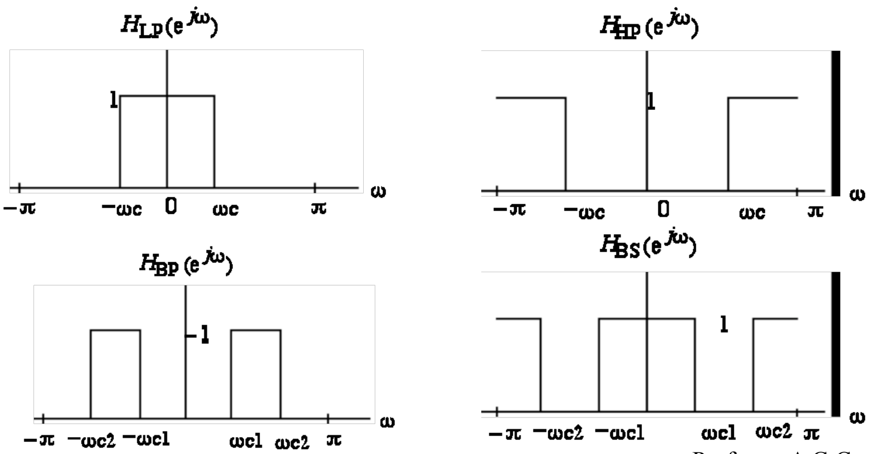
# DIGITAL FILTERS

- Finite Impulse Response (FIR)
- Infinite Impulse Response (IIR)
- Background
- Matlab functions



# Digital Filter Specifications

- Only the magnitude approximation problem
- Four basic types of ideal filters with magnitude responses as shown below (Piecewise flat)



# Digital Filter Specifications

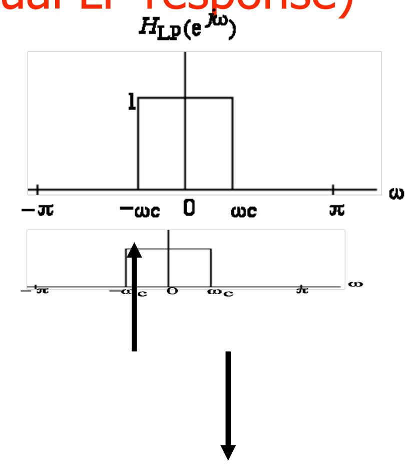
- These filters are unealisable because (one of the following is sufficient)
  - their impulse responses infinitely long non-causal
  - Their amplitude responses cannot be equal to a constant over a band of frequencies

  Another perspective that provides some understanding can be obtained by looking at the ideal amplitude squared.



# Digital Filter Specifications

- Consider the ideal LP response squared (same as actual LP response)



## Digital Filter Specifications

- The realisable squared amplitude response transfer function (and its differential) is continuous in $\omega$
- Such functions
  - if IIR can be infinite at point but around that point cannot be zero.
  - if FIR cannot be infinite anywhere.
- Hence previous defferential of ideal response is unrealisable

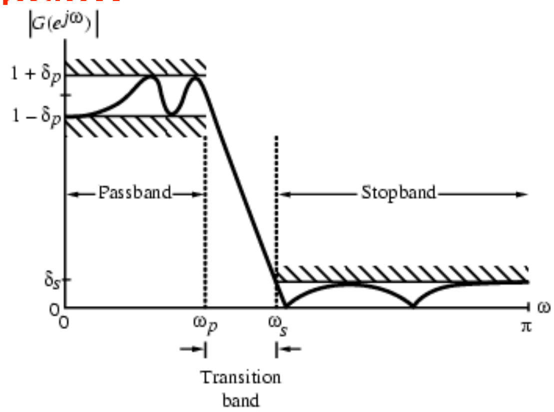


## Digital Filter Specifications

- A realisable response would effectively need to have an approximation of the *delta functions* in the differential
- This is a necessary condition

## Digital Filter Specifications

- For example the magnitude response of a digital lowpass filter may be given as indicated below






## Digital Filter Specifications

- In the **passband** $0 \le \omega \le \omega_p$ we require that $|G(e^{j\omega})| \cong 1$ with a deviation $\pm\delta_p$

$$1-\delta_p \le |G(e^{j\omega})| \le 1+\delta_p, \quad |\omega| \le \omega_p$$

- In the **stopband** $\omega_s \le \omega \le \pi$ we require that $|G(e^{j\omega})| \cong 0$ with a deviation $\delta_s$

$$|G(e^{j\omega})| \le \delta_s, \quad \omega_s \le |\omega| \le \pi$$

## Digital Filter Specifications

Filter specification parameters

- $\omega_p$ - **passband edge frequency**
- $\omega_s$ - **stopband edge frequency**
- $\delta_p$ - **peak ripple value** in the passband
- $\delta_s$ - **peak ripple value** in the stopband



## Digital Filter Specifications

- Practical specifications are often given in terms of **loss function (in dB)**
- $$\mathcal{G}(\omega) = -20\log_{10}\left|G(e^{j\omega})\right|$$
- **Peak passband ripple**
  $$\alpha_p = -20\log_{10}(1-\delta_p) \quad \text{dB}$$
- **Minimum stopband attenuation**
  $$\alpha_s = -20\log_{10}(\delta_s) \quad \text{dB}$$

# Digital Filter Specifications

- In practice, passband edge frequency $F_p$ and stopband edge frequency $F_s$ are specified in Hz
- For digital filter design, normalized bandedge frequencies need to be computed from specifications in Hz using

$$\omega_p = \frac{\Omega_p}{F_T} = \frac{2\pi F_p}{F_T} = 2\pi F_p T$$

$$\omega_s = \frac{\Omega_s}{F_T} = \frac{2\pi F_s}{F_T} = 2\pi F_s T$$



# Digital Filter Specifications

- Example - Let $F_p = 7$ kHz, $F_s = 3$ kHz, and $F_T = 25$ kHz
- Then

$$\omega_p = \frac{2\pi(7\times 10^3)}{25\times 10^3} = 0.56\pi$$

$$\omega_s = \frac{2\pi(3\times 10^3)}{25\times 10^3} = 0.24\pi$$

# Selection of Filter Type

- The transfer function $H(z)$ meeting the specifications must be a causal transfer function
- For IIR real digital filter the transfer function is a real rational function of $z^{-1}$

$$H(z) = \frac{p_0 + p_1 z^{-1} + p_2 z^{-2} + \cdots + p_M z^{-M}}{d_0 + d_1 z^{-1} + d_2 z^{-2} + \cdots + d_N z^{-N}}$$

- $H(z)$ must be stable and of lowest order $N$ or $M$ for reduced computational complexity



# Selection of Filter Type

- FIR real digital filter transfer function is a polynomial in $z^{-1}$ (order $N$) with real coefficients

$$H(z) = \sum_{n=0}^{N} h[n] z^{-n}$$

- For reduced computational complexity, degree $N$ of $H(z)$ must be as small as possible
- If a linear phase is desired then we must have: $h[n] = \pm h[N-n]$
- (More on this later)

# Selection of Filter Type

- Advantages in using an FIR filter -

  (1) Can be designed with exact linear phase

  (2) Filter structure always stable with quantised coefficients
- Disadvantages in using an FIR filter - Order of an FIR filter is considerably higher than that of an equivalent IIR filter meeting the same specifications; this leads to higher computational complexity for FIR



# FIR Design

## FIR Digital Filter Design

Three commonly used approaches to FIR filter design -

(1) Windowed Fourier series approach

(2) Frequency sampling approach

(3) Computer-based optimization methods

## Finite Impulse Response Filters

- The transfer function is given by

$$H(z)=\sum_{n=0}^{N-1} h(n).z^{-n}$$

- The length of Impulse Response is N
- All poles are at $z=0$ .
- Zeros can be placed anywhere on the z-plane



## FIR: Linear phase

For phase linearity the FIR transfer function **must** have zeros outside the unit circle

# FIR: Linear phase

- To develop expression for phase response set transfer function (order *n*)

$$H(z) = h_0 + h_1 z^{-1} + h_2 z^{-2} + \ldots + h_n z^{-n}$$

- In factored form

$$H(z) = K \prod_{i=1}^{n_1} (1 - \alpha_i z^{-1}) . \prod_{i=1}^{n_2} (1 - \beta_i z^{-1})$$

- Where $|\alpha_i| < 1, \quad |\beta_i| > 1 \quad K$ , is real & zeros occur in conjugates



# FIR: Linear phase

- Let $H(z) = KN_1(z)N_2(z)$

  where

$$N_1(z) = \prod_{i=1}^{n_1} (1 - \alpha_i z^{-1}) \qquad N_2(z) = \prod_{i=1}^{n_2} (1 - \beta_i z^{-1})$$

- Thus

$$\ln(H(z)) = \ln(K) + \sum_{i=1}^{n_1} \ln(1 - \alpha_i z^{-1}) + \sum_{i=1}^{n_2} \ln(1 - \beta_i z^{-1})$$

# FIR: Linear phase

- Expand in a Laurent Series convergent within the unit circle
- To do so modify the second sum as

$$\sum_{i=1}^{n_2} \ln(1-\beta_i z^{-1}) = \sum_{i=1}^{n_2} \ln(-\beta_i z^{-1}) + \sum_{i=1}^{n_2} \ln(1-\frac{1}{\beta_i} z)$$



# FIR: Linear phase

- So that

$$\ln(H(z)) = \ln(\overline{K}) - n_2 \ln(z) + \sum_{i=1}^{n_1} \ln(1-\alpha_i z^{-1}) + \sum_{i=1}^{n_2} \ln(1-\frac{1}{\beta_i} z)$$

- Thus

$$\ln(H(z)) = \ln(\overline{K}) - n_2 \ln(z) + \sum_{m=1}^{\infty} \frac{s_m^{N_1}}{m} z^{-m} + \frac{s_{-m}^{N_2}}{m} z^{m}$$

- where

$$s_m^{N_1} = \sum_{i=1}^{n_1} \alpha_i^m \qquad s_{-m}^{N_2} = \sum_{i=1}^{n_1} \beta_i^{-m}$$

# FIR: Linear phase

- $s_m^{N1}$ are the root moments of the minimum phase component
- $s_{-m}^{N2}$ are the inverse root moments of the maximum phase component
- Now on the unit circle we have $z = e^{j\theta}$ and $H(e^{j\theta}) = A(\theta)e^{j\phi(\theta)}$



# Fundamental Relationships

$$\ln(H(e^{j\theta})) = \ln(\overline{K}) - jn_2\theta + \sum_{m=1}^{\infty} \frac{s_m^{N1}}{m} e^{-jm\theta} + \frac{s_{-m}^{N2}}{m} e^{jm\theta}$$

$$\ln(H(e^{j\theta})) = \ln(A(\theta)e^{j\phi(\theta)}) = \ln(A(\theta)) + j\phi(\theta)$$

- hence (note Fourier form)

$$\ln(A(\theta)) = \ln(\overline{K}) + \sum_{m=1}^{\infty} \left(\frac{s_m^{N1}}{m} + \frac{s_{-m}^{N2}}{m}\right)\cos m\theta$$

$$\phi(\theta) = -n_2\theta - \sum_{m=1}^{\infty} \left(\frac{s_m^{N1}}{m} - \frac{s_{-m}^{N2}}{m}\right)\sin m\theta$$

## FIR: Linear phase

- Thus for linear phase the second term in the fundamental phase relationship must be identically zero for all index values.
- Hence
- 1) the maximum phase factor has zeros which are the inverses of the those of the minimum phase factor
- 2) the phase response is linear with group delay (normalised) equal to the number of zeros outside the unit circle



## FIR: Linear phase

- It follows that zeros of linear phase FIR trasfer functions not on the circumference of the unit circle occur in the form $\left[\rho_i e^{\pm j\theta_i}\right]^{\pm 1}$

## FIR: Linear phase

- For Linear Phase t.f. (order *N-1*)
- $$h(n) = \pm h(N-1-n)$$
- so that for *N* even:

$$H(z) = \sum_{n=0}^{N/2-1} h(n).z^{-n} \pm \sum_{n=N/2}^{N-1} h(n).z^{-n}$$

$$= \sum_{n=0}^{N/2-1} h(n).z^{-n} \pm \sum_{n=0}^{N/2-1} h(N-1-n).z^{-(N-1-n)}$$

$$= \sum_{n=0}^{N/2-1} h(n)\left[z^{-n} \pm z^{-m}\right] \qquad m = N-1-n$$



## FIR: Linear phase

- for *N* odd:

$$H(z) = \sum_{n=0}^{\frac{N-1}{2}-1} h(n).\left[z^{-n} \pm z^{-m}\right] + h\left(\frac{N-1}{2}\right)z^{-\left(\frac{N-1}{2}\right)}$$

- I) On $C: |z| = 1$ we have for N even, and +ve sign

$$H(e^{j\omega T}) = e^{-j\omega T\left(\frac{N-1}{2}\right)} . \sum_{n=0}^{N/2-1} 2h(n).\cos\left(\omega T\left(n - \frac{N-1}{2}\right)\right)$$

## FIR: Linear phase

- II) While for –ve sign

$$H(e^{j\omega T}) = e^{-j\omega T\left(\frac{N-1}{2}\right)} . \sum_{n=0}^{N/2-1} j2h(n).\sin\left(\omega T\left(n-\frac{N-1}{2}\right)\right)$$

- [Note: antisymmetric case adds $\pi/2$ rads to phase, with discontinuity at $\omega = 0$ ]
- III) For N odd with +ve sign

$$H(e^{j\omega T}) = e^{-j\omega T\left[\frac{N-1}{2}\right]}\left\{h\left(\frac{N-1}{2}\right) + \sum_{n=0}^{\frac{N-3}{2}} 2h(n).\cos\left[\omega T\left(n-\frac{N-1}{2}\right)\right]\right\}$$



## FIR: Linear phase

- IV) While with a –ve sign

$$H(e^{j\omega T}) = e^{-j\omega T\left[\frac{N-1}{2}\right]}\left\{\sum_{n=0}^{\frac{N-3}{2}} 2j.h(n).\sin\left[\omega T\left(n-\frac{N-1}{2}\right)\right]\right\}$$

- [Notice that for the antisymmetric case to have linear phase we require

$$h\left(\frac{N-1}{2}\right) = 0.$$

The phase discontinuity is as for *N* even]

## FIR: Linear phase

- The cases most commonly used in filter design are (I) and (III), for which the amplitude characteristic can be written as a polynomial in

$$\cos\frac{\omega T}{2}$$



## Design of FIR filters: Windows

(i) Start with ideal infinite duration $\{h(n)\}$

(ii) Truncate to finite length. (This produces unwanted ripples increasing in height near discontinuity.)

(iii) Modify to $\tilde{h}(n) = h(n).w(n)$

Weight $w(n)$ is the window

# Windows

Commonly used windows

- Rectangular $1-\frac{2|n|}{N}$ $\quad |n| < \frac{N-1}{2}$
- Bartlett $1+\cos\left(\frac{2\pi n}{N}\right)$
- Hann
- Hamming $0.54+0.46\cos\left(\frac{2\pi n}{N}\right)$
-
- Blackman $0.42+0.5\cos\left(\frac{2\pi n}{N}\right)+0.08\cos\left(\frac{4\pi n}{N}\right)$
-
- Kaiser $J_0\left[\beta\sqrt{1-\left(\frac{2n}{N-1}\right)^2}\right]\Big/ J_0(\beta)$



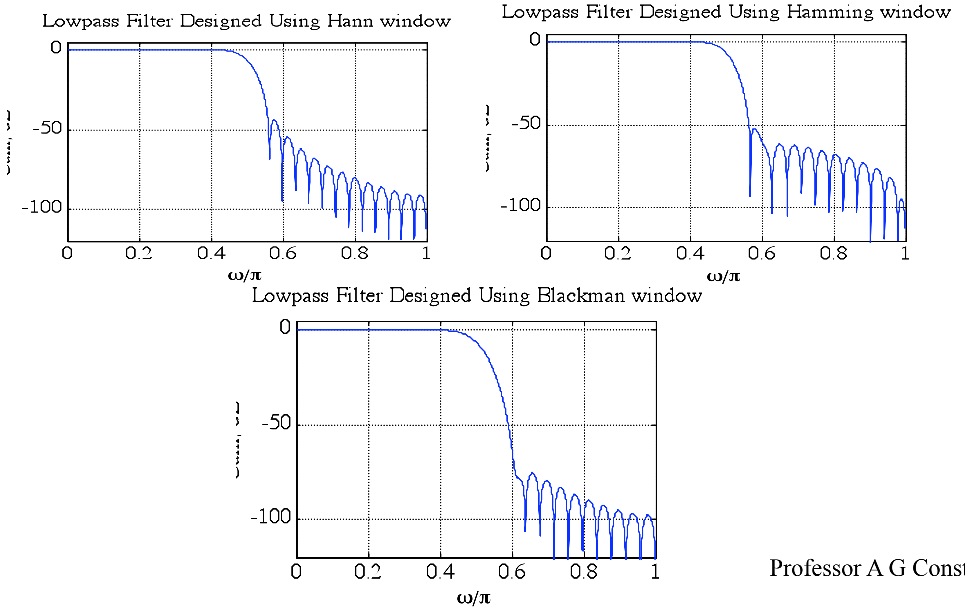
# Kaiser window

- Kaiser window

| β | Transition width (Hz) | Min. stop attn dB |
|---|---|---|
| 2.12 | 1.5/N | 30 |
| 4.54 | 2.9/N | 50 |
| 6.76 | 4.3/N | 70 |
| 8.96 | 5.7/N | 90 |

# Example

- Lowpass filter of length 51 and $\omega_c = \pi / 2$






# Frequency Sampling Method

- In this approach we are given $H(k)$ and need to find $H(z)$
- This is an interpolation problem and the solution is given in the DFT part of the course

$$H(z) = \frac{1}{N} \sum_{k=0}^{N-1} H(k) . \frac{1 - z^{-N}}{1 - e^{j\frac{2\pi}{N}k} . z^{-1}}$$

- It has similar problems to the windowing approach

## Linear-Phase FIR Filter Design by Optimisation

- Amplitude response for all 4 types of linear-phase FIR filters can be expressed as $\breve{H}(\omega) = Q(\omega)A(\omega)$

where

$$Q(\omega) = \begin{cases} 1, & \text{for Type 1} \\ \cos(\omega/2), & \text{for Type 2} \\ \sin(\omega), & \text{for Type 3} \\ \sin(\omega/2), & \text{for Type 4} \end{cases}$$



## Linear-Phase FIR Filter Design by Optimisation

- Modified form of weighted error function

$$\begin{aligned} \mathcal{E}(\omega) &= W(\omega)[Q(\omega)A(\omega) - D(\omega)] \\ &= W(\omega)Q(\omega)[A(\omega) - \tfrac{D(\omega)}{Q(\omega)}] \\ &= \tilde{W}(\omega)[A(\omega) - \tilde{D}(\omega)] \end{aligned}$$

where

$$\tilde{W}(\omega) = W(\omega)Q(\omega)$$

$$\tilde{D}(\omega) = D(\omega)/Q(\omega)$$

# Linear-Phase FIR Filter Design by Optimisation

- Optimisation Problem - Determine $\tilde{a}[k]$ which minimise the peak absolute value of $\mathcal{E}(\omega) = \tilde{W}(\omega)[\sum_{k=0}^{L} \tilde{a}[k]\cos(\omega k) - \tilde{D}(\omega)]$

  over the specified frequency bands $\omega \in R$
- After $\tilde{a}[k]$ has been determined, construct the original $A(e^{j\omega})$ and hence $h[n]$



# Linear-Phase FIR Filter Design by Optimisation

Solution is obtained via the Alternation Theorem

The optimal solution has equiripple behaviour consistent with the total number of available parameters.

Parks and McClellan used the Remez algorithm to develop a procedure for designing linear FIR digital filters.

# FIR Digital Filter Order Estimation

Kaiser's Formula:

$$N \cong \frac{-20\log_{10}(\sqrt{\delta_p\delta_s})}{14.6(\omega_s-\omega_p)/2\pi}$$

- ie $N$ is inversely proportional to transition band width and not on transition band location



# FIR Digital Filter Order Estimation

- Hermann-Rabiner-Chan's Formula:

$$N \cong \frac{D_\infty(\delta_p,\delta_s)-F(\delta_p,\delta_s)[(\omega_s-\omega_p)/2\pi]^2}{(\omega_s-\omega_p)/2\pi}$$

where

$$D_\infty(\delta_p,\delta_s)=[a_1(\log_{10}\delta_p)^2+a_2(\log_{10}\delta_p)+a_3]\log_{10}\delta_s$$
$$+[a_4(\log_{10}\delta_p)^2+a_5(\log_{10}\delta_p)+a_6]$$
$$F(\delta_p,\delta_s)=b_1+b_2[\log_{10}\delta_p-\log_{10}\delta_s]$$

with $a_1=0.005309,\ a_2=0.07114,\ a_3=-0.4761$

$a_4=0.00266,\ a_5=0.5941,\ a_6=0.4278$

$b_1=11.01217,\quad b_2=0.51244$

## FIR Digital Filter Order Estimation

- Formula valid for $\delta_p \geq \delta_s$
- For $\delta_p < \delta_s$ , formula to be used is obtained by interchanging $\delta_p$ and $\delta_s$
- Both formulae provide only an estimate of the required filter order *N*
- If specifications are not met, increase filter order until they are met



## FIR Digital Filter Order Estimation

- Fred Harris' guide:

$$N \cong \frac{A}{20(\omega_s - \omega_p)/2\pi}$$

where *A* is the attenuation in dB

- Then add about 10% to it

# MATLAB Resources

- Filter design toolbox (FIR and IIR)
- Functions
  - Filter
  - Filtfilt
  - Rand and randn
  - Xcorr
  - Freqz, invfreqz, fvtool
  - Poly, roots, conv, polyval,



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# IIR Digital Filter Design

Standard approach

(1) Convert the digital filter specifications into an analogue prototype lowpass filter specifications

(2) Determine the analogue lowpass filter transfer function $H_a(s)$

(3) Transform $H_a(s)$ by replacing the complex variable to the digital transfer function $G(z)$



# IIR Digital Filter Design

- This approach has been widely used for the following reasons:

  (1) Analogue approximation techniques are highly advanced

  (2) They usually yield closed-form solutions

  (3) Extensive tables are available for analogue filter design

  (4) Very often applications require digital simulation of analogue systems

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# IIR Digital Filter Design

- Let an analogue transfer function be

$$H_a(s) = \frac{P_a(s)}{D_a(s)}$$

where the subscript "*a*" indicates the analogue domain

- A digital transfer function derived from this is denoted as

$$G(z) = \frac{P(z)}{D(z)}$$



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# IIR Digital Filter Design

- Basic idea behind the conversion of $H_a(s)$ into $G(z)$ is to apply a mapping from the *s*-domain to the *z*-domain so that essential properties of the analogue frequency response are preserved
- Thus mapping function should be such that
  - Imaginary $(j\Omega)$ axis in the *s*-plane be mapped onto the unit circle of the *z*-plane
  - A stable analogue transfer function be mapped into a stable digital transfer function

# IIR Digital Filter: The bilinear transformation

- To obtain $G(z)$ replace $s$ by $f(z)$ in $H(s)$
- Start with requirements on $G(z)$

| $G(z)$ | Available $H(s)$ |
|---|---|
| Stable | Stable |
| Real and Rational in $z$ | Real and Rational in $s$ |
| Order $n$ | Order $n$ |
| L.P. (lowpass) cutoff $\Omega_c$ | L.P. cutoff $\omega_c T$ |



# IIR Digital Filter

- Hence $f(z)$ is real and rational in $z$ of order one
- i.e. $$f(z) = \frac{az+b}{cz+d}$$
- For LP to LP transformation we require

$$s = 0 \quad \rightarrow \quad z = 1 \quad f(1) = 0 \rightarrow a + b = 0$$
$$s = \pm j\infty \rightarrow \quad z = -1 \quad f(-1) = \pm j\infty \rightarrow c - d = 0$$

- Thus $$f(z) = \left(\frac{a}{c}\right) \cdot \frac{z-1}{z+1}$$

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# IIR Digital Filter

- The quantity $\left(\frac{a}{c}\right)$ is fixed from $\omega_c T \leftrightarrow \Omega_c$
- ie on $C : |z| = 1$ $\quad f(z)\big|_c = \left(\frac{a}{c}\right).j\tan\frac{\omega T}{2}$
- Or

$$j\Omega_c = \left(\frac{a}{c}\right).j\tan\frac{\omega_c T}{2}$$

- and

$$s = \left(\frac{\Omega_c}{\tan\left(\frac{\omega_c T}{2}\right)}\right).\frac{1-z^{-1}}{1+z^{-1}}$$



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# Bilinear Transformation

- Transformation is unaffected by scaling. Consider inverse transformation with scale factor equal to unity

$$z = \frac{1+s}{1-s}$$

- For

$$s = \sigma_o + j\Omega_o$$

$$z = \frac{(1+\sigma_o) + j\Omega_o}{(1-\sigma_o) - j\Omega_o} \Rightarrow |z|^2 = \frac{(1+\sigma_o)^2 + \Omega_o^2}{(1-\sigma_o)^2 + \Omega_o^2}$$

- and so

$$\sigma_o = 0 \rightarrow |z| = 1$$

$$\sigma_o < 0 \rightarrow |z| < 1$$

$$\sigma_o > 0 \rightarrow |z| > 1$$

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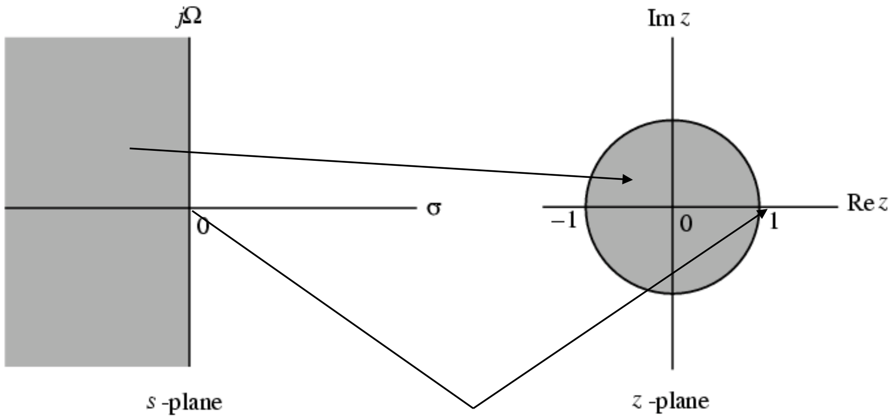
# Bilinear Transformation

- Mapping of *s*-plane into the *z*-plane



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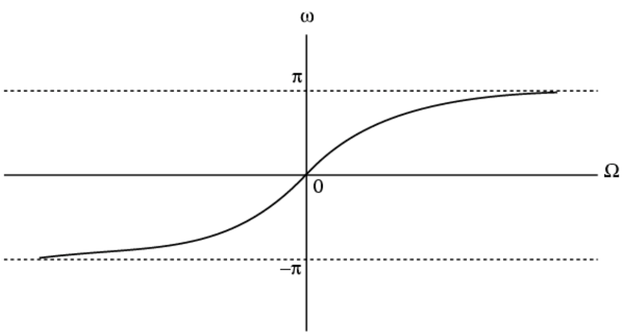
# Bilinear Transformation

- For $z = e^{j\omega}$ with *unity scalar* we have

$$j\Omega = \frac{1 - e^{-j\omega}}{1 + e^{-j\omega}} = j\tan(\omega/2)$$

or $$\Omega = \tan(\omega/2)$$



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# Bilinear Transformation

- Mapping is highly nonlinear
- Complete negative imaginary axis in the *s*-plane from $\Omega = -\infty$ to $\Omega = 0$ is mapped into the lower half of the unit circle in the *z*-plane from $z = -1$ to $z = 1$
- Complete positive imaginary axis in the *s*-plane from $\Omega = 0$ to $\Omega = \infty$ is mapped into the upper half of the unit circle in the *z*-plane from $z = 1$ to $z = -1$



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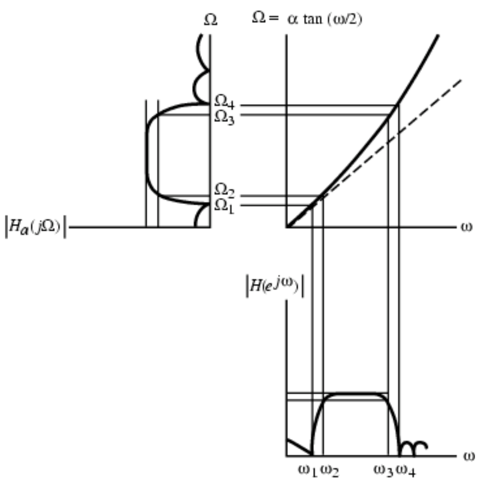
# Bilinear Transformation

- Nonlinear mapping introduces a distortion in the frequency axis called **frequency warping**
- Effect of warping shown below



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# Spectral Transformations

- To transform $G_L(z)$ a given lowpass transfer function to another transfer function $G_D(\hat{z})$ that may be a lowpass, highpass, bandpass or bandstop filter (solutions given by Constantinides)
- $z^{-1}$ has been used to denote the unit delay in the prototype lowpass filter $G_L(z)$ and $\hat{z}^{-1}$ to denote the unit delay in the transformed filter $G_D(\hat{z})$ to avoid confusion



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# Spectral Transformations

- Unit circles in *z*- and $\hat{z}$-planes defined by $z = e^{j\omega}$ , $\hat{z} = e^{j\hat{\omega}}$
- Transformation from *z*-domain to $\hat{z}$-domain given by
$$z = F(\hat{z})$$
- Then $G_D(\hat{z}) = G_L\{F(\hat{z})\}$

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# Spectral Transformations

- From $z = F(\hat{z})$ , thus $|z| = |F(\hat{z})|$ , hence

$$|F(\hat{z})| \begin{cases} >1, & \text{if } |z| > 1 \\ =1, & \text{if } |z| = 1 \\ <1, & \text{if } |z| < 1 \end{cases}$$

- Therefore $1/F(\hat{z})$ must be a stable allpass function

$$\frac{1}{F(\hat{z})} = \pm \prod_{\ell=1}^{L} \left( \frac{1 - \alpha_\ell^* \hat{z}}{\hat{z} - \alpha_\ell} \right), \quad |\alpha_\ell| < 1$$



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# Lowpass-to-Lowpass Spectral Transformation

- To transform a lowpass filter $G_L(z)$ with a cutoff frequency $\omega_c$ to another lowpass filter $G_D(\hat{z})$ with a cutoff frequency $\hat{\omega}_c$ , the transformation is

$$z^{-1} = \frac{1}{F(\hat{z})} = \frac{1 - \alpha \hat{z}}{\hat{z} - \alpha}$$

- On the unit circle we have

$$e^{-j\omega} = \frac{e^{-j\hat{\omega}} - \alpha}{1 - \alpha e^{-j\hat{\omega}}}$$

which yields

$$\tan(\omega/2) = \left( \frac{1+\alpha}{1-\alpha} \right) \tan(\hat{\omega}/2)$$

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# Lowpass-to-Lowpass Spectral Transformation

- Solving we get

$$\alpha = \frac{\sin\left((\omega_c - \hat{\omega}_c)/2\right)}{\sin\left((\omega_c + \hat{\omega}_c)/2\right)}$$

- Example - Consider the lowpass digital filter

$$G_L(z) = \frac{0.0662(1+z^{-1})^3}{(1-0.2593z^{-1})(1-0.6763z^{-1}+0.3917z^{-2})}$$

which has a passband from dc to $0.25\pi$ with a 0.5 dB ripple

- Redesign the above filter to move the passband edge to $0.35\pi$



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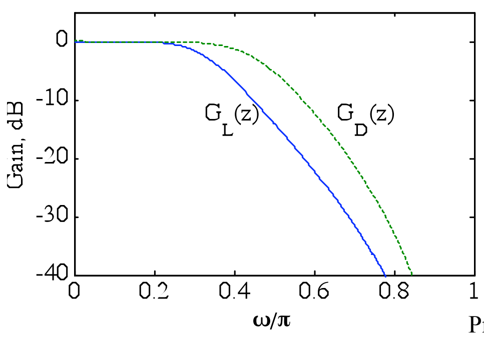
# Lowpass-to-Lowpass Spectral Transformation

- Here

$$\alpha = -\frac{\sin(0.05\pi)}{\sin(0.3\pi)} = -0.1934$$

- Hence, the desired lowpass transfer function is $G_D(\hat{z}) = G_L(z)\Big|_{z^{-1} = \frac{\hat{z}^{-1}+0.1934}{1+0.1934\hat{z}^{-1}}}$



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# Lowpass-to-Lowpass Spectral Transformation

- The lowpass-to-lowpass transformation

$$z^{-1} = \frac{1}{F(\hat{z})} = \frac{1-\alpha\,\hat{z}}{\hat{z}-\alpha}$$

can also be used as highpass-to-highpass, bandpass-to-bandpass and bandstop-to-bandstop transformations



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# Lowpass-to-Highpass Spectral Transformation

- Desired transformation

$$z^{-1} = -\frac{\hat{z}^{-1}+\alpha}{1+\alpha\,\hat{z}^{-1}}$$

- The transformation parameter $\alpha$ is given by

$$\alpha = -\frac{\cos\left((\omega_c+\hat{\omega}_c)/2\right)}{\cos\left((\omega_c-\hat{\omega}_c)/2\right)}$$

where $\omega_c$ is the cutoff frequency of the lowpass filter and $\hat{\omega}_c$ is the cutoff frequency of the desired highpass filter

# Lowpass-to-Highpass Spectral Transformation

- Example - Transform the lowpass filter

$$G_L(z) = \frac{0.0662(1+z^{-1})^3}{(1-0.2593\,z^{-1})(1-0.6763\,z^{-1}+0.3917z^{-2})}$$

- with a passband edge at $0.25\pi$ to a highpass filter with a passband edge at $0.55\pi$
- Here $\alpha = -\cos(0.4\pi)/\cos(0.15\pi) = -0.3468$
- The desired transformation is

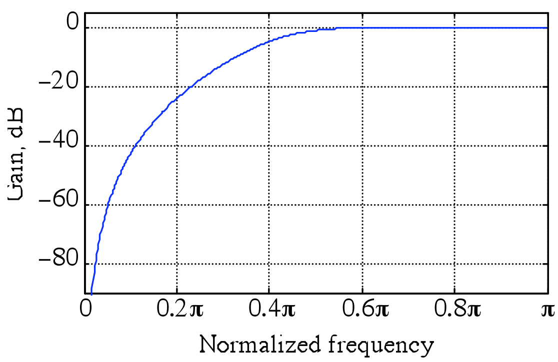
$$z^{-1} = -\frac{\hat{z}^{-1} - 0.3468}{1 - 0.3468\hat{z}^{-1}}$$



# Lowpass-to-Highpass Spectral Transformation

- The desired highpass filter is

$$G_D(\hat{z}) = G(z)\Big|_{z^{-1} = -\frac{\hat{z}^{-1}-0.3468}{1-0.3468\,\hat{z}^{-1}}}$$



# Lowpass-to-Highpass Spectral Transformation

- The lowpass-to-highpass transformation can also be used to transform a highpass filter with a cutoff at $\omega_c$ to a lowpass filter with a cutoff at $\hat{\omega}_c$
- and transform a bandpass filter with a center frequency at $\omega_o$ to a bandstop filter with a center frequency at $\hat{\omega}_o$



# Lowpass-to-Bandpass Spectral Transformation

- Desired transformation

$$z^{-1} = -\frac{\hat{z}^{-2} - \frac{2\alpha\beta}{\beta+1}\hat{z}^{-1} + \frac{\beta-1}{\beta+1}}{\frac{\beta-1}{\beta+1}\hat{z}^{-2} - \frac{2\alpha\beta}{\beta+1}\hat{z}^{-1} + 1}$$

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# Lowpass-to-Bandpass Spectral Transformation

- The parameters $\alpha$ and $\beta$ are given by

$$\alpha = \frac{\cos\left((\hat{\omega}_{c2} + \hat{\omega}_{c1})/2\right)}{\cos\left((\hat{\omega}_{c2} - \hat{\omega}_{c1})/2\right)}$$

$$\beta = \cot\left((\hat{\omega}_{c2} - \hat{\omega}_{c1})/2\right)\tan(\omega_c/2)$$

where $\omega_c$ is the cutoff frequency of the lowpass filter, and $\hat{\omega}_{c1}$ and $\hat{\omega}_{c2}$ are the desired upper and lower cutoff frequencies of the bandpass filter



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# Lowpass-to-Bandpass Spectral Transformation

- Special Case - The transformation can be simplified if $\omega_c = \hat{\omega}_{c2} - \hat{\omega}_{c1}$
- Then the transformation reduces to

$$z^{-1} = -\hat{z}^{-1}\frac{\hat{z}^{-1} - \alpha}{1 - \alpha\,\hat{z}^{-1}}$$

where $\alpha = \cos\hat{\omega}_o$ with $\hat{\omega}_o$ denoting the desired center frequency of the bandpass filter

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# Lowpass-to-Bandstop Spectral Transformation

- Desired transformation

$$z^{-1} = \frac{\hat{z}^{-2} - \frac{2\alpha\beta}{1+\beta}\hat{z}^{-1} + \frac{1-\beta}{1+\beta}}{\frac{1-\beta}{1+\beta}\hat{z}^{-2} - \frac{2\alpha\beta}{1+\beta}\hat{z}^{-1} + 1}$$



AGC DSP

# Lowpass-to-Bandstop Spectral Transformation

- The parameters $\alpha$ and $\beta$ are given by

$$\alpha = \frac{\cos\left((\hat{\omega}_{c2} + \hat{\omega}_{c1})/2\right)}{\cos\left((\hat{\omega}_{c2} - \hat{\omega}_{c1})/2\right)}$$

$$\beta = \tan\left((\hat{\omega}_{c2} - \hat{\omega}_{c1})/2\right)\tan(\omega_c/2)$$

where $\omega_c$ is the cutoff frequency of the lowpass filter, and $\hat{\omega}_{c1}$ and $\hat{\omega}_{c2}$ are the desired upper and lower cutoff frequencies of the bandstop filter

# Digital Filters

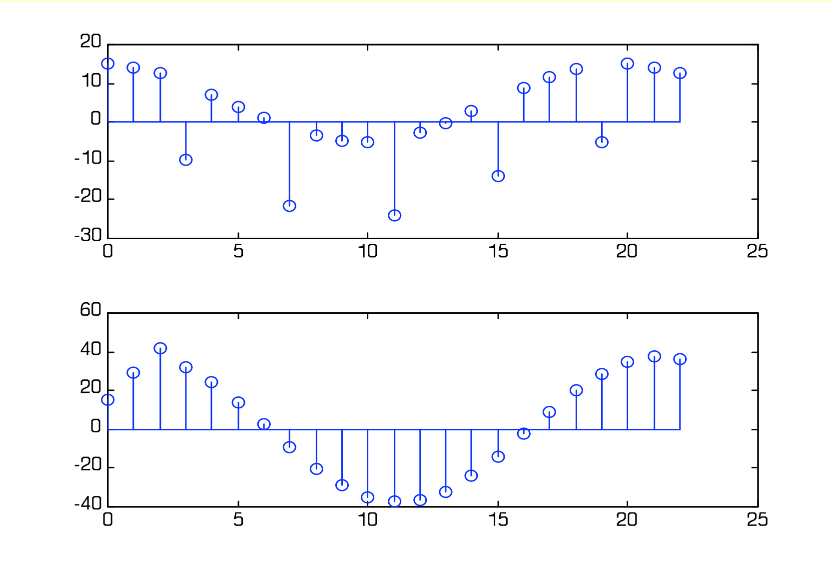
- Filtering operation

| Time k | Given signal | OPERATION | | |
|---|---|---|---|---|
| 0 | | | | |
| 1 | 4.0 | | 14.0 | |
| 2 | 1.1 | ADD | 2.5 | |
| 3 | -21.6 | 4.0 + 1.1 - 21.6 - 3.6 = | -9.3 | |
| 4 | -3.6 | | -20.1 | |
| 5 | -4.7 | | -28.8 | |



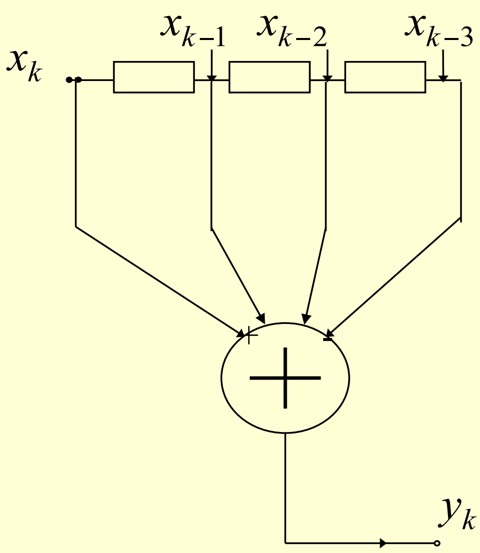

# Digital Filters

Filtering



# Digital Filters

Filtering








# Digital Filters

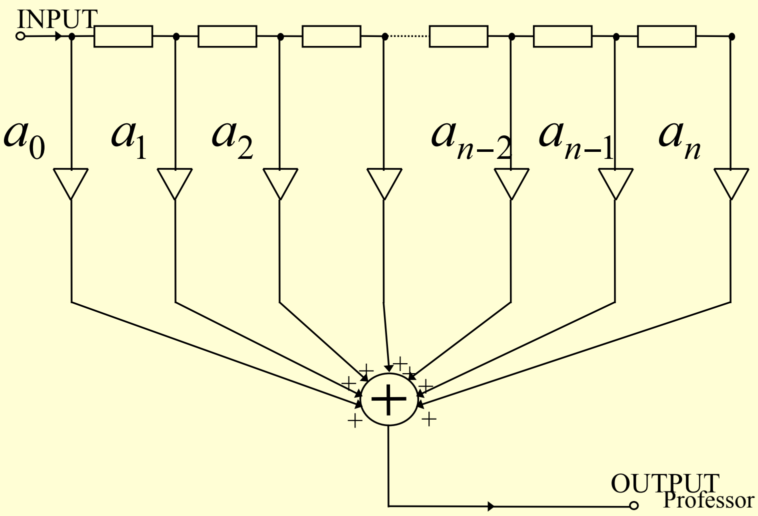
Filtering $y_k = x_k + x_{k-1} + x_{k-2} + x_{k-3}$

- Basic operations required
- (a) Delay
- (b) Addition
- (c) Multiplication (Scaling)

# Digital Filters

Filtering: More general operation

$$y_k = a_0 x_k + a_1 x_{k-1} + a_2 x_{k-2} + .... + a_n x_{k-n}$$






# Digital Filters

Impulse response

$$\{a_0,\ a_1,\ a_2,\ a_3,\ ....\ ,\ a_n,\ 0,\ 0,\ ....\}$$

- Most general linear form

$$y_k = a_0 x_k + a_1 x_{k-1} + .... + a_n x_{k-n}$$
$$- b_1 y_{k-1} - b_2 y_{k-2} - .... - b_m y_{k-m}$$

- Recursive or Infinite Impulse Response (IIR) filters

# Digital Filters

A simple first order $y_k = x_k - by_{k-1}$

$$k = 0 \quad y_0 = 1$$

$$k = 1 \quad y_1 = 0 - b.1 = -b$$

$$k = 2 \quad y_2 = 0 - b(-b) = b^2$$

$$k = n \quad y_n = (-b)^n$$





# Digital Filters

Transfer function

$$G(z) = \frac{\text{Output z - transform } Y(z)}{\text{Input z} - \text{transform } X(z)}$$

- For FIR $y_k = a_0 x_k + a_1 x_{k-1} + a_2 x_{k-2} + .... + a_n x_{k-n}$

$$Y(z) = a_0 X(z) + a_1 z^{-1} X(z) + .... + a_n z^{-n} X(z)$$

$$G(z) = a_0 + a_1 z^{-1} + .... + a_n z^{-n}$$

# Digital Filters-Stability

For IIR
$$G(z) = \frac{a_0 + a_1 z^{-1} + \ldots + a_n z^{-n}}{1 + b_1 z^{-1} + b_2 z^{-2} + \ldots + b_m z^{-m}}$$
$$= \sum_i \frac{A_i . z}{z - \alpha_i}$$

- Stability: Note that $y_k = x_k - b y_{k-1} \rightarrow (-b)^k$

$$Y(z) = X(z) - bz^{-1} Y(z)$$
$$\frac{Y(z)}{X(z)} = \frac{1}{1 + bz^{-1}} = \frac{z}{z + b}$$





# Digital Filters

Thus there is a pole at $z = -a$

- if its magnitude is more than 1 then the impulse response increases without bound
- if its magnitude is less than 1 decreases exponentially to zero
- Frequency Response: Set $x_k = e^{j\omega kT}$ and $y_k = A(\omega) e^{j\omega kT}$

# Digital Filters

So that

$$A(\omega)e^{j\omega kT} = a_0 e^{j\omega kT} + a_1 e^{j\omega(k-1)T} + .... + a_n e^{j\omega(k-n)T}$$

$$- b_1 A(\omega)e^{j\omega(k-1)T} - .... - b_m A(\omega)e^{j\omega(k-m)T}$$

- And hence

$$A(\omega) = \frac{a_0 + a_1 e^{-j\omega T} + .... + a_n e^{-j\omega nT}}{1 + b_1 e^{-j\omega T} + .... + b_m e^{-j\omega mT}}$$

- Compare with transfer function

$$z^{-1} \Rightarrow \exp(-j\omega T)$$





# Digital Filters

In the initial example $y_k = x_k + x_{k-1} + x_{k-2} + x_{k-3}$

or

$$\frac{Y(z)}{X(z)} = 1 + z^{-1} + z^{-2} + z^{-3}$$

- And thus

$$A(\omega) = 1 + e^{-j\omega T} + e^{-j2\omega T} + e^{-j3\omega T}$$

$$A(\omega) = e^{-j\frac{3\omega T}{2}}\left[e^{j\frac{3\omega T}{2}} + e^{j\frac{\omega T}{2}} + e^{-j\frac{\omega T}{2}} + e^{-j\frac{3\omega T}{2}}\right]$$

$$|A(\omega)| = \left|2\left(\cos\frac{3\omega T}{2} + \cos\frac{\omega T}{2}\right)\right| \qquad \phi(\omega) = -\frac{3\omega T}{2}$$

# 2-D z tranform

2-D z-transform

$$X(z_1,z_2)=\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}x(n_1,n_2)z_1^{-n_1}.z_2^{-n_2}$$

- Example : $\{x(n_1,n_2)\}=\{\alpha_1^{k_1}.\alpha_2^{k_2}$

$$X(z_1,z_2)=\sum_{k_1}\sum_{k_2}\alpha_1^{k_1}\alpha_2^{k_2}.z_1^{-k_1}z_2^{-k_2}$$

$$=\sum_{k_1}\alpha_1^{k_1}z_1^{-k_1}\sum_{k_2}\alpha_2^{k_2}z_2^{-k_2}$$





# 2-D z tranform

- And hence

$$X(z_1,z_2)=\left(\frac{1}{1-\alpha_1 z_1^{-1}}\right)\cdot\left(\frac{1}{1-\alpha_2 z_2^{-1}}\right)$$

(i) Separable transforms.

(ii) Non-separable transforms.

# 2-D Digital Filters

2-D filters

$$y(k_1,k_2) = \sum_{n_1=0}^{N_1} \sum_{n_2=0}^{N_2} a(n_1,n_2)x(k_1-n_1,k_2-n_2)$$

$$-\sum_{n_1=0}^{M_1} \sum_{\substack{n_2=0 \\ n_1+n_2\neq 0}}^{M_2} b(n_1,n_2).y(k_1-n_1,k_2-n_2)$$

- Thus we can have
- (a) FIR 2-D filters and
- (b) IIR 2-D filters



# 2-D Digital Filters

Transfer function

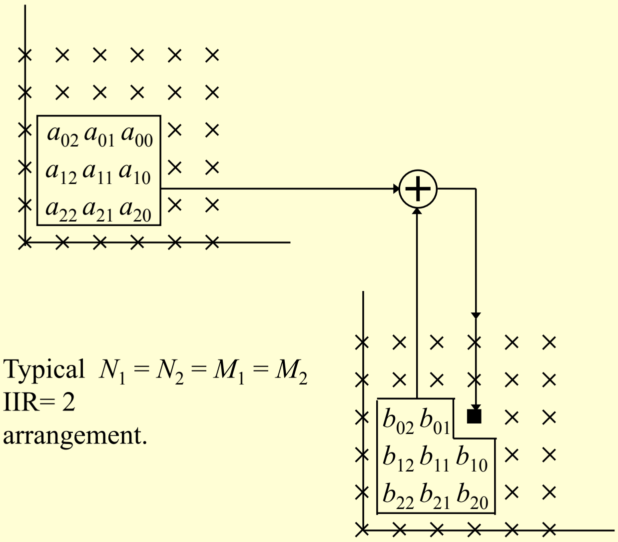
$$G(z_1,z_2) = \frac{\sum_{n_1=0}^{N_1} \sum_{n_2=0}^{N_2} a(n_1,n_2).z_1^{-n_1}.z_2^{-n_2}}{1+\sum_{n_1=0}^{M_1} \sum_{n_2=0}^{M_2} b(n_1,n_2).z_1^{-n_1}.z_2^{-n_2}}$$

- For convolution set $Y(z_1,z_2) = G(z_1,z_2).X(z_1,z_2)$

$$y(k_1,k_2) = \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} g(n_1,n_2).x(k_1-n_1,k_2-n_2)$$

# 2-D Digital Filters

Filtering



Typical $N_1 = N_2 = M_1 = M_2$
IIR= 2
arrangement.





# 2-D Digital Filters

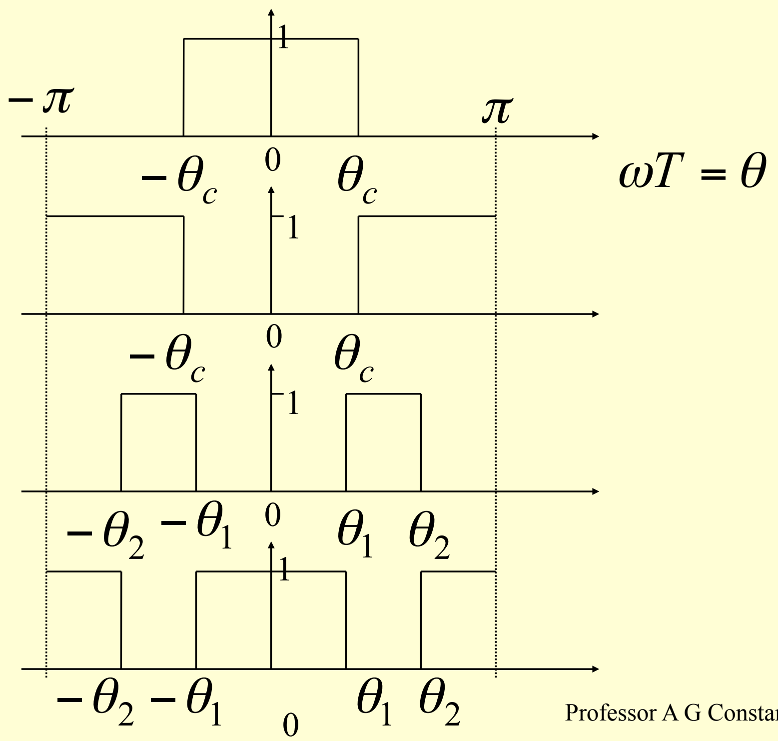
(a) Separable filters

$$G(z_1, z_2) = G_1(z_1).G_2(z_2)$$

(b) Non-separable filters $G(z_1, z_2)$

is not expressible as a product of separate and independent factors

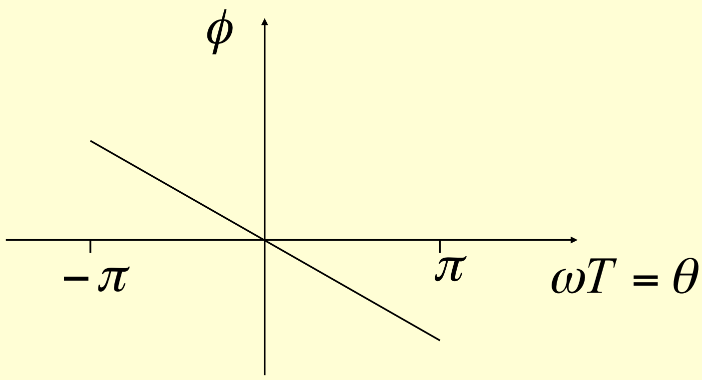
# Ideal filters








# Ideal filters



# Z - transform

- Defined as power series

$$F(z) = \sum_{k=0}^{\infty} f_k . z^{-k}$$

$$F(z) = f_0 + f_1 z^{-1} + f_2 z^{-2} + \ldots. + f_r z^{-r} + \ldots\ldots$$

- Examples:

$$\{f_k\} = \{1\} \quad F(z) = 1 + 1.z^{-1} + 1.z^{-2} + 1.z^{-3} + \ldots.$$




# Z - transform

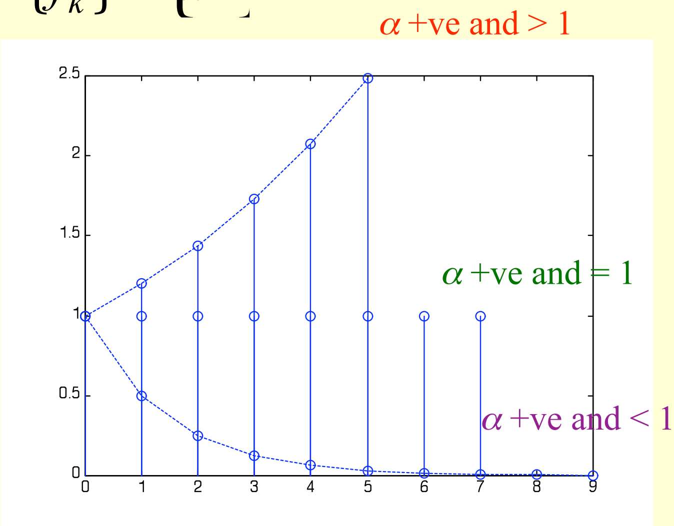
- And since

$$z^{-1}F(z) = z^{-1} + z^{-2} + z^{-3} + \ldots.$$

- We get $\quad F(z) = \dfrac{1}{1 - z^{-1}}$

# Z - transform

- Define $\{f_k\} = \{\alpha^k\}$








# Z - transform

- We have $F(z) = \sum_{k=0}^{\infty} \alpha^k z^{-k}$

$$= 1 + (\alpha \ z^{-1}) + (\alpha \ z^{-1})^2 + (\alpha \ z^{-1})^3 + \ldots.$$

- ie $F(z) = \dfrac{1}{1-\alpha z^{-1}} = \dfrac{z}{z-\alpha} \qquad |\alpha z^{-1}| < 1$

- Note that $F(z)$ has a pole at $z = \alpha$ on the z-plane.

# Z - transform

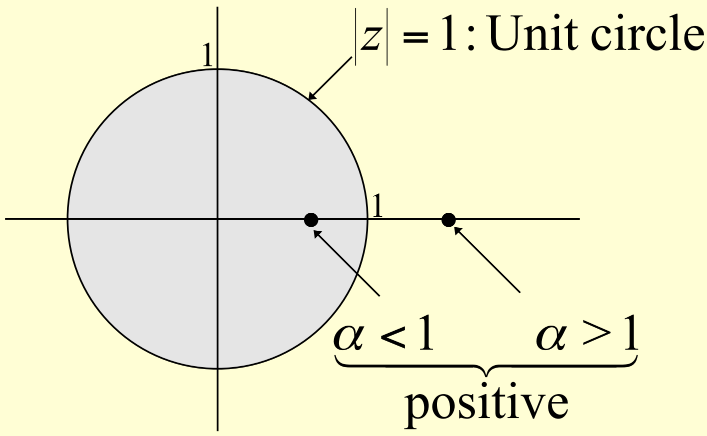
Note:

- (i) If magnitude of pole is $> 1$ then $\{f_k\}$ increases without bound
- (ii) If magnitude of pole is $< 1$ then $\{f_k\}$ has a bounded variation
- i.e. the contour $|z| = 1$ on the z-plane is of crucial significance.
- It is called the Unit circle





# The unit circle



# Z – transform properties

(i) Linearity

- The z-transform operation is linear
- $\mathcal{Z}\left[c_1\{f_k^{(1)}\}+c_2\{f_k^{(2)}\}\right]=c_1F_1(z)+c_2F_2(z)$
- Where $F_i(z)=\mathcal{Z}\left[\{f_k^{(i)}\}\right]$, $i = 1, 2$

(ii) Shift Theorem

$$\mathcal{Z}\left[\{f_{k-m}\}\right]=z^{-m}F(z)$$





# Z - transform

- Let $F(z)=f_0+f_1z^{-1}+f_2z^{-2}+\ldots.$
- $\mathcal{Z}\left[\{f_{k-m}\}\right]=f_{-m}+f_{-m+1}.z^{-1}+f_{-m+2}.z^{-2}+\ldots.$

$$\ldots.+f_0z^{-m}+f_1z^{-m-1}+f_2z^{-m-2}+\ldots.$$

- But $f_i\equiv 0$ for negative $i$.

# Z - transform

Examples:

- (i) Consider generation of new discrete time signal from $\{f_k\}$ via $\{g_k\}$

$$g_k = f_k - f_{k-1}$$

- Recall linearity and shift

$$G(z) = (1 - z^{-1})F(z)$$

- (ii) $\mathcal{Z}[\{\cos\omega_0 kT\}]$ write

$$\cos\omega_0 kT = \frac{e^{j\omega_0 kT}}{2} + \frac{e^{-j\omega_0 kT}}{2}$$





# Z - transform

- From $[\{\cos\omega_0 kT\}] = \mathrm{Re}[\{e^{j\omega_0 kT}\}]$
- With $e^{j\omega_0 T} = \alpha$ from earlier result

$$\mathcal{Z}[\{\alpha^k\}] = \frac{1}{1 - \alpha z^{-1}}$$

- We obtain $\mathcal{Z}[\{\cos\omega_0 kT\}] = \mathrm{Re}\frac{1}{1 - e^{j\omega_0 T}.z^{-1}}$

$$= \frac{1 - (\cos\omega_0 T).z^{-1}}{1 - 2(\cos\omega_0 T).z^{-1} + z^{-2}}$$

# Inverse Z - transform

- Given $F(z)$ to determine $\{f_k\}$ .
- Basic relationship is

$$F(z) = f_0 + f_1 z^{-1} + f_2 z^{-2} + f_3 z^{-3} + \ldots.$$

- $\{f_k\}$ may be obtained by power series expansion. It suffers from cumulative errors





# Inverse Z - transform

- Alternatively

$$z^{n-1}F(z) = z^{n-1}f_0 + z^{n-2}f_1 + z^{n-3}f_2 + \ldots.$$

$$\ldots + f_n z^{-1} + \ldots$$

- Use $\oint_\Gamma z^m dz = 2\pi j$ for $m$ = -1
- $= 0$ otherwise
- where closed contour $\Gamma$ encloses origin

# Inverse Z - transform

- Integrate to yield

$$\frac{1}{2\pi j}\oint_{\Gamma} z^n F(z).\frac{dz}{z} = f_n$$

Examples

- (i) $F(z) = \dfrac{1}{1-0.5z^{-1}}$

write

$$F(z) = \frac{z}{z-0.5}$$





# Inverse Z - transform

- And hence $f_k = \dfrac{1}{2\pi j}\oint_{\Gamma}\dfrac{z}{z-0.5}.z^k.\dfrac{dz}{z}$

Pole at $z = 0.5$ of Residue $(0.5)^k$

(ii) Let $Y(z) = G(z).X(z)$

where $G(z) = \mathcal{Z}[\{g_k\}]$

and $X(z) = \mathcal{Z}[\{x_k\}]$

- To determine $\{y_k\} = \mathcal{Z}^{-1}[Y(z)]$

## Inverse Z - transform

- From inversion formula

$$y_k = \frac{1}{2\pi j} \oint_{\Gamma} Y(z).z^k.\frac{dz}{z}$$

$$y_k = \frac{1}{2\pi j} \oint_{\Gamma} G(z).X(z).z^k.\frac{dz}{z}$$

- But $G(z) = \sum_{n=0}^{\infty} g_n z^{-n}$



## Inverse Z - transform

- Hence $y_k = \frac{1}{2\pi j} \oint_{\Gamma} \left( \sum_{n=0}^{\infty} g_n z^{-n} \right) X(z).z^k.\frac{dz}{z}$

$$y_k = \sum_{n=0}^{\infty} g_n.\underbrace{\frac{1}{2\pi j} \oint_{\Gamma} X(z).z^{k-n}.\frac{dz}{z}}_{x_{k-n}}$$

- Thus $y_k = \sum_{n=0}^{\infty} g_n.x_{k-n}$

# Inverse Z - transform

Note:

- (i) For causal signals $x_i \equiv 0$ for negative $i$. Thus upper convolution summation limit is in this case equal to $k$.
- (ii) Frequency representation of a discrete-time signal is obtained from its z-transform by replacing $z^{-1} \to e^{-j\omega T}$ where $T$ is the sampling period of interest. (Justification will be given later.)