The Fourier Transform & the DFT

- Fourier transform $$F(j\omega) = \int_{-\infty}^{+\infty} f(t).e^{-j\omega t}.dt$$
- Take N samples of $f(t)$ from 0 to $.(N\text{-}1)T$
- Can $F(j\omega)$ be estimated from these?
- Estimate based on rectangular approximation of integral

$$\hat{F}(j\omega) = \sum_{n=0}^{N-1} f(nT).e^{-j\omega nT}.T$$

The Fourier Transform & the DFT

- Estimate based on rectangular approximation of integral
$$\hat{F}(j\omega) = \sum_{n=0}^{N-1} f(nT).e^{-j\omega nT}.T$$
- Now take N samples of $\hat{F}(j\omega)$ at multiples of $$\omega_0 = \frac{2\pi}{NT}$$
- Note that $\hat{F}(j\omega)$ is of period $\dfrac{2\pi}{T}$

The Fourier Transform & the DFT

- Then

$$\hat{F}(jk\omega_0) = \sum_{n=0}^{N-1} T.f(nT).e^{-j\frac{2\pi}{NT}.k.n.T}$$

$$= T.\sum_{n=0}^{N-1} f(nT).W_N^{-nk} \qquad W_N = \exp\left(j\frac{2\pi}{N}\right)$$

- i.e. DFT relationship where

$$d(n) \overset{\Delta}{=} T.f(T.[n \bmod N])$$

- has a DFT $D(k)$ such that

$$D(k) = \hat{F}(jk\omega_0)$$

The Fourier Transform & the DFT

- How good is the estimate?
- Let period of DFT operation be $T_0 = NT$
- Hence $T_0 = \dfrac{2\pi}{\omega_0}$
- Similarly $N\omega_0 = \Omega_0 = \dfrac{2\pi}{T}$
- There are two approximations involved in estimating $F(j\omega)$
- (i) $f(t)$ is sampled leading to aliasing for non-bandlimited signals.
- (ii) N samples only are retained

The Fourier Transform & the DFT

- Point (i) : Sampling yields a new signal

$$g(n) \overset{\Delta}{=} T.f(nT)$$

- With Fourier Transform

$$G(j\omega) = \sum_{k=-\infty}^{+\infty} F(j\omega + jk.\Omega_0)$$

- i.e. it may be a poor approximation to $F(j\omega)$
- Point (ii) : Retaining samples *0, N-1* is effectively windowing the data by

$$\omega(n) = 1 \qquad 0 \le n \le N-1$$

The Fourier Transform & the DFT

- Fourier Transform of window

$$W(j\omega) = e^{-j\frac{N-1}{2}.\omega T} . \frac{\sin(N\omega T/2)}{\sin(\omega T/2)}$$

- Actual signal used in DFT is $h(n) = g(n).w(n)$
- i.e. this leads to convolution in frequency domain

$$H(j\omega) = \frac{T}{2\pi} \int_0^{\Omega_0} G(j\phi).W(j\omega - j\phi).d\phi$$

The Fourier Transform & the DFT

- Note that

$$H(j\omega) = \sum_{n=-\infty}^{+\infty} T.f(nT).w(n).e^{-j\omega nT}$$

$$= \sum_{n=0}^{N-1} T.f(nT).w(n).e^{-j\omega nT} = \hat{F}(j\omega)$$

- Ie $H(j\omega)$ is the estimate of

$$\hat{F}(j\omega) = \frac{T}{2\pi}\int_0^{\Omega_0}\left[\sum_{k=-\infty}^{+\infty} F(j\phi + jk\Omega_0)\right].W[j\omega - j\phi].d\phi$$

The Fourier Transform & the DFT

- Since $W(j\omega)$ is periodic with period Ω_0 its contributions to above may be taken over the entire domain as

$$\hat{F}(j\omega) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} F(j\phi).\left[T.W[j(\omega-\phi)]\right]d\phi$$

- i.e. the estimate $\hat{F}(j\omega)$ is the convolution between the desired transform and $V(j\omega)$

$$V(j\omega) = T.e^{-j\frac{(N-1)}{2}\omega T}.\frac{\sin(N\omega T/2)}{\sin(\omega T/2)}$$

The Fourier Transform & the DFT

- The main lobe of $V(j\omega)$ is of width $2\times\frac{2\pi}{NT}=2\omega_0$
- hence convolution "smears" or "blurs" a step to the same width.
- For greater bandwidth, Ω_0 must be increased (or T decreased).
- To increase resolution we must decrease ω_0
- for a fixed T this implies N must increase.

The Fourier Transform & the DFT

- Frequency resolution is the Minimum separation between two sinusoids, resolvable in frequency

- Because of the convolution two impulses in frequency will be smeared, so that if they are to be resolvable they must be separated by at least one frequency bin ie $\frac{2\pi}{NT}$

The Fourier Transform & the DFT

- If maximum frequency in the signal is $\omega_{\max}$ and the required resolution is Δ then

$$\frac{2\pi}{2T} > \omega_{\max} \qquad \Delta > 2.\frac{2\pi}{NT}$$

- thus $N > \frac{4\pi}{4T} = \frac{4\omega_{\max}}{\Delta}$
- For example (rad/s)

$$\Delta = 2\pi \times 1 \qquad \omega_{\max} = 2\pi \times 10^3$$

- then $N > 4000$

The Fourier Transform & the DFT

- In practice $\hat{F}(j\omega)$ is required within a prescibed resolution and bandwidth.
- Let $F(j\omega)$ be limited to $\omega_{\max}$, and the required resolution to be Δ or better.
- Then $\Omega_0 > 2\omega_{\max}$ $\quad 2\omega_0 < \Delta$
- Hence $\frac{\Omega_0}{2} = \frac{\pi}{T} > \omega_{\max}$ i.e. $T < \frac{\pi}{\omega_{\max}}$

The Fourier Transform & the DFT

- Ie $2\omega_0 = 2\left[\frac{2\pi}{NT}\right] < \Delta \qquad N > \frac{4\omega_{\max}}{\Delta}$
- If $f(t)$ is assumed to be of duration $t_{\max}$ then again $\frac{\Omega_0}{2} > \omega_{\max} \quad \rightarrow \quad T < \frac{\pi}{\omega_{\max}}$
- Set $T_0 = \frac{2\pi}{\omega_0} = NT$
- and $\frac{T_0}{2} > t_{\max} \quad \rightarrow \quad \frac{T_0}{2} = \frac{NT}{2} > t_{\max}$

The Fourier Transform & the DFT

- i.e. $N > \frac{2t_{\max}}{T}$
- Hence $N > \frac{2t_{\max}.\omega_{\max}}{\pi}$

where $t_{\max}.\omega_{\max}$ time – bandwidth product of signal

Discrete Fourier Transforms

- Consider finite duration signal

$$\{x(n)\} \quad n = 0 \ldots N-1$$

- Its z-tranform is $X(z) = \sum_{n=0}^{N-1} x(n).z^{-n}$
- Evaluate at points on z-plane as

$$X(k) = X(z)\Big|_{z_k} = \sum_{n=0}^{N-1} x(n)e^{-j\frac{2\pi}{N}kn}$$

- We can evaluate N independent points

Discrete Fourier Transforms

$$X(k) = \sum_{n=0}^{N-1} x(n)e^{-j\frac{2\pi}{N}kn}$$

- This is known as the Discrete Fourier Transform (DFT) of $\{x(n)\}$
- Periodic in k ie $X(k+pN) = X(k)$
- This is as expected since the spectrum is periodic in frequency

Discrete Fourier Transforms

- Multiply both sides of the DFT by $e^{j\frac{2\pi}{N}km}$
- And add over the frequency index k

$$\sum_{k=0}^{N-1} X(k) e^{j\frac{2\pi}{N}km} = \sum_{n=0}^{N-1} x(n) \sum_{k=0}^{N-1} e^{j\frac{2\pi}{N}k(m-n)}$$

- From which $x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k).e^{j\frac{2\pi}{N}kn}$

Discrete Fourier Transforms

- This is the inverse DFT

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k).e^{j\frac{2\pi}{N}kn} \qquad x(n+qN) = x(n)$$

- That is a) the DFT assumes that we deal with periodic signals in the time domain

  b) Sampling in one domain produces periodic behaviour in the other domain

Discrete Fourier Transforms

- Effectively by knowing $X(z)\big|_{z=z_k} = X(k)$
- $X(z)$ is known everywhere since

$$X(z) = \sum_{n=0}^{N-1} x(n).z^{-n} = \sum_{n=0}^{N-1} \frac{1}{N} \sum_{k=0}^{N-1} X(k) e^{j\frac{2\pi}{N}kn} z^{-n}$$

- or

$$X(z) = \frac{1}{N} \sum_{k=0}^{N-1} X(k).\frac{1 - z^{-N}}{1 - e^{j\frac{2\pi}{N}k}.z^{-1}}$$

Discrete Fourier Transforms

- The formula

$$X(z) = \frac{1}{N} \sum_{k=0}^{N-1} X(k).\frac{1 - z^{-N}}{1 - e^{j\frac{2\pi}{N}k}.z^{-1}}$$

- This is essentially an interpolation and forms the basis of the *Frequency Sampling Method* for FIR digital filter design

Convolution in DFT

- Consider the following transform pairs

$$\{x_p(n)\} \leftrightarrow X_p(k) \quad \{h_p(n)\} \leftrightarrow H_p(k)$$

- Define $Y_p(k) = H_p(k).X_p(k)$
- Find

$$y_p(n) \overset{\Delta}{=} \text{IDFT}[Y_p(k)]$$

Convolution in DFT

- From IDFT

$$y_p(n) = \frac{1}{N}\sum_{k=0}^{N-1} Y_p(k).e^{j\frac{2\pi}{N}kn} = \frac{1}{N}\sum_{k=0}^{N-1} H_p(k).X_p(k).e^{j\frac{2\pi}{N}kn}$$

- However

$$H_p(k) = \sum_{m=0}^{N-1} h_p(m)e^{-j\frac{2\pi}{N}km}$$

$$y_p(n) = \frac{1}{N}\sum_{k=0}^{N-1}\sum_{m=0}^{N-1} h_p(m).e^{-j\frac{2\pi}{N}km}.X_p(k).e^{j\frac{2\pi}{N}kn}$$

Convolution in DFT

- Or

$$y_p(n) = \sum_{m=0}^{N-1} h_p(m).\underbrace{\frac{1}{N}\sum_{k=0}^{N-1} X_p(k).e^{j\frac{2\pi}{N}k(n-m)}}_{x_p(n-m)}$$

- Thus

$$y_p(n) = \sum_{m=0}^{N-1} h_p(m).x_p(n-m)$$

- This the *Circular Convolution*

Computation of the DFT: The FFT Algorithm

- Computation of DFT requires for every sample N multiplications. There are N samples to be computed i.e. N^2 time consuming operations.
- The Fast Fourier Algorithm:(Decimation in time - DIT, assume even no. of samples)
- set $x_1(n) = x(2n) \quad x_2(n) = x(2n+1)$

$$n = 0,1,\ldots,\frac{N}{2}-1$$

FFT

- Then DFT of $\{x(n)\}$ is written

$$X(k) = \sum_{\substack{n=0 \\ \text{even}}}^{N-1} x(n).W_N^{\ nk} + \sum_{\substack{n=0 \\ \text{odd}}}^{N-1} x(n).W_N^{\ nk}$$

- set

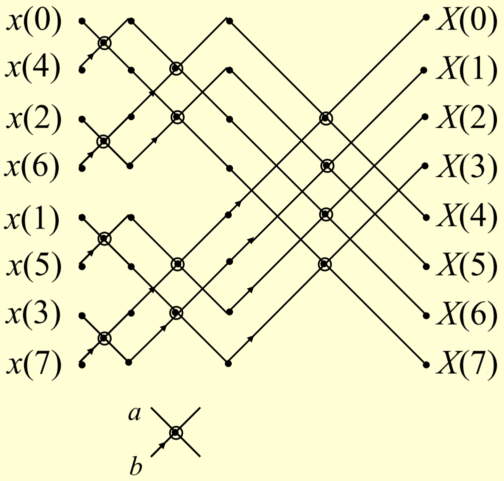
$$W_N = e^{-j\frac{2\pi}{N}} \qquad W_N^{\ 2} = e^{-j2\pi/(N/2)} = W_{N/2}$$

FFT

- ie $$X(k) = \sum_{n=0}^{\frac{N}{2}-1} x_1(n).W_{N/2}^{\ nk} + W_N^{\ k} \sum_{n=0}^{\frac{N}{2}-1} x_2(n).W_{N/2}^{\ nk}$$
- Or $$X(k) = X_1(k) + W_N^{\ k} X_2(k)$$
- Computations of each of summations is now of order $(N/2)^2$, and thus total computational effort is reduced to $N^2/2$.
- Continuation of “divide-&-compute” reduces effort to Nlog(N)

8-point FFT

- 8-point Signal Flow Diagram



FFT times

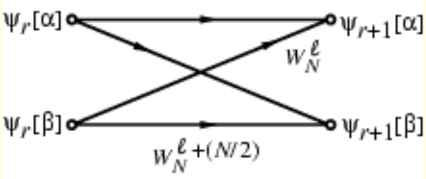
- Time (1 multiplication per microsec)

	N	Direct DFT	FFT
2^6	64	.02 sec	.002 sec
2^9	512	1	.02 sec
2^{12}	4096	67	.2
2^{15}	32768	1 hr 11 mins	2
2^{18}	262144	3 days 4 hrs	19

Decimation-in-Time FFT Algorithm

- In the basic module two output variables are generated by a weighted combination of two input variables as indicated below

  where $r = 1, 2, \ldots, \mu$ and $\alpha, \beta = 0, 1, \ldots, N-1$



- Basic computational module is called a **butterfly computation**

Decimation-in-Time FFT Algorithm

- Input-output relations of the basic module are:

$$\Psi_{r+1}[\alpha] = \Psi_r[\alpha] + W_N^{\ell}\Psi_r[\beta]$$

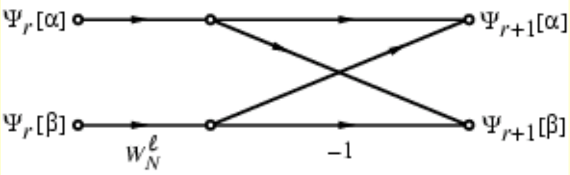
$$\Psi_{r+1}[\beta] = \Psi_r[\alpha] + W_N^{\ell+(N/2)}\Psi_r[\beta]$$

- Substituting $W_N^{\ell+(N/2)} = -W_N^{\ell}$ in the second equation given above we get

$$\Psi_{r+1}[\beta] = \Psi_r[\alpha] - W_N^{\ell}\Psi_r[\beta]$$

Decimation-in-Time FFT Algorithm

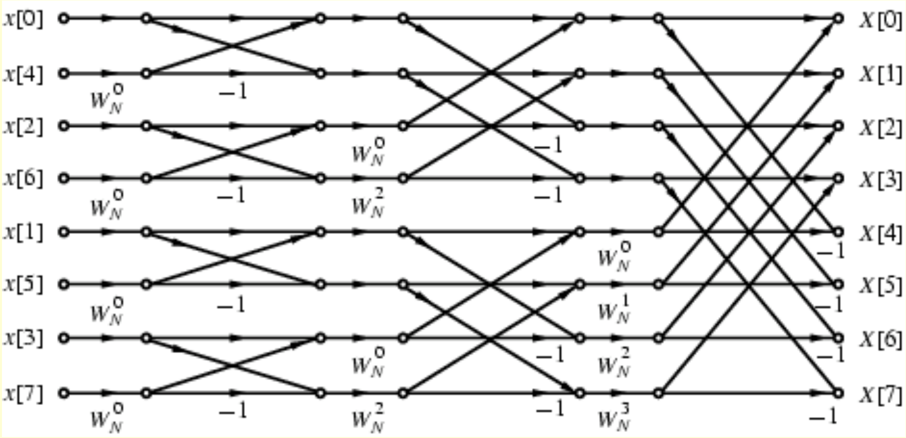
- Modified butterfly computation requires only one complex multiplication as indicated below



- Use of the above modified butterfly computation module reduces the total number of complex multiplications by 50%

Decimation-in-Time FFT Algorithm

- New flow-graph using the modified butterfly computational module for $N = 8$



Decimation-in-Time FFT Algorithm

- Computational complexity can be reduced further by avoiding multiplications by $W_N^0 = 1$, $W_N^{N/2} = -1$, $W_N^{N/4} = j$, and $W_N^{3N/4} = -j$
- The DFT computation algorithm described here also is efficient with regard to memory requirements
- Note: Each stage employs the same butterfly computation to compute $\Psi_{r+1}[\alpha]$ and $\Psi_{r+1}[\beta]$ from $\Psi_r[\alpha]$ and $\Psi_r[\beta]$