Communications II
Lecture 4: Effects of Noise on AM

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Noise in Analog Communication Systems

• How do various analog modulation schemes perform in the presence of noise?

• Which scheme performs best?

• How can we measure its performance?
We must find a way to quantify (=to measure) the performance of a modulation scheme.

We use the signal-to-noise ratio (SNR) at the output of the receiver:

$$SNR_0 \equiv \frac{\text{average power of message signal at the receiver output}}{\text{average power of noise at the receiver output}} = \frac{P_S}{P_N}$$
Model of an analog communication system
$P_T$: The transmitted power

Limited by: equipment capability, cost, government restrictions, interference with other channels, etc

The higher it is, the more the received power ($PS$), the higher the $SNR$

For a fair comparison between different modulation schemes: $P_T$ should be the same for all

We use the **baseband** signal to noise ratio $SNR_{baseband}$ to calibrate the $SNR$ values we obtain
A Baseband Communication System

• It does not use modulation

• It is suitable for transmission over wires

• The power it transmits is identical to the message power: $P_T = P$

• The results carry over to band-pass systems
Average signal(=message) power:
\[ P = \text{the area under the triangular curve} \]

Assume:
Additive, white noise with power spectral density \( PSD = N_0/2 \)

Average noise power at the receiver:
\[ P_N = \text{area under the straight line} = 2W \times N_0/2 = WN_0 \]
SNR at the receiver output:

\[ SNR_{\text{baseband}} = \frac{P_T}{N_0 W} \]

Note: Assume no propagation loss \( P_T = P_S \)

Improve the SNR by:

(a) increasing the transmitted power \( P_T \uparrow \),
(b) restricting the message bandwidth \( W \downarrow \),
(c) making the receiver less noisy \( N_0 \downarrow \).
REVISION: Amplitude Modulation

General form of an AM signal:

\[ s(t)_{AM} = [A + m(t)] \cos(2\pi f_c t) \]

- \( A \): the amplitude of the carrier
- \( f_c \): the carrier frequency
- \( m(t) \): the message signal
Modulation index:

\[ \mu \equiv \frac{m_p}{A} \]

\( m_p \): the peak amplitude of \( m(t) \), i.e., \( m_p = \max |m(t)| \)
Signal recovery

1) $\mu \leq 1 \Rightarrow A \geq m_p$ : use an envelope detector

2) Otherwise: use synchronous detection = product demodulation = coherent detection
Synchronous detection

- Multiply the waveform at the receiver with a local carrier of the same frequency (and phase) as the carrier used at the transmitter:

\[
\cos(2\pi f_c t)s(t)_{AM} = [A + m(t)] \cos^2(2\pi f_c t)
\]

\[
= [A + m(t)] \frac{1 + \cos(4\pi f_c t)}{2}
\]

\[
= [A + m(t)] \left\{ \frac{1}{2} + \frac{\cos(4\pi f_c t)}{2} \right\}
\]

\[
= m(t) \frac{1}{2} + A \frac{1}{2} + \ldots
\]

- Use a LPF to recover \((m(t) + A)/2\) and finally \(m(t)\)

- **Problem**: At the receiver you need a signal perfectly synchronised with the transmitted carrier
REVISION Amplitude Modulation: Double-sideband suppressed carrier (DSB-SC)

\[ s(t)_{DSBS} = A m(t) \cos(2\pi f_c t) \]

Signal recovery: With synchronous detection only
Noise in DSB-SC

The true signal received is:

\[ x(t) = s(t) + n(t) \]
\[ = s(t) + n_c(t) \cos(2\pi f_c \, t) - n_s(t) \sin(2\pi f_c \, t) \]
\[ = A m(t) \cos(2\pi f_c \, t) + n_c(t) \cos(2\pi f_c \, t) - n_s(t) \sin(2\pi f_c \, t) \]
\[ = [A m(t) + n_c(t)] \cos(2\pi f_c \, t) - n_s(t) \sin(2\pi f_c \, t) \]
For **synchronous** detection:

- multiply with $2\cos(2\pi f_c t)$:

$$y(t) = 2\cos(2\pi f_c t)x(t)$$

$$= A_m(t)\cos^2(2\pi f_c t) + n_c(t)2\cos^2(2\pi f_c t) - n_s(t)\sin(4\pi f_c t)$$

$$= A_m(t)[1 + \cos(4\pi f_c t)] + n_c(t)[1 + \cos(4\pi f_c t)]$$

$$- n_s(t)\sin(4\pi f_c t)$$

- Use a LPF to keep

$$\tilde{y} = A_m(t) + n_c(t)$$
Signal power at the receiver output:

\[ P_S = E\{A^2m^2(t)\} = A^2 E\{m^2(t)\} = A^2 P \]

Power of the noise signal \( n_c(t) \):

\[ P_N = \int_{-W}^{W} N_0 df = 2N_0W \]
$SNR$ at the receiver output:

$$SNR_0 = \frac{A^2 P}{2N_0 W}$$

To which transmitted power does this correspond?

$$P_T = E\{A^2 m(t)^2 \cos^2(2\pi f_c t)\} = \frac{A^2 P}{2}$$
So

\[ SNR_0 = \frac{P_T}{N_0W} = SNR_{DSB-SC} \]

Comparison with

\[ SNR_{baseband} = \frac{P_T}{N_0W} \]

\[ SNR_{DSB-SC} = SNR_{baseband} \]

**Conclusion:** a DSB-SC system provides no SNR performance gain over a baseband system.
Pre-detection signal:

\[ x(t) = [A + m(t)] \cos(2\pi f_c t) + n(t) \]
\[ = [A + m(t)] \cos(2\pi f_c t) + n_c(t) \cos(2\pi f_c t) - n_s(t) \sin(2\pi f_c t) \]
\[ = [A + m(t) + n_c(t)] \cos(2\pi f_c t) - n_s(t) \sin(2\pi f_c t) \]
Signal Recovery:

- Multiply with $2 \cos(2 \pi f_c t)$:

$$y(t) = A[1 + \cos(4\pi f_c t)] + m(t)[1 + \cos(4\pi f_c t)]$$
$$+ n_c(t)[1 + \cos(4\pi f_c t)] - n_s(t)\sin(4\pi f_c t)$$

- LPF

$$\tilde{y} = A + m(t) + n_c(t)$$
Signal power at the receiver output:

\[ P_S = E\{m^2(t)\} = P \]

Noise power:

\[ P_N = 2N_0P \]
SNR at the receiver output:

\[ SNR_0 = \frac{P}{2N_0W} = SNR_{AM} \]

Transmitted power:

\[ P_T = \frac{A^2}{2} + \frac{P}{2} = \frac{A^2 + P}{2} \]

SNR of a baseband signal with the same transmitted power:

\[ SNR_{baseband} = \frac{A^2 + P}{2N_0W} \]
Thus:

\[ SNR_{AM} = \frac{P}{A^2 + P} SNR_{baseband} \]

Note:

\[ \frac{P}{A^2 + P} < 1 \]

**Conclusion:** the performance of standard AM with synchronous recovery is worse than that of a baseband system.
Noise in standard AM, Envelope Detection

Phasor diagram of the signals present at an AM receiver

\[ E_i(t) : \text{receiver output} = y(t) \]
\[ y(t) = \text{envelope of } x(t) \]
\[ = \sqrt{[A + m(t) + n_c(t)]^2 + n_s(t)^2} \]  \hspace{1cm} (95)

Equation too complicated

Must use limiting cases to put it in a form where noise and message are added
1st Approximation: (a) Small Noise Case

\[ n(t) \ll [A + m(t)] \]

Then

\[ n_s(t) \ll [A + m(t) + n_c(t)] \]

Then

\[ y(t) \approx [A + m(t) + n_c(t)] \]
Thus

\[ SNR_0 = \frac{P}{2N_0W} \approx SNR_{\text{env}} \]

And in terms of baseband SNR:

\[ SNR_{\text{env}} \approx \frac{P}{A^2 + P} SNR_{\text{baseband}} \]

**Valid for small noise only!**
2nd Approximation: (b) Large Noise Case

\[ n(t) \gg [A + m(t)] \]

Isolate the small quantity in (95):

\[
y^2(t) = [A + m(t) + n_c(t)]^2 + n_s^2(t)
\]

\[
= (A + m(t))^2 + n_c^2(t) + 2(A + m(t))n_c(t) + n_s^2(t)
\]

\[
= [n_c^2(t) + n_s^2(t)]\left[1 + \frac{(A + m(t))^2}{n_c^2(t) + n_s^2(t)} + \frac{2(A + m(t))n_c(t)}{n_c^2(t) + n_s^2(t)}\right]
\]
\[ y^2(t) \approx \left[ n_c^2(t) + n_s^2(t) \right] \left( 1 + \frac{2[A + m(t)]n_c(t)}{n_c^2(t) + n_s^2(t)} \right) \]

\[ = E_n^2(t) \left( 1 + \frac{2[A + m(t)]n_c(t)}{E_n^2(t)} \right) \]

where

\[ En(t) \equiv \sqrt{n_c^2(t) + n_s^2(t)} \quad : \text{the envelope of the noise} \]
From the phasor diagram: $n_c(t) = E_n(t) \cos \theta_n(t)$

Then:

$$y(t) \approx E_n(t) \sqrt{1 + \frac{2[A + m(t)] \cos \theta_n(t)}{E_n(t)}}$$

Use $\sqrt{1 + x} \approx 1 + \frac{x}{2}$ for $x \ll 1$:

$$y(t) \approx E_n(t) \left(1 + \frac{[A + m(t)] \cos \theta_n(t)}{E_n(t)} \right)$$

$$= E_n(t) + [A + m(t)] \cos \theta_n(t)$$
Noise is multiplicative here!

No term proportional to the message!

Result: a threshold effect, as below some carrier power level (very low $A$), the performance of the detector deteriorates very rapidly.
SSB modulation

Single (lower) sideband AM:

\[ s(t)_{SSB} = \frac{A}{2} m(t) \cos 2\pi f_c t + \frac{A}{2} \hat{m}(t) \sin 2\pi f_c t \]

where \( \hat{m}(t) \) is the Hilbert transform of \( m(t) \).

\( \hat{m}(t) \) is obtained by passing \( m(t) \) through a linear filter with transfer function \( -j \text{sgn}(f) \).

\( \hat{m}(t) \) and \( m(t) \) have the same power \( P \).

The average power is \( A^2 P/4 \).
Noise in SSB

Receiver signal $x(t) = s(t) + n(t)$.

Apply a band-pass filter on the lower sideband. Using coherent detection:

\[
y(t) = x(t) \times 2 \cos(2\pi f_c t) = \left( \frac{A}{2} m(t) + n_c(t) \right) + \left( \frac{A}{2} m(t) + n_c(t) \right) \cos(4\pi f_c t) + \left( \frac{A}{2} \hat{m}(t) + n_s(t) \right) \sin(4\pi f_c t)
\]

After low-pass filtering,

\[
y(t) = \left( \frac{A}{2} m(t) + n_c(t) \right)
\]
Signal power $A^2P/4$

Noise power for $nc(t) =$ that for band-pass noise $= N_0W$

SNR at output

$$SNR_{SSB} = \frac{A^2P}{4N_0W}$$

For a baseband system with the same transmitted power $A^2P/4$

$$SNR_{baseband} = \frac{A^2P}{4N_0W}$$

**Conclusion:** SSB achieves the same SNR performance as DSB-SC (and the baseband model) but only requires half the band-width.
## Summary

<table>
<thead>
<tr>
<th>(De-) Modulation Format</th>
<th>Output SNR</th>
<th>Transmitted Power</th>
<th>Baseband Reference SNR</th>
<th>Figure of Merit (= Output SNR / Reference SNR)</th>
</tr>
</thead>
<tbody>
<tr>
<td>AM Coherent Detection</td>
<td>$\frac{P}{2N_0W}$</td>
<td>$\frac{A^2 + P}{2}$</td>
<td>$\frac{A^2 + P}{2N_0W}$</td>
<td>$\frac{P}{A^2 + P} &lt; 1$</td>
</tr>
<tr>
<td>DSB-SC Coherent Detection</td>
<td>$\frac{A^2P}{2N_0W}$</td>
<td>$\frac{A^2P}{2}$</td>
<td>$\frac{A^2P}{2N_0W}$</td>
<td>1</td>
</tr>
<tr>
<td>SSB Coherent Detection</td>
<td>$\frac{A^2P}{4N_0W}$</td>
<td>$\frac{A^2P}{4}$</td>
<td>$\frac{A^2P}{4N_0W}$</td>
<td>1</td>
</tr>
<tr>
<td>AM Envelope Detection (Small Noise)</td>
<td>$\frac{P}{2N_0W}$</td>
<td>$\frac{A^2 + P}{2}$</td>
<td>$\frac{A^2 + P}{2N_0W}$</td>
<td>$\frac{P}{A^2 + P} &lt; 1$</td>
</tr>
<tr>
<td>AM Envelope Detection (Large Noise)</td>
<td>Poor</td>
<td>$\frac{A^2 + P}{2}$</td>
<td>$\frac{A^2 + P}{2N_0W}$</td>
<td>Poor</td>
</tr>
</tbody>
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