Digital Signal Processing & Digital Filters

An Introductory Course

D. Mandic and L. Arnaut
d.mandic@imperial.ac.uk, Luk.Arnaut@npl.co.uk

URL: www.commsp.ee.ic.ac.uk/~mandic

(based on the notes by Prof. A G Constantinides)

Digital Signal Processing & Digital Filters

Contents

Digital Signal Processing & Digital Filters

BOOKS

- **Main Course text books:** Digital Signal Processing: A computer Based Approach, S K Mitra, McGraw Hill
- Mathematical Methods and Algorithms for Signal Processing, Todd Moon, Addison Wesley

- **Other books:**
- Digital Signal Processing, Roberts & Mullis, Addison Wesley
- Digital Filters, Antoniou, McGraw Hill

DIGITAL FILTERS

Analogue Vs Digital Signal Processing

Reliability:

Analogue system performance degrades due to:

- Long term drift (ageing)
- Short term drift (temperature?)
- Sensitivity to voltage instability.
- Batch-to-Batch component variation.
- High discrete component count

  Interconnection failures

DIGITAL FILTERS

Digital Systems:

- No short or long term drifts.
- Relative immunity to minor power supply variations.
- Virtually identical components.
- IC's have > 15 year lifetime
- Development costs
- System changes at design/development stage only software changes.
- Digital system simulation is realistic.

Power aspects

- Size, Dissipation
- DSP chips available as well as ASIC/FPGA realisations

Applications

Radar systems & Sonar systems

- Doppler filters.
- Clutter Suppression.
- Matched filters.
- Target tracking.
- Identification

DIGITAL FILTERS

Image Processing

- Image data compression.
- Image filtering.
- Image enhancement.
- Spectral Analysis.
- Scene Analysis / Pattern recognition.

DIGITAL FILTERS

Biomedical Signal Analysis

- Spatial image enhancement. (X-rays)
- Spectral Analysis.
- 3-D reconstruction from projections.
- Digital filtering and Data compression.

DIGITAL FILTERS

Music

- Music recording.
- Multi-track “mixing”.
- CD and DAT.
- Filtering / Synthesis / Special effects.

DIGITAL FILTERS

Seismic Signal Analysis

- Bandpass Filtering for S/N improvement.
- Predictive deconvolution to extract reverberation characteristics.
- Optimal filtering. (Wiener and Kalman.)

DIGITAL FILTERS

Telecommunications and Consumer Products

These are the largest and most pervasive applications of DSP and Digital Filtering

- Mobile Communications
- Digital Recording
- Digital Cameras
- Blue Tooth or similar

Background:

Discrete-Time Signals - Time-Domain Representation

Based on notes by Dr Tania Stathaki

What is a signal ?

A signal is a function of an independent variable such as time, distance, position, temperature, pressure, etc.

For example…

- **Electrical Engineering**
  voltages/currents in a circuit
  speech signals
  image signals
- **Physics**
  radiation
- **Mechanical Engineering**
  vibration studies
- **Astronomy**
  space photos

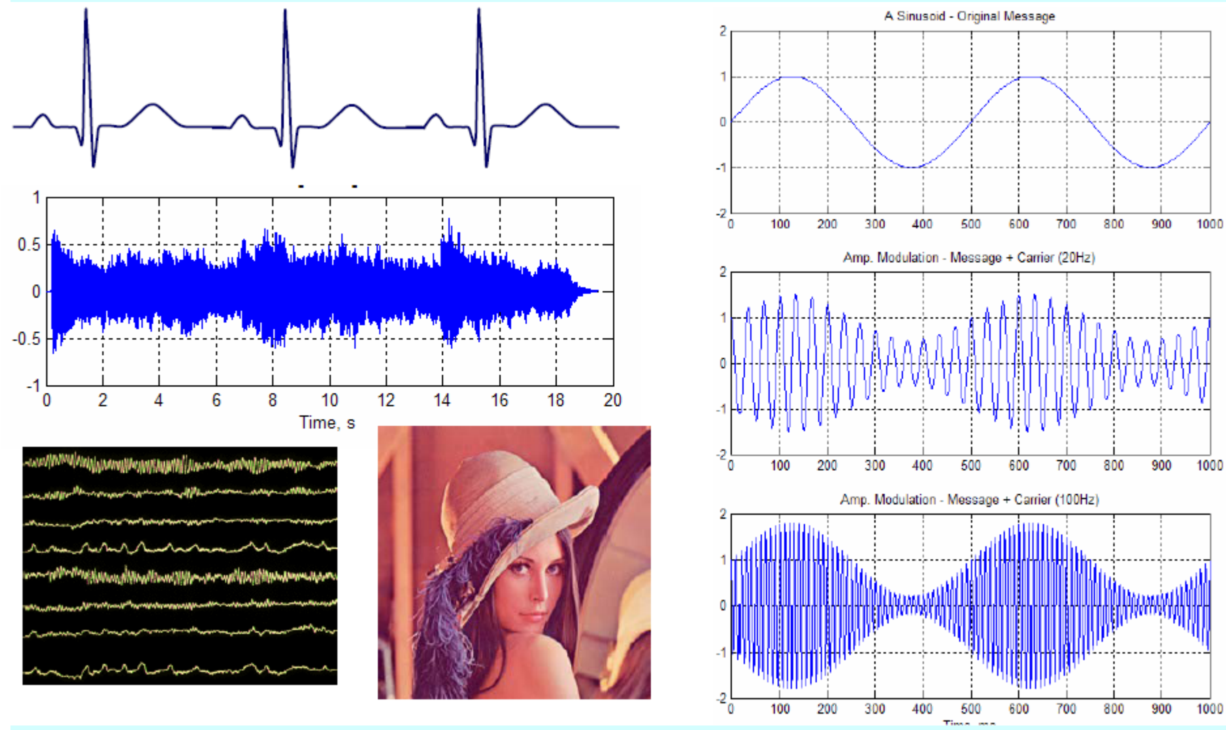
or

- **Biomedicine**

  EEG, ECG, MRI, X-Rays, Ultrasounds
- **Seismology**

  tectonic plate movement, earthquake prediction
- **Economics**

  stock market data



What is DSP?

Mathematical and algorithmic manipulation of **discretized and quantized** or **naturally digital** signals in order to extract the most relevant and pertinent information that is carried by the signal.

What is a signal?
What is a system?
What is processing?

Signals can be characterized in several ways

Continuous time signals vs. discrete time signals ($x(t)$, $x[n]$).

Temperature in London / signal on a CD-ROM.

Continuous valued signals vs. discrete signals.

Amount of current drawn by a device / average scores of TOEFL in a school over years.

–Continuous time and continuous valued : **Analog signal.**

–Continuous time and discrete valued: **Quantized signal.**

–Discrete time and continuous valued: **Sampled signal.**

–Discrete time and discrete values: **Digital signal.**

Real valued signals vs. complex valued signals.

Resident use electric power / industrial use reactive power.

Scalar signals vs. vector valued (multi-channel) signals.

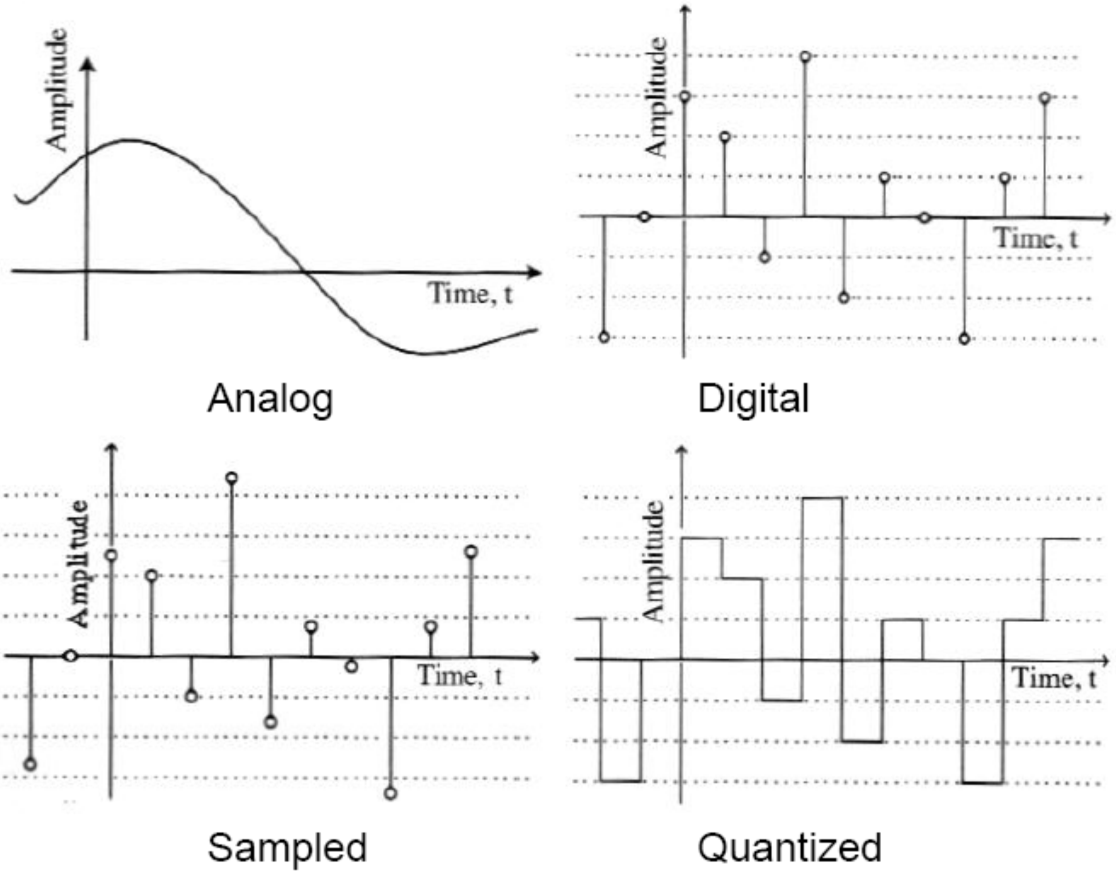
Blood pressure signal / 128 channel EEG.

Deterministic vs. random signal:

Recorded audio / noise.

One-dimensional vs. two dimensional vs. multidimensional signals.

Speech / still image / video.



Systems

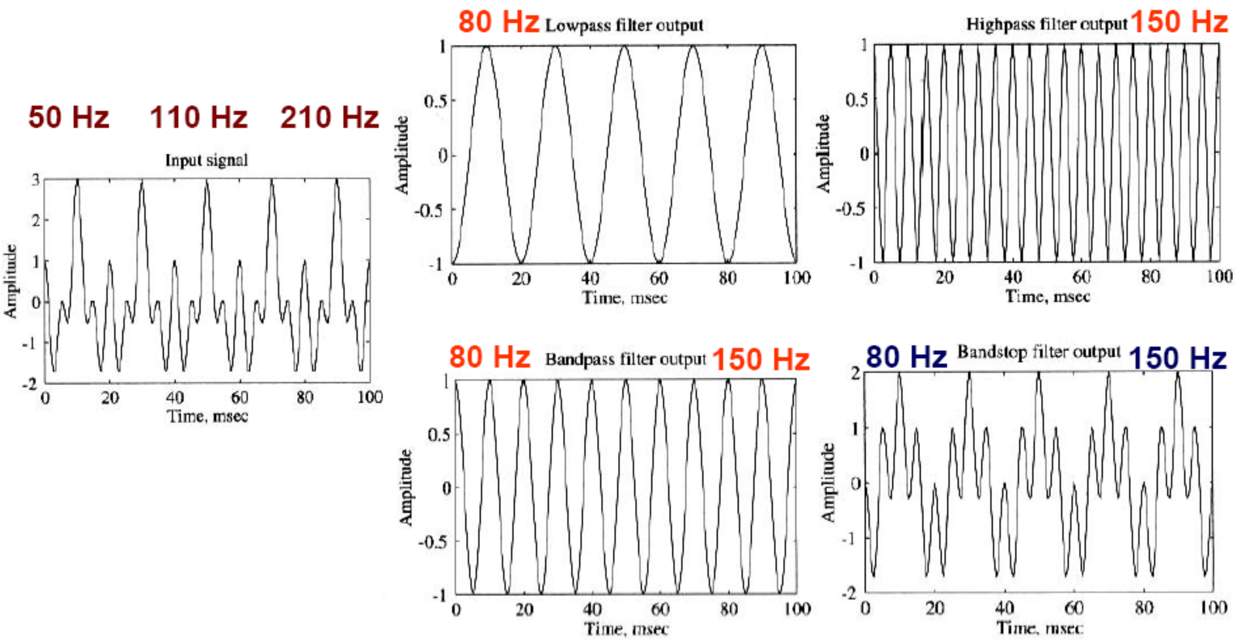
- For our purposes, a DSP system is one that can ***mathematically manipulate (e.g., change, record, transmit, transform) digital signals***.
- Furthermore, we are not interested in processing analog signals either, even though most signals in nature are analog signals.

Filtering

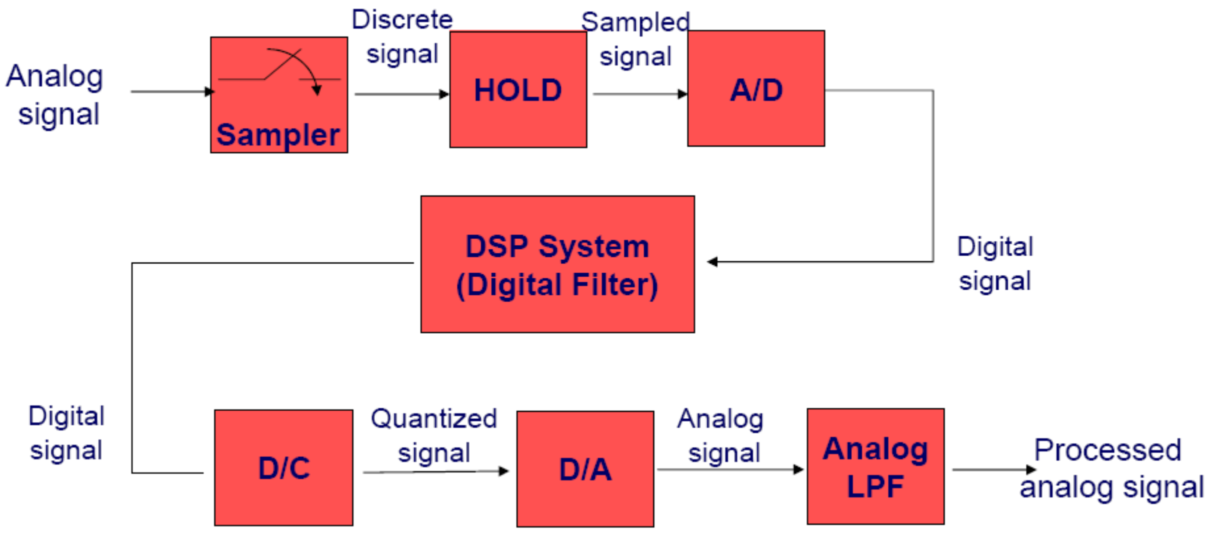
- **By far the most commonly used DSP operation**
  Filtering refers to deliberately changing the frequency content of the signal, typically, by removing certain frequencies from the signals.
  For de-noising applications, the (frequency) filter removes those frequencies in the signal that correspond to noise.
  In various applications, filtering is used to focus to that part of the spectrum that is of interest, that is, the part that carries the information.

- **Typically we have the following types of filters**
  Low-pass (LPF) –removes high frequencies, and retains (passes) low frequencies.
  High-pass (HPF) –removes low frequencies, and retains high frequencies.
  Band-pass (BPF) –retains an interval of frequencies within a band, removes others.
  Band-stop (BSF) –removes an interval of frequencies within a band, retains others.
  Notch filter –removes a specific frequency.

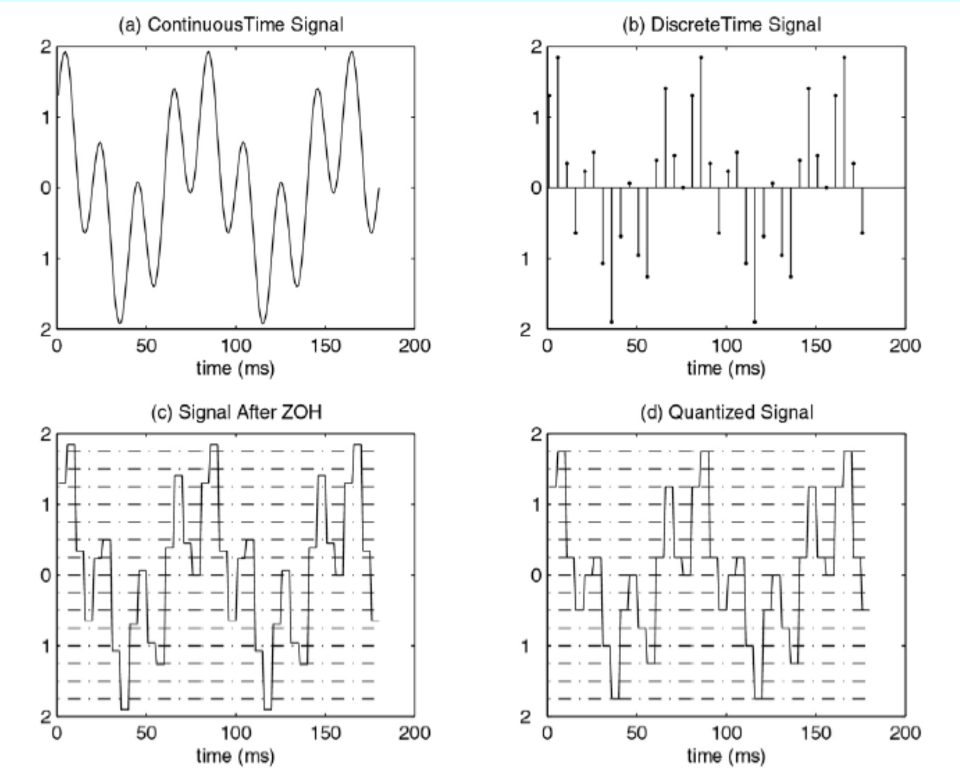
A Common Application: Filtering



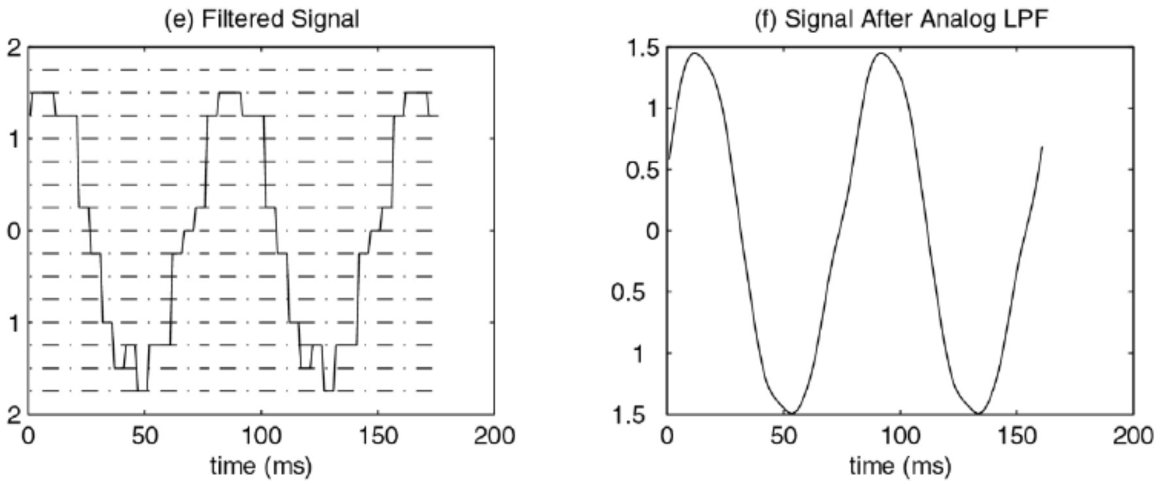
Components of a DSP System



Components of a DSP System



Components of a DSP System



Analog-to-Digital-to-Analog…?

- Why not just process the signals in continuous time domain? Isn't it just a waste of time, money and resources to convert to digital and back to analog?

- Why DSP? We digitally process the signals in discrete domain, because it is
  – More flexible, more accurate, easier to mass produce.
  – Easier to design.
    - System characteristics can easily be changed by programming.
    - Any level of accuracy can be obtained by use of appropriate number of bits.
  – More deterministic and reproducible-less sensitive to component values, etc.
  – Many things that cannot be done using analog processors can be done digitally.
    - Allows multiplexing, time sharing, multi-channel processing, adaptive filtering.
    - Easy to cascade, no loading effects, signals can be stored indefinitely w/o loss.
    - Allows processing of very low frequency signals, which requires unpractical component values in analog world.

Analog-to-Digital-to-Analog…?

- **On the other hand, it can be**
  - **Slower, sampling issues.**
  - **More expensive, increased system complexity,**

    consumes more power.
- **Yet, the advantages far outweigh the disadvantages. Today, most continuous time signals are in fact processed in discrete time using digital signal processors.**

Analog-Digital

Examples of analog technology

- photocopiers
- telephones
- audio tapes
- televisions (intensity and color info per scan line)
- VCRs (same as TV)

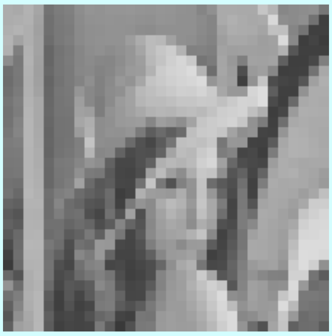
Examples of digital technology

- Digital computers!

2 Dimensions
From Continuous to Discrete: Sampling

256x256

64x64



Discrete (Sampled) and Digital (Quantized) Image

256x256 256 levels

256x256 32 levels

Discrete (Sampled) and Digital (Quantized) Image

256x256 256 levels

256x256 2 levels

Sampling Rate Alteration

- Employed to generate a new sequence $y[n]$ with a sampling rate F_T' higher or lower than that of the sampling rate F_T of a given sequence $x[n]$
- **Sampling rate alteration ratio** is $R = \frac{F_T'}{F_T}$
- If $R > 1$, the process called **interpolation**
- If $R < 1$, the process called **decimation**

Sampling Rate Alteration

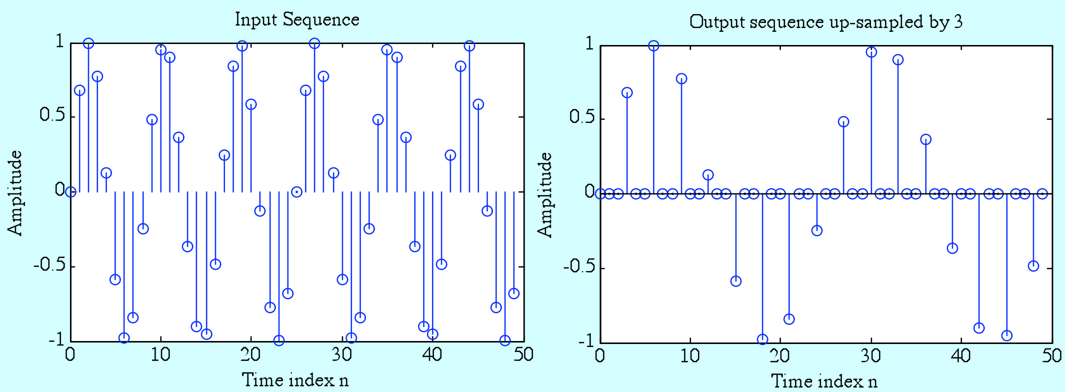
- In **up-sampling** by an integer factor $L > 1$, $L-1$ equidistant zero-valued samples are inserted by the **up-sampler** between each two consecutive samples of the input sequence $x[n]$:

$$x_u[n] = \begin{cases} x[n/L], & n = 0, \pm L, \pm 2L, \cdots \\ 0, & \text{otherwise} \end{cases}$$

$$x[n] \longrightarrow \boxed{\uparrow L} \longrightarrow x_u[n]$$

Sampling Rate Alteration

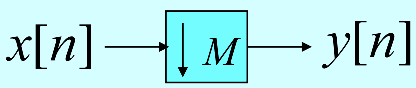
- An example of the up-sampling operation



Sampling Rate Alteration

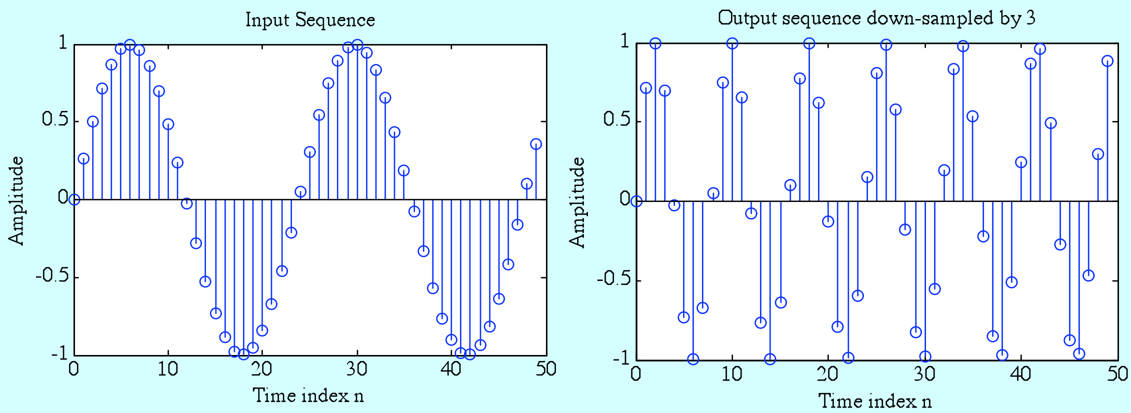
- In **down-sampling** by an integer factor $M > 1$, every M-th samples of the input sequence are kept and $M - 1$ in-between samples are removed:

$$y[n] = x[nM]$$



Sampling Rate Alteration

- An example of the down-sampling operation



BACKGROUND: Signal Spaces

- The purpose of this part of the course is to introduce the basic concepts behind generalised Fourier Analysis
- The approach is taken via vector spaces and least squares approximation
- Modern Signal Processing is based in a substantial way on the theory of vector spaces. In this course we shall be concerned with the discrete–time case only

Signal Spaces

- In order to compare two signals we need to define a measure of "distance" known as a metric.
- A metric is a function $d(x,y)$ that produces a scalar value from two signals such that
  - 1) $d(x,y)=d(y,x)$
  - 2) $d(y,x)\geq 0$
  - 3) $d(x,x)=0$
  - 4) $d(x,z)\leq d(x,y)+d(y,z)$

Signal Spaces

- There are many metrics that can be used.
- Examples:
- 1) If we have a set of finite numbers representing signals then a metric may be

$$d_1(x,y) = \sum_{i=1}^{n} |x_i - y_i|$$

- This is known as the l_1 metric or the Manhattan distance.

Signal Spaces

- 2) Another metric for the same set is

$$d_2(x,y) = \left[\sum_{i=1}^{n} (x_i - y_i)^2 \right]^{1/2}$$

This is called the l_2 metric or the Euclidean distance

- 3) Yet another form is the l_p metric

$$d_p(x,y) = \left[\sum_{i=1}^{n} (x_i - y_i)^p \right]^{1/p}$$

Signal Spaces

- 4) As the integer $p \to \infty$ the last form becomes

$$d_\infty(x, y) = \max_{i=1,2,\ldots,n} |x_i - y_i|$$

This is called the l_∞ metric or distance.

- In channel coding we use the Hamming Distance as a metric

$$d_H(x, y) = \sum_{i=1}^{n} (x_i \oplus y_i)$$

where $\oplus$ is the modulo-2 addition of two binary vectors

Signal Spaces

- When the set of vectors which we use is defined along with an appropriate metric then we say we have a metric space.
- There are many metric spaces as seen from the earlier discussion on metrics.
- (In the case of continuous time signals we say we have function spaces)

Vector Spaces

- We think of a vector $\mathbf{x}$ as an assembly of elements arranged as x_i $i = 1,2,3,4,...$

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ . \\ x_n \end{bmatrix}$$

- The length may be in certain cases infinite

Vector Spaces

- A **linear vector space** *S* over a set of scalars *R* is formed when a collection of vectors is taken together with an addition operation and a scalar multiplication, and satisfies the following:
- 1) *S* is a group under addition ie the following are satisfied
  - a) for any $\mathbf{x}$ and $\mathbf{y}$ in *S* then $\mathbf{x} + \mathbf{y}$ is also in *S*
  - b) there is a zero identity element ie
  - c) for every element there is another such that their sum is zero

$$\mathbf{x} + \mathbf{0} = \mathbf{x}$$

  - d) addition is associative ie

$$(\mathbf{x} + \mathbf{y}) + \mathbf{z} = \mathbf{x} + (\mathbf{y} + \mathbf{z})$$

Vector Spaces

- 2) For any pair of scalars $a, b \in R$ and $\mathbf{x}, \mathbf{y} \in S$

$$a\mathbf{x} \in S$$

$$a(b\mathbf{x}) = (ab)\mathbf{x}$$

$$(a+b)\mathbf{x} = a\mathbf{x} + b\mathbf{x}$$

$$a(\mathbf{x}+\mathbf{y}) = a\mathbf{x} + a\mathbf{y}$$

- 3) There is a unit element in the set of scalars *R* such that $1\mathbf{x} = \mathbf{x}$

(The set of scalars is usually taken as the set of real numbers)

Linear Combination

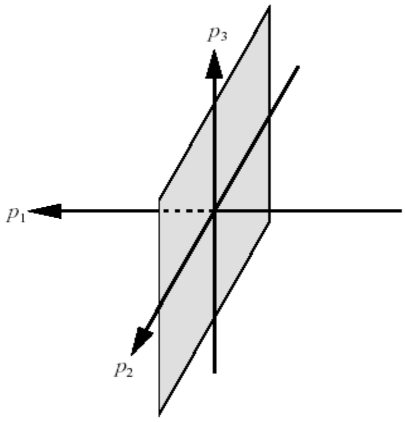
- Often we think of a signal as being composed of other simpler (or more convenient to deal with) signals. The simplest composition is a linear combination of the form

$$x[n] = \sum_{i=1}^{m} c_i p_i[n]$$

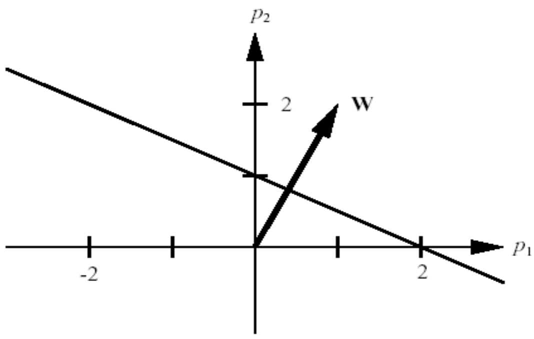
- Where $\{p_i[n]\}$ $i = 1,2,3,4,\ldots$ are the simpler signals, and the coefficients are in the scalar set.

Vector space …

Is the p_2, p_3 plane a vector space?



Is the line $p_1 + 2p_2 - 2 = 0$ a vector space?



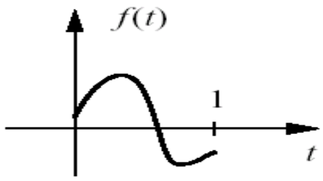
Vector space …

Polynomials of degree 2 or less.

$$x = 2 + t + 4t^2$$

$$y = 1 + 5t$$

Continuous functions in the interval [0,1].



Linear Independence

- If there is no nonzero set of coefficients

$$\{c_i\} \qquad i = 1,2,3,4,\ldots$$

such that $\sum_{i=1}^{m} c_i p_i[n] = 0$

then we say that the set of vectors

$$\{p_i[n]\} \qquad i = 1,2,3,4,\ldots$$

is linearly dependent

Linear Independence

- Examples:
- 1)

$$\mathbf{p_1} = [2 \quad -3 \quad 4]^T$$
$$\mathbf{p_2} = [-1 \quad 6 \quad -2]^T$$
$$\mathbf{p_3} = [1 \quad 6 \quad 2]^T$$

Observe that $4\mathbf{p_1} + 5\mathbf{p_2} + 3\mathbf{p_3} = 0$

ie the set is linearly dependent

Linear Independence

- Examples
- 2) $p_1 = t$

$$p_2 = 1 + t$$

the set is linearly independent

Linear Independence

$$\mathbf{p}_1 = \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix} \qquad \mathbf{p}_2 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

Let

$$a_1\mathbf{p}_1 + a_2\mathbf{p}_2 = \mathbf{0}$$

$$\begin{bmatrix} -a_1 + a_2 \\ a_1 + a_2 \\ -a_1 + (-a_2) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

This can only be true if

$$a_1 = a_2 = 0$$

Therefore the vectors are independent.

The Span of Vectors

- The Span of a vector set is the set of all possible vectors that can be reached (ie made through linear combinations) from the given set.

- That is there exists a set $\{c_i\}$ $i = 1,2,3,4,...$ such that

$$x[n] = \sum_{i=1}^{m} c_i p_i[n]$$

The Span of Vectors

- Example: The vectors below are in 3-D real vector space.

$$\mathbf{p}_1 = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix}^T$$

$$\mathbf{p}_2 = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}^T$$

- Their linear combination forms

$$\mathbf{x} = \begin{bmatrix} c_1 & c_1 + c_2 & 0 \end{bmatrix}^T$$

which is essentially a vector in the plane of the given two vectors.

Basis and Dimension

- If $\{p_i[n]\}$ $i = 1,2,3,4,...$ is a selection of linearly independent vectors from a vector space such that they span the entire space then we say the selected vectors form a (complete) basis.
- The number of elements in the basis is the cardinality of the basis
- The least number of independent vectors to span a given vector space S is called the dimension of the vector space, usually designated as

$$\dim(S)$$

IMPORTANT!

- Every vector space has a basis.
- *Thus for many purposes whatever operations we want to do in the vector space can be done with the basis.*

Basis

Polynomials of degree 2 or less.

Basis A:

$u_1 = 1 \qquad u_2 = t \qquad u_3 = t^2$

Basis B:

$u_1 = 1 - t \qquad u_2 = 1 + t \qquad u_3 = 1 + t + t^2$

(Any three linearly independent vectors in the space will work.)

How can you represent the vector $x = 1+2t$ using both basis sets?

Vector Spaces

- Let us start with the intuitive notion that we can represent a signal as

$$x[n] = \sum_{i=0}^{\infty} c_i p_i[n]$$

- This representation is called a projection of $x[n]$, the signal, into the linear vector space

$$\{p_i[n]\} \qquad i = 1,2,3,4,\ldots$$

- The vectors above are linearly independent and can span any signal in the space

Vector Spaces

- Examples are seen in Matrix Theory and typically in Fourier Analysis.
- The length of a vector is known as the norm.
- We can select any convenient norm, but for mathematical tractability and convenience we select the second order norm.
- A real valued function $\|\mathbf{x}\|$ is the norm of $\mathbf{x}$

Norm

- A real valued function $\mathbf{x}$ is the norm of $\|\mathbf{x}\|$ when it satisfies
  - Positivity $\|\mathbf{x}\| \geq 0$
  - Zero length $\|\mathbf{x}\| = 0 \quad if \quad \mathbf{x} = 0$
  - Scaling $\|\alpha\mathbf{x}\| = \alpha\|\mathbf{x}\|$
  - Triangle inequality $\|\mathbf{x}+\mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$

Induced norm/Cauchy Schwartz inequality

- Induced norm of the space follows from the inner product as

$$\langle \mathbf{x}, \mathbf{x} \rangle = \|\mathbf{x}\|^2$$

- The l_2 norm is represented as $\|\mathbf{x}\|_2$
- The following condition (Cauchy-Schwartz) is satisfied by an induced norm (eg l_2)

$$|\langle \mathbf{x}, y \rangle| \leq \|\mathbf{x}\|_2 \|\mathbf{y}\|_2$$

Inner Product

- The inner product of two vectors in a scalar, and has the following properties
  - $\langle \mathbf{x}, \mathbf{y} \rangle = \langle \mathbf{x}, \mathbf{y} \rangle^*$
  - if the vectors are real then $\langle \mathbf{x}, \mathbf{y} \rangle = \langle \mathbf{y}, \mathbf{x} \rangle$
  - $\langle \alpha \mathbf{x}, \mathbf{y} \rangle = \alpha \langle \mathbf{y}, \mathbf{x} \rangle$

Inner Product

- $$\langle \mathbf{x}+\mathbf{y},\mathbf{z}\rangle = \langle \mathbf{x},\mathbf{z}\rangle + \langle \mathbf{y},\mathbf{z}\rangle$$
- $$\langle \mathbf{x},\mathbf{x}\rangle > 0 \quad \mathbf{x} \neq 0$$
- $$\langle \mathbf{x},\mathbf{x}\rangle = 0 \quad \mathbf{x} = 0$$
- In finite-dimensional real space

$$\langle \mathbf{x},\mathbf{y}\rangle = \mathbf{x}^T\mathbf{y} = \sum x_i y_i$$

Direction of Vectors

- From the two and three dimensional cases we define the angle θ between two vectors to be given from

$$\cos\theta = \frac{\langle \mathbf{x},\mathbf{y}\rangle}{\|\mathbf{x}\|_2 \|\mathbf{y}\|_2}$$

- If $\mathbf{y} = a\mathbf{x}$ the vectors are colinear
- If $\langle \mathbf{x},\mathbf{y}\rangle = 0$ the vectors are orthogonal $\mathbf{x} \perp \mathbf{y}$
  (the zero vector is orthogonal to all vectors)

Orthonormal

- A set of vectors $[\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \ldots \mathbf{p}_m]$ is orthonormal when

$$\langle \mathbf{p}_i, \mathbf{p}_j \rangle = \delta_{ij}$$

- (Pythagoras)
  If $\mathbf{x} \perp \mathbf{y}$ then the indiced norms satisfy

$$\|\mathbf{x} + \mathbf{y}\|^2 = \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2$$

Weighted inner product

- Very often we want a weighted inner product which we define as

$$\langle \mathbf{x}, \mathbf{y} \rangle_W = \mathbf{y}^H \mathbf{W} \mathbf{x}$$

where $\mathbf{W}$ is a Hermitian matrix

For the induced norm to be positive for $\mathbf{y} = \mathbf{x}$ we must have for all non-zero vectors

$$\langle \mathbf{x}, \mathbf{x} \rangle_W = \mathbf{x}^H \mathbf{W} \mathbf{x} > 0$$

This means that $\mathbf{W}$ must be positive definite

Example

- Let

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad \mathbf{x}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

- Clearly

$$\mathbf{x}_1^T \mathbf{x}_2 = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = 3 \neq 0$$

- while

$$\mathbf{x}_1^T \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix} \mathbf{x}_2 = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = 0$$

Example

- Note that the inner product in the previous example cannot serve as a norm as for any

$$\mathbf{x} = \begin{bmatrix} a \\ a \end{bmatrix}$$

we have

$$\mathbf{x}^T \mathbf{W} \mathbf{x} = 0$$

- This violates the conditions for a norm

Complete spaces

- If every signal in a signal space is reachable (ie can be spanned) by a set of vectors then the space is complete and the reaching a complete set
- This means that there will be no left over part of a given signal expressed as an appropriate linear combination of basis vectors
- For example a Fourier series reaches all periodic signals in the least square sense, and hence the set of all complex exponentials is a complete set

Hilbert spaces

- Complete linear vector spaces with induced l_2 norms are known as Hilbert Spaces
- In signal processing we are mainly interested in finite energy signals ie in Hilbert spaces
- If the norm above is orther than the second then we have Banach Spaces. (Such norms are useful in approximating signals and system functions as we shall see later)

Orthogonal subspaces

- Let *S* be a inner product signal (vector) space and *V* and *W* be subspaces of *S*.
- Then *V* and *W* are orthogonal if for every $\mathbf{v} \in V$ and $\mathbf{w} \in W$ we have $\langle \mathbf{v}, \mathbf{w} \rangle = 0$
- In above the set of all vectors orthogonal to a subspace V is called the orthogonal complement of the of the subspace denoted by $V^{\perp}$

Inner-sum spaces

- Let *S* be a inner product signal (vector) space and *V* and *W* be subspaces of *S* with $\mathbf{v} \in V$ and $\mathbf{w} \in W$
- Then *V* +*W* is the inner sum space that consists of all vectors $\mathbf{x} = \mathbf{w} + \mathbf{v}$
- Example: Let *S* be the set of all 3-tuples in *GF(2)* and

  $$\mathbf{v} = \lfloor 1 \quad 0 \quad 1 \rfloor \in S \quad \mathbf{w} = \lfloor 0 \quad 0 \quad 1 \rfloor \in S$$

  Then

  $$\mathbf{x} = \mathbf{w} + \mathbf{v} = \lfloor 1 \quad 0 \quad 0 \rfloor \in S$$

Example …

- Let $\mathbf{A} = span\{[0 \quad 1 \quad 0]\}$

  $\mathbf{B} = span\{[1 \quad 0 \quad 0]\}$

- Then

$$\mathbf{A} = \{[0 \quad 0 \quad 0],[0 \quad 1 \quad 0]\}$$

$$\mathbf{B} = \{[0 \quad 0 \quad 0],[1 \quad 0 \quad 0]\}$$

- Note that since vectors in one of these are pairwise orthogonal to the vectors in the other the two subspaces are are orthogonal

Example …

- The orthogonal complement of $\mathbf{A}$ is found by observing the the second member of the set is orthogonal to all vectors having as their first entry a zero. Thus

$$\mathbf{A}^{\perp} = \{[0 \quad 0 \quad 0],[0 \quad 1 \quad 0],[0 \quad 0 \quad 1],[0 \quad 1 \quad 1]\}$$

- Thus observe that $\mathbf{B} \subset \mathbf{A}^{\perp}$
- And the pair-wise sum

$$\mathbf{A} + \mathbf{B} = \{[0 \quad 0 \quad 0],[0 \quad 1 \quad 0],[1 \quad 0 \quad 0],[1 \quad 1 \quad 0]\}$$

Disjoint spaces

- If two linear vector spaces of the same dimensionality have only the zero vector in common they are called disjoint.
- Two disjoint spaces are such that one is the algebraic complement of the other
- Their sum is the entire vector space

Disjoint spaces …

- Let *S* be a inner product signal (vector) space and *V* and *W* be subspaces of *S*
- Then for every $\mathbf{x} \in W + V$ there exist unique vectors $\mathbf{v} \in V$ $\mathbf{w} \in W$

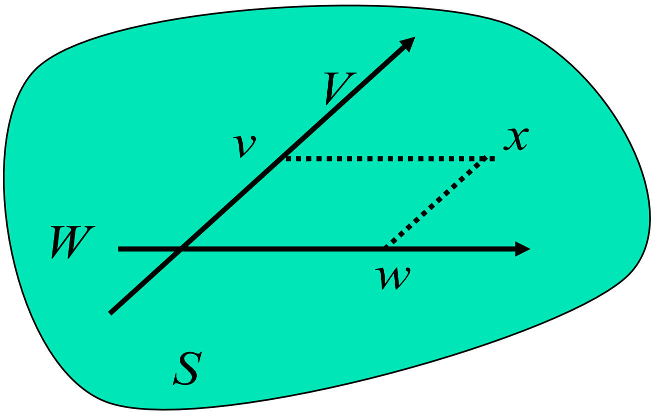
  such that $\mathbf{x} = \mathbf{w} + \mathbf{v}$

  if and only if the two sbspaces are disjoint.

  (ie if they are not disjoint a vector may be generated form a pair-wise sum of more than one pair)

Projections

- From above a pictorial representation can be produced as



Projections

- Let S be a inner product signal (vector) space and V and W be subspaces of S
- We can think of $\mathbf{v} \in V$ and $\mathbf{w} \in W$ as being the projections of $\mathbf{x} = \mathbf{w} + \mathbf{v}$ in the component sets.
- We introduce the projection operator $P : S \rightarrow V$ such that for any $\mathbf{x} \in S$ we have

$$\mathbf{Px} = \mathbf{v}$$

- That is the operation returns that component of $\mathbf{x}$ that lies in V

Projections

- Thus if $\mathbf{x}$ is already in V the operation does not change the value of $\mathbf{x}$
- Thus

$$\mathbf{P}(\mathbf{Px}) = \mathbf{Px}$$

- This gives us the definition

A linear tranformation $\mathbf{P}$ is a projection if

$$\mathbf{P}^2 = \mathbf{P}$$

(Idempotent operator)

BACKGROUND: Hilbert Spaces

Linear Transformations and Least Squares: Hilbert Spaces

Linear Transformations

- A transformation from a vector space X to a vector space Y with the same scalar field denoted by
  $$L : X \to Y$$
  is linear when
- $$L(ax) = aL(x)$$
  $$L(x_1 + x_2) = x_1 + x_2$$
- Where $x, x_1, x_2 \in X$
- We can think of the transformation as an operator

Linear Transformations ...

- Example: Mapping a vector space from R^n to R^m can be expressed as a *mxn* matrix.
- Thus the transformation
  $$L(x_1, x_2.x_3) = (x_1 + 2x_2, 3x_2 + 4x_3)$$
  can be written as
  $$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 3 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Range space & Null space

- The range space of a transformation $L : X \to Y$ is the set of all vectors that can be reached by the transformation

$$R(L) = \{\mathbf{y} = L(\mathbf{x}) : \mathbf{x} \in X\}$$

- The null space of the transformation is the set of all vectors in X that are transformed to the null vector in Y.

$$N(L) = \{L(\mathbf{x}) = \mathbf{0} : \mathbf{x} \in X\}$$

Range space & Null space ...

- If $\mathbf{P}$ is a projection operator then so is $\mathbf{I} - \mathbf{P}$
- Hence we have

$$\mathbf{Px} + (\mathbf{I} - \mathbf{P})\mathbf{x} = \mathbf{x}$$

- Thus the vector $\mathbf{x}$ is decomposed into two disjoint parts. These parts are not necessarily orthogonal

- If the range and null space are orthogonal then the projections is said to be orthogonal

Linear Transformations

- Example: Let $\mathbf{x} = [x_1 \quad x_2 \quad x_3 \quad \ldots \quad x_m]^T$
  and let the transformation a *nxm* matrix

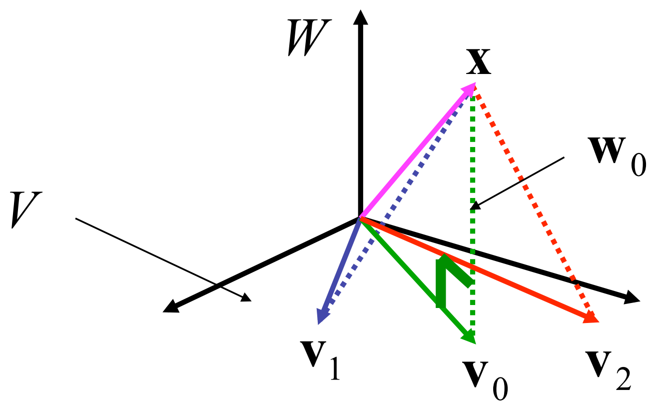
Then

$$\mathbf{Ax} = x_1\mathbf{p}_1 + x_2\mathbf{p}_2 + x_3\mathbf{p}_3 + \ldots + x_m\mathbf{p}_m$$

Thus, the range of the linear transformation (or column space of the matrix $\mathbf{A}$) is the span of the basis vectors.
The null space is the set which yields $\mathbf{Ax} = \mathbf{0}$

A Problem

- Given a signal vector $\mathbf{x}$ in the vector space *S*, we want to find a point $\mathbf{v}$ in the subset *V* of the space , nearest to $\mathbf{x}$



$$\mathbf{w}_0 = \mathbf{x} - \mathbf{v}_0$$

A Problem ...

- Let us agree that "nearest to" in the figure is taken in the Euclidean distance sense.
- The projection $\mathbf{v}_0$ orthogonal to the set *V* gives the desired solution.
- Moreover the error of representation is

$$\mathbf{w}_0 = \mathbf{x} - \mathbf{v}_0$$

- This vector is clearly orthogonal to the set *V* *(More on this later)*

Another perspective ...

- We can look at the above problem as seeking to find a solution $\mathbf{v}$ to the set of linear equations

$$\mathbf{A}\mathbf{v} = \mathbf{x}$$

where the given vector $\mathbf{x}$ is not in the range of $\mathbf{A}$ as is the case with an overspecified set of equations.

- There is no exact solution. If we project orthogonally the given vector into the range of $\mathbf{A}$ then we have the "shortest norm solution" in terms of the Euclidean distance of the "error".

Another perspective …

- The least error is then orthogonal to the data into which we are projecting
- Set

$$\mathbf{A} = [\mathbf{p}_1 \quad \mathbf{p}_2 \quad \mathbf{p}_3 \quad \cdots \quad \mathbf{p}_m]$$

$$\mathbf{v} = v_1\mathbf{p}_1 + v_2\mathbf{p}_2 + v_3\mathbf{p}_3 + \ldots + v_m\mathbf{p}_m$$

- Then as in the above figure we can write

$$\mathbf{x} = \mathbf{A}\mathbf{v} + \mathbf{w}$$

- Where $\mathbf{w}$ is the error, which is orthogonal to each of the members of $\mathbf{A}$ above.

Another perspective …

- Thus we can write

$$\langle \mathbf{x} - \mathbf{A}\mathbf{v}, \mathbf{p}_j \rangle = 0, \qquad j = 1,2,3,\ldots,m$$

- Or

$$\begin{bmatrix} \mathbf{p}_1^H \\ \mathbf{p}_2^H \\ . \\ \mathbf{p}_m^H \end{bmatrix} (\mathbf{x} - \mathbf{A}\mathbf{v}) = \mathbf{0} \qquad \mathbf{A}^H(\mathbf{x} - \mathbf{A}\mathbf{v}) = \mathbf{0}$$

$$\mathbf{v} = (\mathbf{A}^H\mathbf{A})^{-1}\mathbf{A}^H\mathbf{x}$$

Another perspective …

- Thus

$$\mathbf{v} = (\mathbf{A}^H \mathbf{A})^{-1} \mathbf{A}^H \mathbf{x} = \mathbf{P}\mathbf{x}$$

and hence the projection matrix is

$$\mathbf{P} = (\mathbf{A}^H \mathbf{A})^{-1} \mathbf{A}^H$$

ie this is the matrix that projects orthogonally into the column space of $\mathbf{A}$

Another perspective …

- If we adopt the weighted form $\langle \mathbf{x}, \mathbf{y} \rangle_{\mathbf{W}} = \mathbf{x}^H \mathbf{W} \mathbf{y}$
- The induced norm is

$$\|\mathbf{x}\|_{\mathbf{W}}^2 = \mathbf{x}^H \mathbf{W} \mathbf{x}$$

- Then the projection matrix is

$$\mathbf{P} = \mathbf{A}(\mathbf{A}^H \mathbf{W} \mathbf{A})^{-1} \mathbf{A}^H \mathbf{W}$$

- Where $\mathbf{W}$ is positive definite

Least Squares Projection

PROJECTION THEOREM

In a Hilbert space the orthogonal projection of a signal into a smaller dimensional space minimises the norm of the error, and the error vector is orthogonal to the data (ie the smaller dimensional space).

Orthogonality Principle

- Let $\{\mathbf{p}_1 \quad \mathbf{p}_2 \quad \mathbf{p}_3 \quad \cdots \quad \mathbf{p}_\mathrm{m}\}$ be a set of independent vectors in a vector space S.
- We wish to express any vector $\mathbf{x}$ in S as

$$\mathbf{x} = x_1\mathbf{p}_1 + x_2\mathbf{p}_2 + x_3\mathbf{p}_3 + \ldots + x_\mathrm{m}\mathbf{p}_\mathrm{m}$$

- If $\mathbf{x}$ is in the span of the independent vectors then the representation will be exact.
- If on the other hand it is not then there will be an error

Orthogonality Principle

- In the latter case we can write $\mathbf{x} = \hat{\mathbf{x}} + \mathbf{e}$
- Where $\hat{\mathbf{x}} = \sum_{i=1}^{m} x_i \mathbf{p}_i$

  is an approximation to given vector with error $\mathbf{e}$
- We wish to find that approximation which minimises the Euclidean error norm (squared)

$$J(x_1, \quad \dots \quad , x_m) = \left\langle \mathbf{x} - \sum_{i=1}^{m} x_i \mathbf{p}_i , \quad \mathbf{x} - \sum_{i=1}^{m} x_i \mathbf{p}_i \right\rangle$$

Orthogonality Principle

- Expand to

$$J(x_1, \quad \dots \quad , x_m) = \langle \mathbf{x}, \mathbf{x} \rangle - 2\,\mathrm{Re}\left(\sum_{i=1}^{m} x_i^* \langle \mathbf{x}, \mathbf{p}_i \rangle \right) + \sum_{i=1}^{m} \sum_{j=1}^{m} x_j x_i^* \langle \mathbf{p}_j , \mathbf{p}_i \rangle$$

$$J(x_1, \quad \dots \quad , x_m) = \|\mathbf{x}\|^2 - 2\,\mathrm{Re}(\boldsymbol{\chi}^H \mathbf{p}) + \boldsymbol{\chi}^H \mathbf{R}^T \boldsymbol{\chi}$$

- Where $\boldsymbol{\chi} = [x_1 \quad x_2 \quad \dots \quad x_m]^T$

  $\mathbf{p} = [\langle \mathbf{x}, \mathbf{p}_1 \rangle \quad \langle \mathbf{x}, \mathbf{p}_2 \rangle \quad \dots \quad \langle \mathbf{x}, \mathbf{p}_m \rangle]^T$

Reminders

$$\frac{\partial}{\partial \mathbf{x}^*}\mathbf{a}\mathbf{x} = \mathbf{0} \qquad \frac{\partial}{\partial \mathbf{x}^*}\mathbf{x}^H\mathbf{a} = \mathbf{a}$$

$$\frac{\partial}{\partial \mathbf{x}^*}\mathrm{Re}(\mathbf{x}^H\mathbf{a}) = \mathbf{a}/2 \qquad \frac{\partial}{\partial \mathbf{x}^*}\mathbf{x}^H\mathbf{R}\mathbf{x} = \mathbf{R}\mathbf{x}$$

Orthogonality Principle

$$\frac{\partial}{\partial \chi^*}\left(\|\mathbf{x}\|^2 - 2\,\mathrm{Re}\left(\chi^H\mathbf{p}\right) + \chi^H\mathbf{R}^T\chi\right) = -\mathbf{p} + \mathbf{R}\chi$$

- On setting this to zero we obtain the solution

$$\chi = \mathbf{R}^{-1}\mathbf{p}$$

- This is a minimum because on differentiating we have a positive definite matrix

Alternatively …

- The norm squared of the error is

$$J = \mathbf{e}^T\mathbf{e}$$

- where

- We note that $$\frac{\partial}{\partial x_i}\mathbf{e} = -\mathbf{p}_i$$

  and $$\frac{\partial}{\partial x_i}J = 2\left(\frac{\partial}{\partial x_i}\mathbf{e}\right)^T\mathbf{e}$$

Orthogonality Principle

- At the minimum

$$\frac{\partial}{\partial x_i}J = 2\left(\frac{\partial}{\partial x_i}\mathbf{e}\right)^T\mathbf{e} = -2\mathbf{p}_i^T\mathbf{e} = 0$$

- Thus we have and hence

$$\langle \mathbf{p}_i, \mathbf{e} \rangle = \langle \mathbf{p}_i, \mathbf{x} - \hat{\mathbf{x}} \rangle = 0$$

Thus,

1) At the optimum the error is orthogonal to the data (Principle of orthogonality)

2) $$\langle \mathbf{p}_i, \mathbf{x} \rangle = \langle \mathbf{p}_i, \hat{\mathbf{x}} \rangle = \sum_{j=1}^{m} \langle \mathbf{p}_i, \mathbf{p}_j \rangle x_j \qquad i = 1, \ldots, m$$

Orthogonality Principle

- Thus for $\mathbf{p} = \left[\langle \mathbf{x}, \mathbf{p}_1 \rangle \quad \langle \mathbf{x}, \mathbf{p}_2 \rangle \quad \ldots \quad \langle \mathbf{x}, \mathbf{p}_m \rangle \right]^T$

$$\chi = \left[x_1 \quad x_2 \quad \ldots \quad x_m \right]^T$$

$$\mathbf{R} = \begin{bmatrix} \langle \mathbf{p}_1, \mathbf{p}_1 \rangle & \langle \mathbf{p}_1, \mathbf{p}_2 \rangle & .. & \langle \mathbf{p}_1, \mathbf{p}_m \rangle \\ \langle \mathbf{p}_2, \mathbf{p}_1 \rangle & \langle \mathbf{p}_2, \mathbf{p}_2 \rangle & .. & \langle \mathbf{p}_2, \mathbf{p}_m \rangle \\ .. & .. & .. & .. \\ \langle \mathbf{p}_m, \mathbf{p}_1 \rangle & \langle \mathbf{p}_m, \mathbf{p}_2 \rangle & .. & \langle \mathbf{p}_m, \mathbf{p}_m \rangle \end{bmatrix}$$

Hence $\mathbf{p} = \mathbf{R}\chi$ or $\chi = \mathbf{R}^{-1}\mathbf{p}$

Orthogonalisation

- A signal may be projected into any linear space.
- The computation of its coefficients in the various vectors of the selected space is easier when the vectors in the space are orthogonal in that they are then non-interacting, ie the evaluation of one such coefficient will not influence the others
- The error norm is easier to compute
- Thus it makes sense to use an orthogonal set of vectors in the space into which we are to project a signal

Orthogonalisation

- Given any set of linearly independent vectors that span a certain space, there is another set of independent vectors of the same cardinality, pair-wise orthogonal, that spans the same space
- We can think of the given set as a linear combination of orthogonal vectors
- Hence because of independence, the orthogonal vectors is a linear combination of the given vectors
- This is the basic idea behind the Gram-Schmidt procedure

Gram-Schmidt Orthogonalisation

- The problem: (we consider finite dimensional spaces only)
- Given a set of linearly independent vectors $\{\mathbf{x}\}$ to determine a set of vectors $\{\mathbf{p}\}$ that are pair-wise orthogonal
- Write the ith vector as

$$\mathbf{x}_i = x_1^{(i)}\mathbf{p}_1 + x_2^{(i)}\mathbf{p}_2 + x_3^{(i)}\mathbf{p}_3 + \ldots + x_m^{(i)}\mathbf{p}_m \qquad i = 1,\ldots,m$$

Gram-Schmidt Orthogonalisation

- If we knew the orthogonal set $\{\mathbf{p}\}$ then the coefficients of the expression can be determined as the inner product

$$\langle \mathbf{x}_i, \mathbf{p}_j \rangle = x_j \|\mathbf{p}_j\|^2$$

- Step(1) The unknown orthogonal vector can be oriented such that one of its members coincides with one of the members of the given set $\{\mathbf{x}\}$
- Choose $\mathbf{p}_1$ to be coincident with $\mathbf{x}_1$

Gram-Schmidt Orthogonalisation

- Step (2) Each member of $\{\mathbf{x}\}$ has a projection onto $\mathbf{p}_1$ given by

$$\langle \mathbf{x}_i, \mathbf{p}_1 \rangle = x_1^{(i)} \|\mathbf{p}_1\|^2$$

- Step(3) We construct

$$\mathbf{u}_i = \mathbf{x}_i - x_1^{(i)} \mathbf{p}_1 \qquad i = 2, \ldots, m$$

- Step(4) Repeat the above on $\{\mathbf{u}\}$

Gram-Schmidt Orthogonalisation

- Example: Let $\mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\mathbf{x}_2 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$
- Then

$$\mathbf{p}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

- And the projection of $\mathbf{x}_2$ onto $\mathbf{p}_1$ is

$$\begin{bmatrix} -1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 1$$

Gram-Schmidt Orthogonalisation

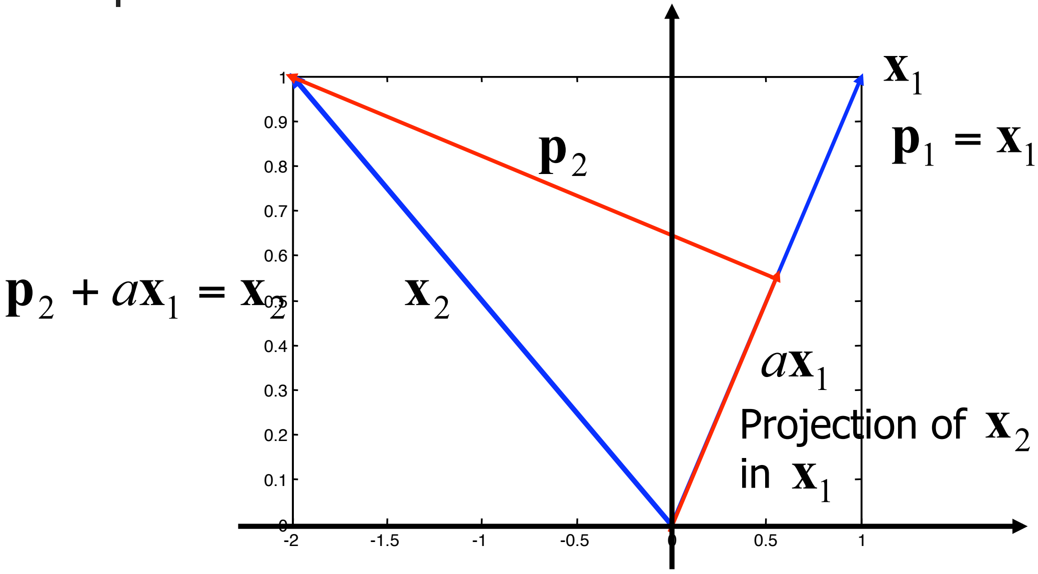
- Form

$$\mathbf{x}_2 - 1\mathbf{p}_1 / \|\mathbf{p}_1\|^2 = \begin{bmatrix} -1 \\ 2 \end{bmatrix} - 1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} / 2 = \begin{bmatrix} -1.5 \\ 1.5 \end{bmatrix}$$

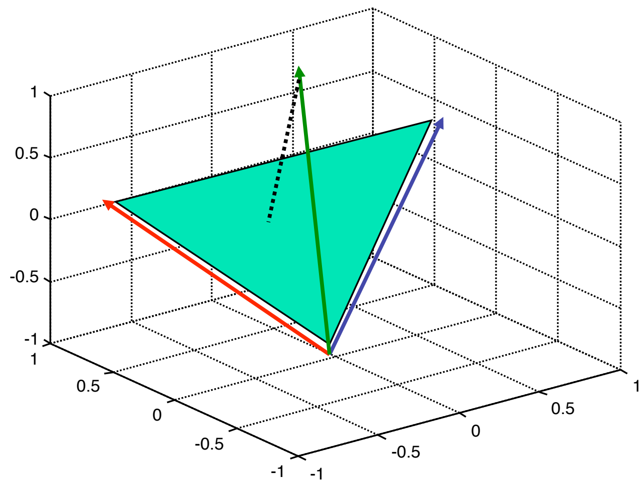
- Then

$$\mathbf{p}_2 = \begin{bmatrix} -1.5 \\ 1.5 \end{bmatrix}$$

Gram-Schmidt



3-D G-S Orthogonalisation



Gram-Schmidt Orthogonalisation

- Note that in the previous 4 steps we have considerable freedom at Step 1 to choose any vector not necessarily coincident with one from the given set of data vectors.
- This enables us to avoid certain numerical ill-conditioning problems that may arise in the Gram-Schmidt case.
- *Can you suggest when we are likely to have ill-conditioning in the G-S procedure?*