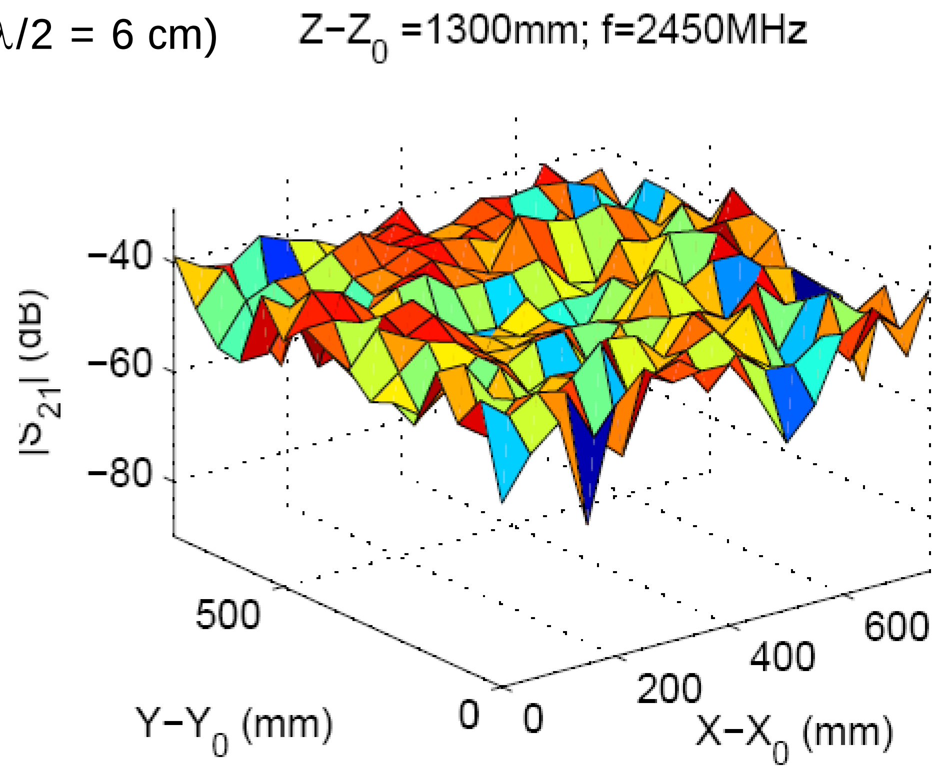
Oversampling: Application

ISM band (2450 MHz)

($\lambda/2$ = 6 cm)



1 cm × 1 cm cell size



5 cm × 5 cm cell size

Multi-Rate DSP

Undersampling

Nyquist Condition

- Alias-free (sub)sampling of (discrete) function

$$x_n(t) = x(t)\,\delta_{T_s}(t) = x(t)\sum_{k=-\infty}^{+\infty}\delta(t-kT_s) = \sum_{k=-\infty}^{+\infty} x(kT_s)\,\delta(t-kT_s)$$

  - Spectrum (Fourier transformation): convolution

$$X_n(f) = \int_{-\infty}^{+\infty} X(f-\varphi)\left[\frac{1}{T_s}\sum_{k=-\infty}^{+\infty}\delta(\varphi-kF_s)\right]d\varphi$$

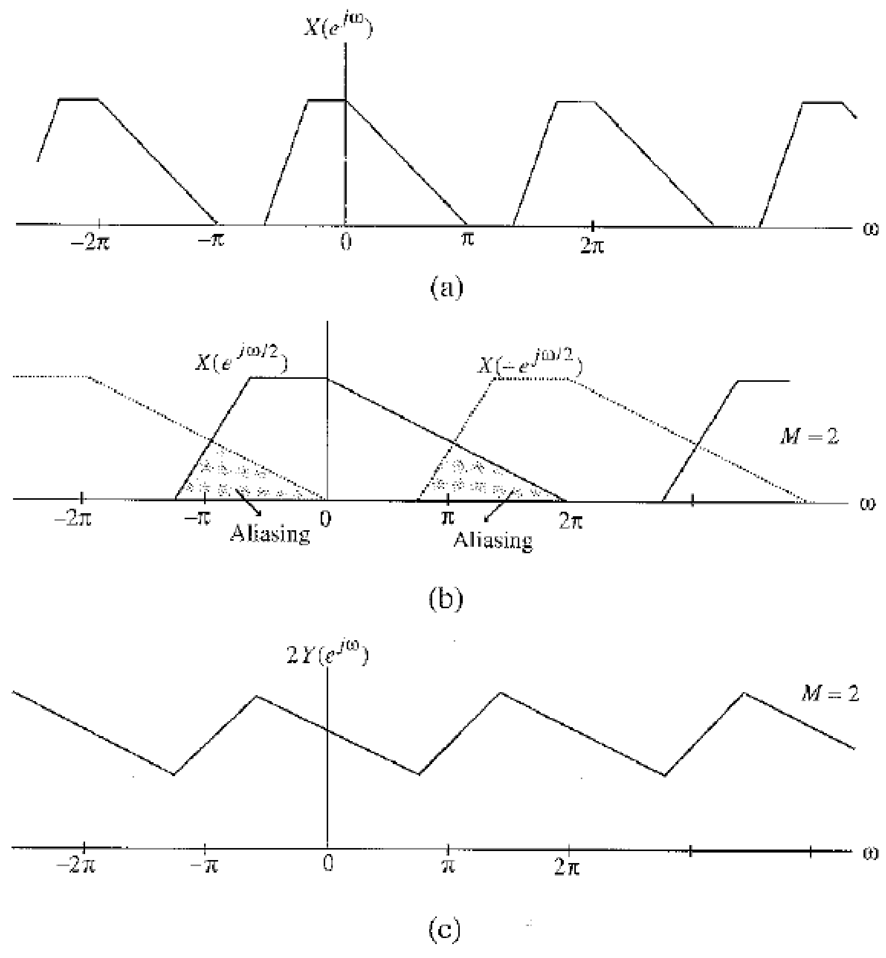
$$= \frac{1}{T_s}\sum_{k=-\infty}^{+\infty}\int_{-\infty}^{+\infty} X(f-\varphi)\,\delta(\phi-kF_s)\,d\varphi$$

Thus,

$$\boxed{X_n(f) = \frac{1}{T_s}\sum_{k=-\infty}^{+\infty} X(f-kF_s)}$$

  - No spectral overlap (aliasing, stroboscopy) iff $\boxed{F_s \geq 2F_B}$

Undersampling: Aliasing



Undersampling: Baseband

- Shannon sampling theorem: $F_s \geq 2F_B$
  - Applies to baseband signals (DC-coupled): F_B is largest frequency component in signal
    - Motivation: avoid spectral overlap of baseband frequency responses that are periodically continued due to sampling operation
  - For bandpass (narrowband modulated) signals (e.g., radio- and optical communications, IF filters, etc.): condition is too conservative: large spectral gaps occur because $F_c >> |F_B - F_c|$

Undersampling: Bandpass

- Aliasing of bandpass signal is avoided if baseband can be folded periodically around carrier frequency without causing overlap
  - Range of permissible sample frequencies:

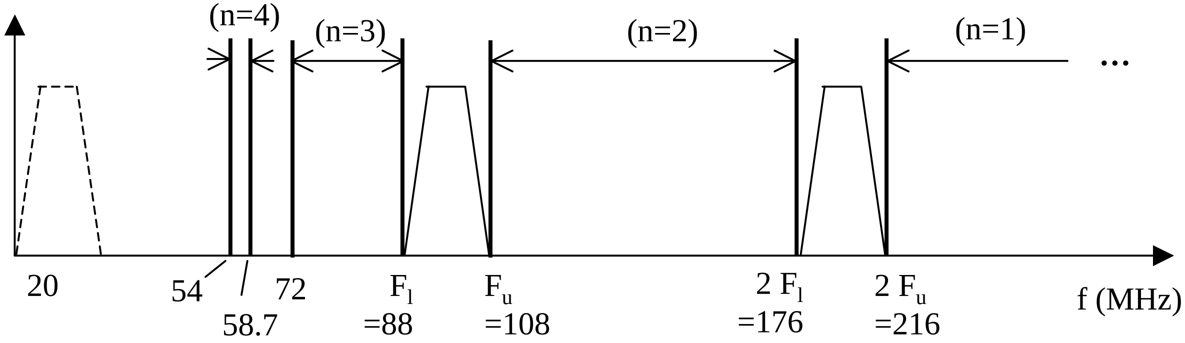
$$\frac{2F_u}{n} \leq F_s \leq \frac{2F_l}{n-1} \quad \text{i.e.,} \quad \boxed{1 \leq n \leq \left\lfloor \frac{F_u}{F_u - F_l} \right\rfloor}$$

- Yields additional permissible lower sampling rates for narrowband signals without aliasing
  - Practically useful (slower computations)

Undersampling: Applications

Example 1: digitization of analogue FM audio

F_l = 88 MHz, F_u = 108 MHz $\Rightarrow$ $1 \le n \le 5$ (nonzero gap)



- n=1: classical (≥ Nyquist rate)
- n=2: between modulated (l_u=1) signal and doubled (l_l=2) signal
- n=3: D_u=2/3, l_l=1
- n=5: $43.2\,\text{MHz} \le F_s \le 44\,\text{MHz} \quad \Rightarrow 86.4\,\text{MHz} \le 2 \ldots 2.5 F_s \le 110\,\text{MHz}$

LRA DSP

Multi-Rate DSP

Quadrature Mirror Filters for Subband Coding

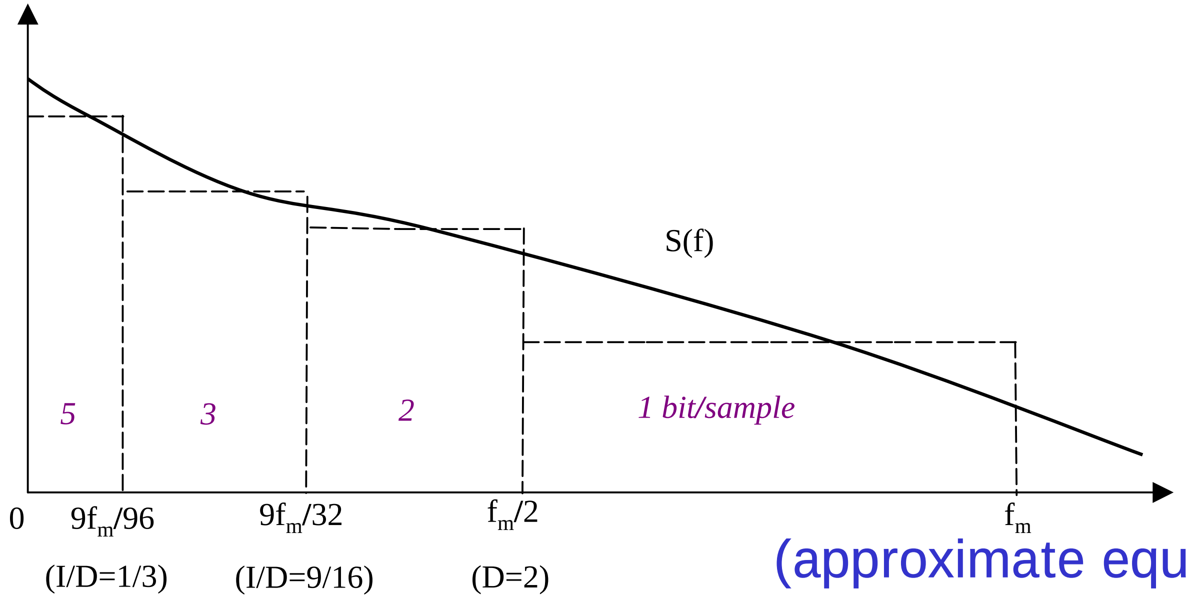
Subband Coding

- Problem statement:
  - Efficient transmission of realistic speech or video signals
    - Contain most energy at relative low frequencies (time/space)
    - Coding scheme to be tailored to assign more bits to LF band

- Solution:
  - *Subband coding:*
    - divide total frequency band in *unequal* subbands;
      - narrowest subband for interval with highest energy (equalization of *power* across band)
    - each subband is encoded separately
    - is alternative to companding (pre-distortion)

Subband Coding

- Example:



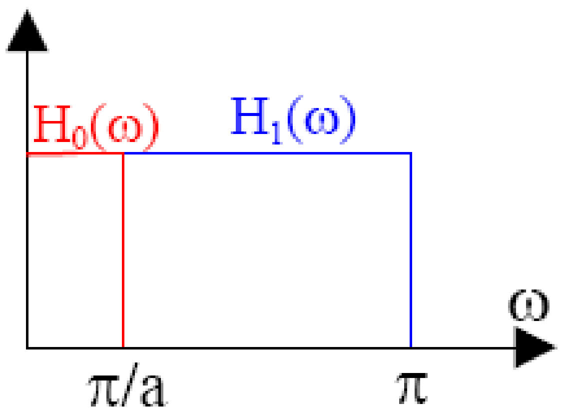
(approximate equalisation)

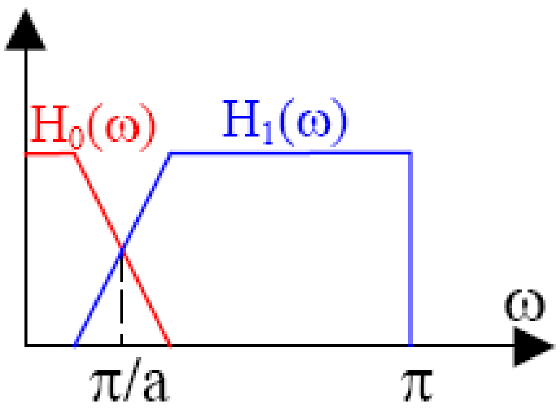
- Multi-rate conversion by factor *I*/*D* after each frequency subdivision (LPF/HPF)

- Reduced bitrate of digitized signal (bandwidth compression) due to nonuniform coding (variable number of bits per sample)

Subband Coding

- Implementation:





Brickwall Filter:

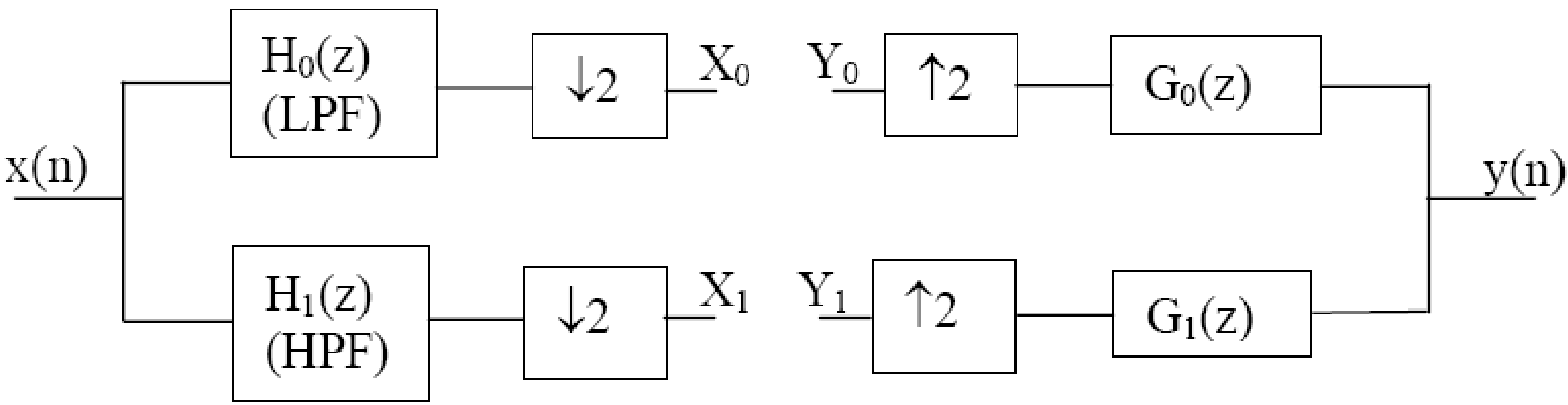
Physically unrealizable

Quadrature Mirror Filter (QMF):

aliasing for decimated subbands can be removed by judicious choice of $H_0(\omega)$ and $H_1(\omega)$

Two-Channel QMF

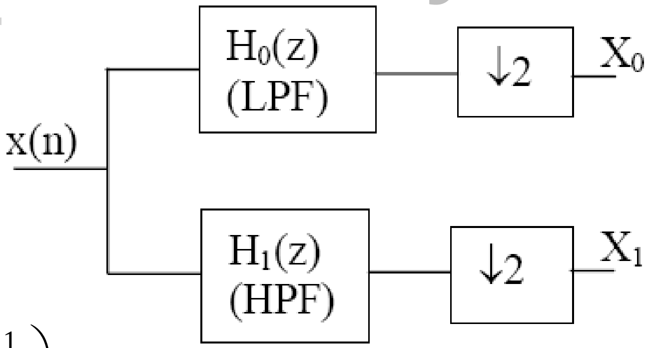
- Implementation (analyzer / synthesizer):



(for I/D=1/2)

Two-Channel QMF: Analysis

- QMF Analyzer:



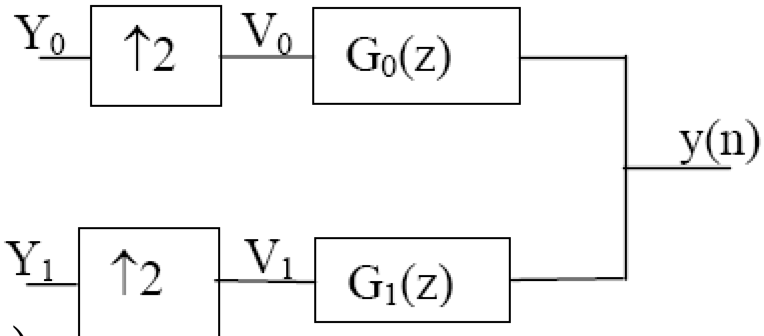
$$X_0(z) = \frac{1}{D}\sum_{k=0}^{D-1} H_0\left(\exp\left(-\frac{j2\pi}{D}k\right)z^{\frac{1}{D}}\right)X\left(\exp\left(-\frac{j2\pi}{D}k\right)z^{\frac{1}{D}}\right), \qquad D = 2$$

$$\Rightarrow \qquad X_0(\omega) = \frac{1}{2}\left[H_0\left(\frac{\omega}{2}\right)X\left(\frac{\omega}{2}\right) + H_0\left(\frac{\omega}{2}-\pi\right)X\left(\frac{\omega}{2}-\pi\right)\right]$$

$$X_1(\omega) = \frac{1}{2}\left[H_1\left(\frac{\omega}{2}\right)X\left(\frac{\omega}{2}\right) + H_1\left(\frac{\omega}{2}-\pi\right)X\left(\frac{\omega}{2}-\pi\right)\right]$$

Two-Channel QMF: Synthesis

- QMF Synthesizer:



$$V_0(z) = Y_0\left(z^I\right), \quad I = 2$$

$$\Rightarrow \; Y(\omega) = G_0(\omega)Y_0(2\omega) + G_1(\omega)Y_1(2\omega)$$

- Cascaded QMF analyzer-synthesizer:

$$Y_0(\omega) = X_0(\omega), \quad Y_1(\omega) = X_1(\omega) \quad \Leftrightarrow$$

Aliasing (k=1)

$$Y(\omega) = \frac{1}{2}\left[H_0(\omega)G_0(\omega) + H_1(\omega)G_1(\omega)\right]X(\omega) + \frac{1}{2}\left[H_0(\omega-\pi)G_0(\omega) + H_1(\omega-\pi)G_1(\omega)\right]X(\omega-\pi)$$

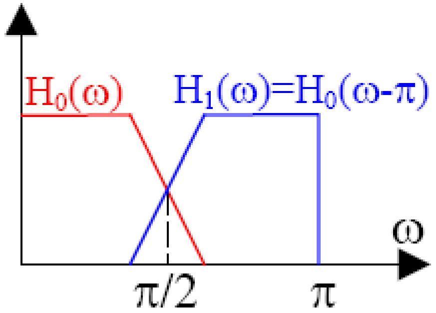
QMF Anti-Aliasing

- Elimination of aliasing for any input signal:

$$H_0(\omega-\pi)G_0(\omega)+H_1(\omega-\pi)G_1(\omega)=0$$

e.g. $$G_0(\omega)=H_1(\omega-\pi),\quad G_1(\omega)=-H_0(\omega-\pi)$$

  - results in time-*invariant* filter
  - example: alias-free symmetric subband coding



$$G_0(\omega)=H_0(\omega),\quad G_1(\omega)=-H_0(\omega-\pi)$$

QMF Perfect Reconstruction

- Distortion-free & alias-free reconstruction:

$$H_0(\omega)G_0(\omega)+H_1(\omega)G_1(\omega)=D\exp(-jk\omega), \qquad D=2$$

$$\Leftrightarrow \quad H_0(\omega)H_1(\omega-\pi)-H_1(\omega)H_0(\omega-\pi)=D\exp(-jk\omega)$$

  - Example: symmetric subband

$$H_0^2(\omega)-H_0^2(\omega-\pi)=D\exp(-jk\omega)$$

    i.e., $\left|H_0^2(\omega)-H_0^2(\omega-\pi)\right|$ independent of ω (all-pass filter), but may exhibit phase distortion!

  - It can be shown: linear-phase FIR QMF causes amplitude distortion

M-Channel QMF Bank

- M branches; $\downarrow M$ in analyzer, $\uparrow M$ in synthesizer
- Output kth analyzer branch (BPF+D):

$$X_k(z) = \frac{1}{M} \sum_{m=0}^{M-1} H_k\left(z^{1/M} \exp\left(-j \frac{2\pi m}{M} \right) \right) X\left(z^{1/M} \exp\left(-j \frac{2\pi m}{M} \right) \right), \qquad (M = D)$$

- Output synthesizer (I+BPF): $Y(z) = \sum_{k=0}^{M-1} G_k(z) Y_k\left(z^M\right)$

$$\Rightarrow Y(z) = \sum_{k=0}^{M-1} G_k(z) \left[\frac{1}{M} \sum_{m=0}^{M-1} H_k\left(z \exp\left(-j \frac{2\pi m}{M} \right) \right) X\left(z \exp\left(-j \frac{2\pi m}{M} \right) \right) \right]$$

$$= \sum_{m=0}^{M-1} L_m(z) X\left(z \exp\left(-j \frac{2\pi m}{M} \right) \right), \qquad \boxed{L_m(z) = \frac{1}{M} \sum_{k=0}^{M-1} H_k\left(z \exp\left(-j \frac{2\pi m}{M} \right) \right) G_k(z)}$$

M-Channel QMF

- Alias-free QMF:

$$Y(\omega) = L_0(\omega)\, X(\omega) \quad \text{iff} \quad \sum_{m=1}^{M-1} L_m(z)\, X\left(z \exp\left(-j \frac{2\pi m}{M} \right) \right) = 0, \qquad \forall X(z)$$

i.e. $\boxed{L_m(z) = 0, \quad 1 \le m \le M-1}$

- Distortion-free & alias-free QMF:

$|L_0(\omega)|$ independent of ω (all-pass filters)