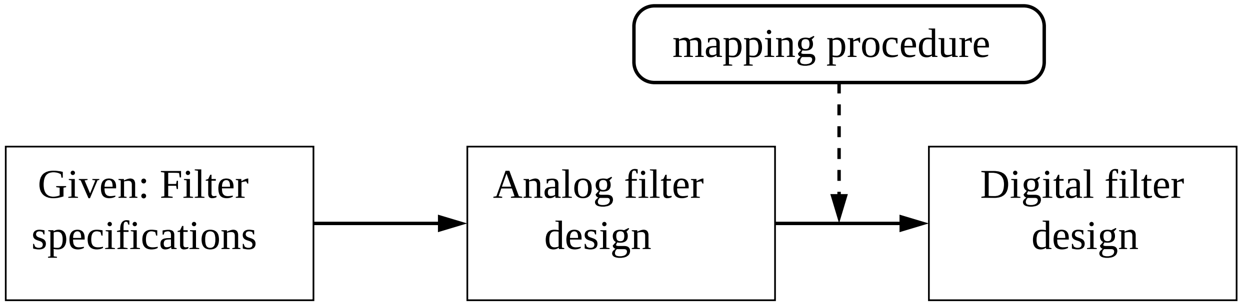
Design of IIR Digital Filters

□ Design an analog filter by obtaining transfer function $\hat{H}(s)$ to meet requirements.

□ Construct a mapping procedure to obtain IIR digital filter $\hat{H}(s) \rightarrow H(z)$

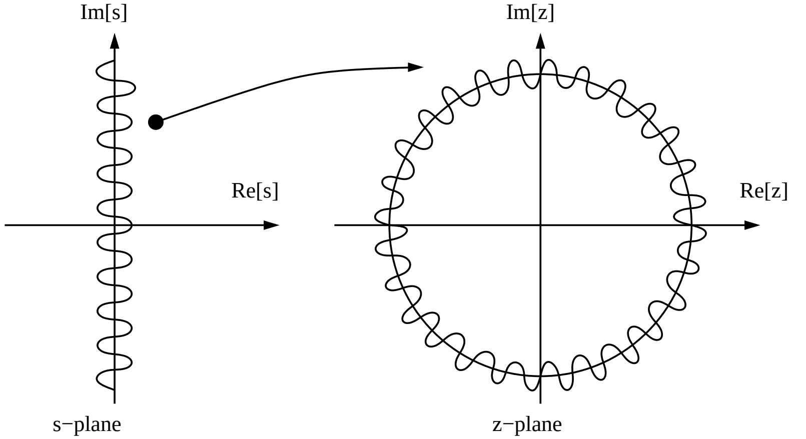


The mapping procedure should satisfy the following conditions:

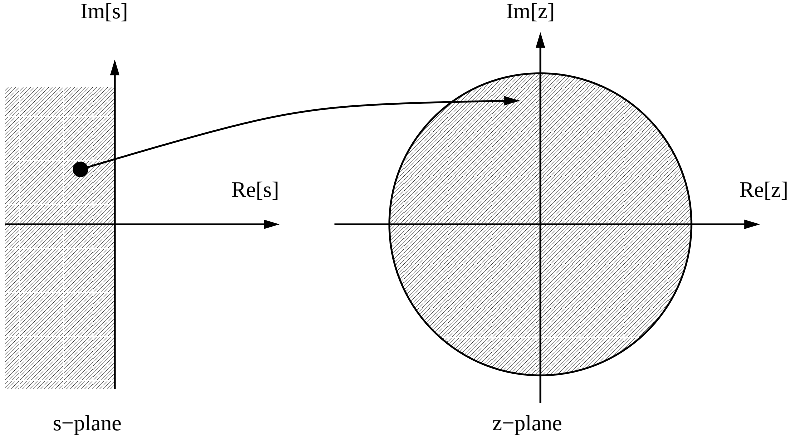
- **Condition 1**: The imaginary axis of the s-plane is mapped onto the unit circle of the z-plane. **This preserves the frequency characteristics.**
- **Condition 2**: The LH s-plane ($\mathrm{Re}[s] < 0$) mapped into the interior of the unit circle of the z-plane. **This preserves the stability properties.**

Design of IIR Digital Filters

Condition 1: $\{s = j\omega | -\infty < \omega < \infty\} \to \{z = e^{j\theta} | -\pi < \theta \leq \pi\}$



Condition 2: $\{s | \mathrm{Re}[s] < 0\} \to \{z || z | < 1\}$



Mapping Procedure: Numerical Integration

A continuous differential equation (characterising an analog filter) can be approximated by a discrete difference equation (characterising a digital filter).

Consider the simplest case, the Euler approximation of a continuous time function $d\hat{y}(t)/dt$ as

$$\left.\frac{d\hat{y}(t)}{dt}\right|_{t=nT} = \frac{y(n) - y(n-1)}{T}$$

where T is the sampling period and $y(k) \triangleq \hat{y}(t)|_{t=kT}$ for all k.

This gives rise to the following mapping procedure

$$\boxed{s = \frac{1 - z^{-1}}{T} \triangleq f(z)}$$

which conversely implies that $z = \frac{1}{1-sT}$

Mapping Procedure: Numerical Integration

What is the quality of this approximation? Are conditions 1 and 2 satisfied?

The imaginary axis of the s-plane is mapped to

$$z = \frac{1}{1 - j\omega T} = \frac{1}{2} + \frac{1}{2}\left(\frac{1 + j\omega T}{1 - j\omega T}\right)$$

$$\Rightarrow z - \frac{1}{2} = \frac{1}{2}\exp\left[j \tan^{-1}\frac{2\omega T}{1 - (\omega T)^2}\right]$$

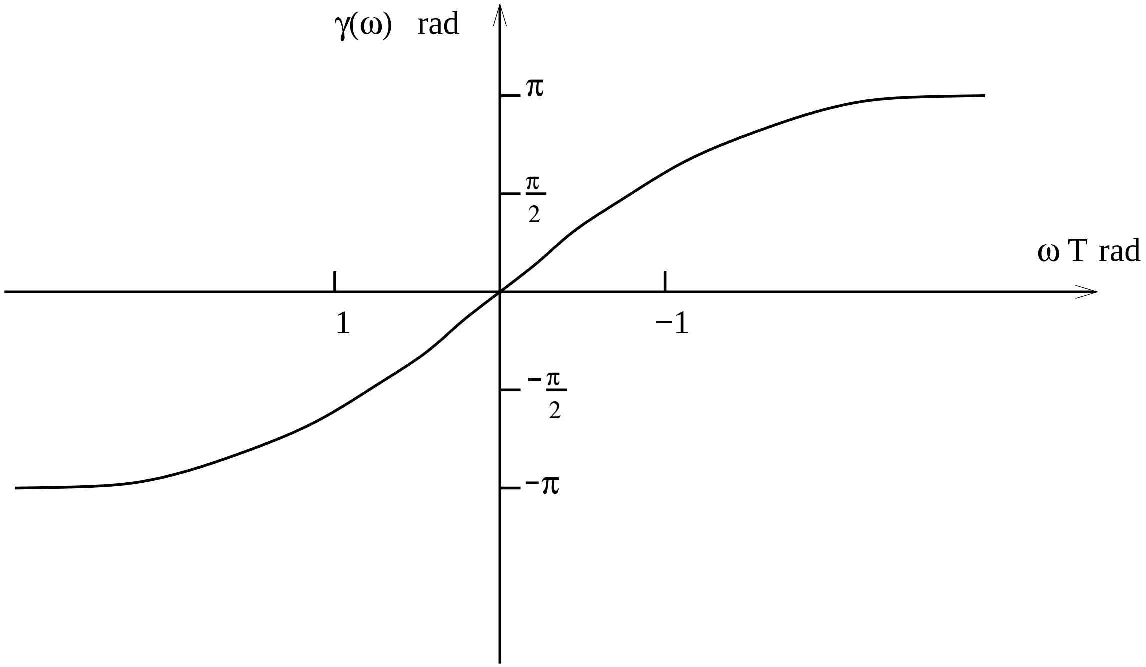
We can conclude the following

1. $|z - 1/2| = 1/2$ for all ω
2. The phase angle ($\gamma(\omega)$) of $[z - 1/2]$ is given by $\tan^{-1}\frac{2\omega T}{1-(\omega T)^2}$

The image of the imaginary axis of the s-plane is a circle in the z-plane with radius 1/2 centered at 1/2.

Mapping Procedure: Numerical Integration

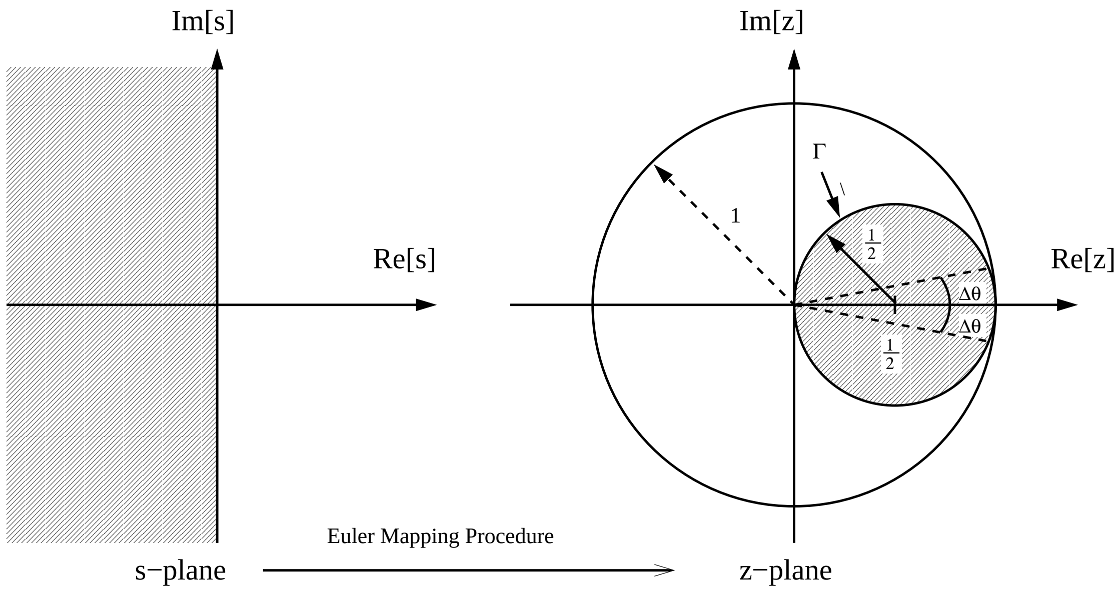
The phase angle $\gamma(\omega)$ goes from $-\pi$ to π as ω goes from $-\infty$ to ∞



When $\mathrm{Re}[s] = \mathrm{Re}[\sigma + j\omega] = \sigma < 0$,

$$|z| = \frac{1}{\sqrt{(1-\sigma T)^2 + (\omega T)^2}} \leq \frac{1}{|1-\sigma T|} \leq 1$$

Mapping Procedure: Numerical Integration



The LH s-plane is mapped into the circle of the z-plane

$\Rightarrow$ **Condition 2 is satisfied !**

Condition 1 is not completely satisfied except for $|\theta|$ small, such as $|\theta| \leq \Delta\theta$.

$\Rightarrow$ **Condition 1 is satisfied for low frequency operations !**