SEQUENTIAL LOCAL FRI SAMPLING OF INFINITE STREAMS OF DIRACS

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ABSTRACT

The theory of sampling signals with finite rate of innovation (FRI) has shown that it is possible to perfectly recover classes of non-bandlimited signals such as streams of Diracs from uniform samples. Most of previous papers, however, have to some extent only focused on the sampling of periodic or finite duration signals.

In this paper we propose a novel method that is able to reconstruct infinite streams of Diracs, even in high noise scenarios. We sequentially process the discrete samples and output locations and amplitudes of the Diracs in real-time. We first establish conditions for perfect reconstruction in the noiseless case and then present the sequential algorithm for the noisy scenario. We also show that we can achieve a high reconstruction accuracy of 1000 Diracs for SNRs as low as 5dB.

***Index Terms*—** Analog-to-digital conversion, finite rate of innovation, sampling theory, annihilating filter.

1. INTRODUCTION

Streams of Diracs are the canonical example of signals with finite rate of innovation (FRI) in that they are completely specified by a finite number of parameters per unit of time. Periodic streams of Diracs are sampled and perfectly reconstructed in [1] using the sinc kernel. Authors in [2, 3] instead propose the use of kernels that reproduce polynomials or exponentials and also propose a sequential algorithm to sample and perfectly reconstruct infinite streams of Diracs. The sequential algorithm, however, was designed to deal only with noiseless samples. The family of sum of sincs (SoS) kernels was introduced in [4] for the sampling of periodic stream of pulses such as Diracs, authors also consider the case of infinite streams of Diracs. However, their method requires that the stream of Diracs be 'bursty'. Specifically, a group of K Diracs must be followed by a long period of absence of Diracs in order for the method to work. They also assume that the reconstruction algorithm is synchronised with the sampling process in order to be automatically in phase with the time window containing the burst of Diracs.

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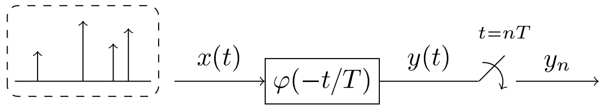


Fig. 1: Acquisition device.

In this paper we propose a novel approach to reconstruct infinite streams of Diracs, in high noise scenarios, with no clear separation between bursts. We sequentially process the discrete samples and output locations and amplitudes of the Diracs in real-time. We first establish conditions for perfect reconstruction in the noiseless case and then present the sequential algorithm for the noisy scenario. We show through simulations that the algorithm is able to process 10000 samples in about 100 seconds and that it can retrieve with high accuracy 1000 Diracs even in very low SNR regimes.

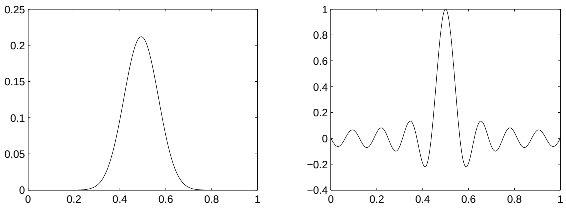
The paper is organised as follows. In Section 2 we review the case of sampling and reconstructing a finite stream of K Diracs as presented in [3]. In Section 3 we explain our sequential algorithm in detail. We treat the noiseless and noisy scenario separately, the former to establish conditions for perfect reconstruction, the latter for application in more realistic situations. To end, we validate our algorithm with simulations in Section 4 and conclude in Section 5.

2. SAMPLING FRI SIGNALS

We consider the case of a stream of K Diracs $x(t) = \sum_{k=1}^{K} a_k \, \delta\left(t - t_k\right), \ a_k, t_k \in \mathbb{R}$, where $\{t_k, a_k\}_{k=1}^{K}$ are unknown delays and amplitudes. The continuous-time signal is filtered with a kernel with impulse response $h(t) = \varphi(-t/T)$ and uniformly sampled at regular intervals of time $t = nT$. The acquisition process is illustrated in Figure 1. The samples y_n can be expressed as

$$y_n = \left\langle x(t) \, , \, \varphi(\tfrac{t}{T} - n) \right\rangle . \tag{1}$$

FRI theory [1–4] shows that for a properly chosen filter $h(t)$, the signal $x(t)$ can be perfectly reconstructed from the samples y_n. We restrict our setup to exponential reproducing kernels [3], which are functions that are able to reproduce exponentials:



(a) E-spline **(b)** Modified E-spline

Fig. 2: Sampling kernels $h(t) = \varphi(-t/T)$ obtained from E-splines of order $P = 15$ and scaled with $T = 1/16$. Note that function $\varphi(t)$ is made anticausal in order to have causal sampling kernels. The modified E-spline (eMOMS) corresponds exactly to half period of the Dirichlet kernel of order $2(P + 1)$.

$$\sum_{n\in\mathbb{Z}} c_{m,n}\, \varphi(t - n) = e^{\alpha_m t}, \quad m = 0, 1, \ldots, P, \qquad (2)$$

where $c_{m,n} \in \mathbb{C}$ and $\{\alpha_m\}_{m=0}^{P}$ are design parameters and are chosen to be purely imaginary and in complex conjugate pairs in order to have a real valued kernel $\varphi(t)$. More specifically, we use $\alpha_m = j\frac{\pi}{P}(m - \frac{P}{2})$ for $m = 0, \ldots, P$. There exist many functions of compact support in time that satisfy the exponential reproducing property (2), such as for example E-splines [3] and the modified E-splines introduced in [5]. The latter, which are a variation of the maximal-order minimal-support kernels presented in [6], are the exponential reproducing kernels that are most resilient to noise [5, 7] and are the kernels of choice for our simulations. They are called eMOMS. Note that these kernels have support $P + 1$.

In order to recover parameters $\{t_k, a_k\}_{k=1}^{K}$, first the samples y_n are linearly combined with coefficients $c_{m,n}$ from (2). This leads to a new set of measurements

$$s_m = \sum_{n\in\mathbb{Z}} c_{m,n}\, y_n, \quad m = 0, 1, \ldots, P. \qquad (3)$$

Combining (1) and (2) it follows that s_m can be expressed as a power sum series [3]:

$$s_m = \left\langle x(t)\, ,\, e^{\alpha_m t/T} \right\rangle = \sum_{k=1}^{K} b_k\, u_k^m, \qquad (4)$$

where $b_k = a_k\, e^{\alpha_0 t_k/T}$ and $u_k = e^{j\pi t_k/PT}$. The FRI recovery is thus turned into an amplitude and frequency estimation problem. This is a well-known problem in spectral analysis and can be solved, for instance, with the annihilating filter method [8]. The critical number of measurements s_m required to recover the $2K$ parameters $\{t_k, a_k\}_{k=1}^{K}$ is exactly $2K$ [3]. It thus follows that $P + 1 \geqslant 2K$ in order to achieve perfect reconstruction.

2.1. FRI reconstruction in the presence of noise

The solution explained so far is only ideal, since noise is generally present in the acquisition process. We assume the only source of noise ε_n is added to the samples y_n. In addition, we consider ε_n are i.i.d. Gaussian random variables, of zero mean and standard deviation σ. Thus, measurements (3) become $\tilde{s}_m = \sum_{n\in\mathbb{Z}} c_{m,n}\, (y_n + \varepsilon_n) = s_m + d_m$, and alternative methods are needed to retrieve the pairs (t_k, a_k). We may solve the problem by using the total least-squares method and Cadzow denoising algorithm [9] introduced for FRI in [10] or matrix pencil method [11] used for FRI in [12]. Further alternative methods can be found in [13–15].

3. SAMPLING AN INFINITE STREAM OF DIRACS

We now consider the case of an infinite train of Diracs

$$x(t) = \sum_{k\in\mathbb{Z}} a_k\, \delta\,(t - t_k)\,. \qquad (5)$$

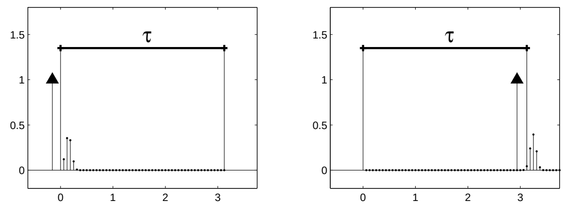
The signal $x(t)$ is assumed to have a finite local rate of innovation, as defined in [1], of $2K/\tau$. This means that, if we consider a sliding window of size τ, the number of Diracs that we see inside the window is always at most K. We propose a sequential algorithm that estimates the locations of the Diracs of (5) by using a sliding window that sequentially covers the interval of time

$$t \in \left]t_i, t_i + \tau\right]. \qquad (6)$$

The sliding window step is of T seconds, which equals the sampling period. We assume that τ is an integer multiple of T, specifically $\tau = NT$. In what follows, we first establish some conditions on the number of samples N, the sampling period T and the order of the E-splines necessary to achieve perfect reconstruction of the infinite stream (5). Second, we present a novel approach that is able to recover Diracs in high noise scenarios processing the stream (5) sequentially.

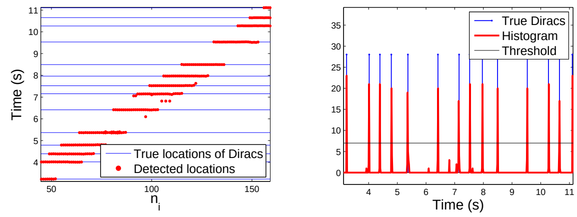
3.1. Noiseless case

We analyze the ideal scenario in the first place to determine the conditions on the number of samples of the sliding window N, the sampling period T and the support $P + 1$ of the sampling kernel that allows our algorithm to be able to provide perfect reconstruction of (5). In our approach we sequentially analyse sets of N samples y_n that cover the temporal interval (6). Due to the fact that we consider a finite number of samples, there exist border effects that may stop us from achieving perfect reconstruction. The sampling kernel $\varphi(t/T)$ has compact support $(P+1)T$. Consequently, a Dirac influences $P+1$ samples. This means that a Dirac located just before the window of interest will generate non-zero samples that will leak inside the window. Moreover, a Dirac located at



(a) **(b)**

Fig. 3: Border effects. (a) A nearby Dirac located before the observation window τ influences the samples y_n of the window. (b) A Dirac inside the window but close to the right border generates non-zero samples outside the window.



(a) Retrieved locations **(b)** Locations histogram

Fig. 4: Noisy scenario. (a) Plot of the sequentially retrieved locations, the horizontal axis indicates the index of the sliding window and the vertical axis the location in time. (b) Histogram of the locations shown in (a). Horizontally aligned dots in (a) lead to peaks in the histogram in (b).

the end of the window of interest will generate non-zero samples beyond the N samples we are considering and therefore cannot be reconstructed. This is illustrated in Figure 3.

Since the algorithm operates sequentially we can assume that when operating on the window $t \in]t_i, t_i + \tau]$ we have already perfectly recovered Diracs up to the time instant t_i. Therefore their contributions can be removed and in this way we can overcome the border effect on the *left* of the window.

To overcome the border effect of the *right* side we determine certain conditions on the number of samples N, the order P of the kernel and the sampling period T. We have seen in Sec. 2 that exact recovery of K Diracs requires $P + 1 \geqslant 2K$. Now, let us put ourselves in the worst case scenario, where we have K Diracs evenly spaced with constant separation τ/K. Each Dirac influences $P+1$ samples due to the support of the kernel. Therefore, K Diracs influence $K(P + 1)$ samples y_n. In this worst case scenario, we have to guarantee that $N \geqslant K(P + 1)$ and combining this condition with $P + 1 \geqslant 2K$ we arrive at

$$N \geqslant 2K^2. \tag{7}$$

As previously mentioned, when a Dirac is near the end of the interval, we are not able to perfectly reconstruct it. The size of this area is PT. Therefore, we can only perfectly recover K Diracs when all of them are in a region of size $(N - P)T$. Again, in the case of constant separation τ/K, we have to guarantee that there will be a position of the sliding window such that the K Diracs are in the perfect reconstruction area. Since they can occupy an interval of maximum size $(K - 1)\frac{\tau}{K}$ and to make sure they are within the perfect reconstruction area for a certain window, we restrict this interval to be smaller than or equal $(N - P - 1)T$. Combining these conditions and choosing the smallest possible number of samples $N = 2K^2$ we obtain

$$\begin{aligned} (K-1)\frac{NT}{K} &\leqslant (N - P - 1)T \\ \Leftrightarrow P &\leqslant 2K - 1. \end{aligned} \tag{8}$$

In addition, we know that P has to satisfy $P \geqslant 2K - 1$. Consequently, we have that

$$P = 2K - 1. \tag{9}$$

The sampling period T is given by the temporal interval $NT = \tau$ and the number of samples $N \geqslant 2K^2$. We thus have $T \leqslant \frac{\tau}{2K^2}$.

The reconstruction algorithm processes the stream of samples sequentially, retrieving the locations of each set of maximum K Diracs from N samples by applying the annihilating filter method. Provided we satisfy the previously described conditions, all Diracs will be located in the perfect reconstruction interval of a certain position of the sliding window, and thus recovered. From the recovered Diracs of the current window, we recalculate the N samples that correspond to this window, and only if the reconstructed samples are identical to the original ones, the Diracs are stored. The maximum number of Diracs K within a window has to be estimated. This is done by trying for all possible values of K, and only when the correct value is estimated the reconstructed samples will coincide with the original ones.

3.2. Noisy scenario

In the presence of noise, perfect reconstruction is not possible and the algorithm previously described becomes unstable. Moreover, the strict conditions on N and P impose critical sampling, since we have exactly $2K$ values of the s_m measurements to retrieve K Diracs. In the noisy case we relax this condition and allow higher values of P. This makes the denoising algorithms mentioned in Sec. 2.1 more effective.

We thus develop a new strategy that is also based on using a sliding window and processing sets of N samples in sequential order. For each window and each group of N samples, we retrieve K Diracs using the algorithm in Sec. 2 coupled with matrix pencil. We then store all the locations and amplitude retrieved in that window. We then slide the window by T and repeat the process. When the found locations correspond to real Diracs, they will be consistent among different positions

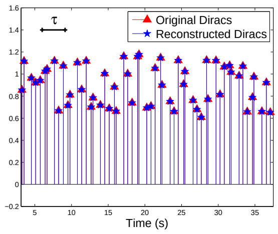
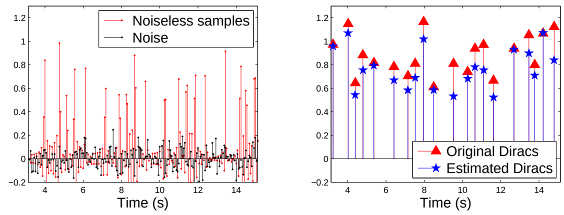


Fig. 5: Sequential perfect reconstruction of a noiseless stream of 1000 Dirac with 10220 y_n samples. Only a small section of the stream is shown. Rate $K = 5$ Diracs per $\tau = 3.125$ s. $N = 50$, $T = 1/16$ and $P = 9$.



(a) y_n samples **(b)** Reconstructed stream

Fig. 6: Noisy samples with a $\mathrm{SNR} = 10$ dB and reconstructed stream from the peaks of the histogram of the retrieved locations. In all these simulations we used eMOMS as sampling kernel.

of the sliding window that capture these Diracs. Otherwise, locations that are not correct and correspond to noise will normally be not consistent. For example, in Figure 4-(a) we plot the retrieved locations for different windows. The horizontal axis represents the index of the window corresponding to a retrieved location, and the vertical axis the Dirac location in time. Consistent locations appear as horizontal alignments of dots, overlapping the blue lines.

In order to detect which locations are consistent, a second step is to construct a histogram of detected locations. Only the peaks of the histogram are assumed to correspond to real Diracs. For a peak in the histogram above a certain threshold, the location of the corresponding Dirac is estimated averaging all the retrieved locations that contribute to this peak. This is illustrated in Figure 4-(b).

Algorithm 1 Sequential FRI retrieval of Diracs

Input: $(y_n)_{n=1}^{N_{TOT}}$: stream of samples
Output: $\{(t_k, a_k)\}$: Dirac locations and amplitudes
1: **for** $n_i = 1$ to $N_{TOT} - N + 1$ **do**
2: Retrieve $\{(t_k^i, a_k^i)\}$ from $(y_n)_{n=n_i}^{n_i+N-1}$
3: **end for**
4: Construct histogram from retrieved locations $\{t_k^i\}$
5: Estimate Diracs from peaks of the histogram

4. SIMULATION RESULTS

We have tested both versions of the algorithm: the noiseless case for which perfect reconstruction is possible; and the noisy scenario, where locations are estimated from the histogram of the retrieved locations. In the noiseless case we always perfectly reconstruct the streams of Diracs with randomly generated locations and amplitudes. This is illustrated in Figure 5. The stream of Diracs is generated to satisfy the maximum rate of K Diracs per τ interval.

In the noisy scenario not all the Diracs are always retrieved, and false positives may also happen. Note also that there is an uncertainty in the retrieved location. A retrieved Dirac is considered to correspond to a true Dirac if the difference between the real location and the estimated location is smaller than a threshold. Here we have set this threshold to $T/2$. We randomly generate the locations of a stream of 1000 Diracs. We then take samples, contaminate them with noise and apply the sequential reconstruction algorithm. Figure 6 shows one realisation of the procedure explained before.

To further analyse the performance variation for different levels of noise we run the algorithm over 100 different realisations of noise for various levels of SNR. Table 1 summarises the obtained performances.

Table 1: Algorithm's performance. Stream of 1000 Diracs (630 seconds) and 10220 samples, $T = 1/16$ s, $N = 50$, $P + 1 = 23$. The detection rate is given in percentage of detected true Diracs. The false positives are the average number of detected Diracs that do not correspond to true Diracs. The precision is the standard deviation of the retrieved locations with respect to the true locations.

SNR (dB)	5	10	15	20
Detection rate	97.69 %	99.97 %	100.00 %	100.00 %
False positives	351.7	37.8	0.5	0.3
Precision (s)	0.0086	0.0049	0.0028	0.0018

The algorithm has been implemented in MATLAB and tested using a commercial laptop (2.5 GHz Intel Core i5 CPU). The average time required to process 10220 samples corresponding to a stream of 630 seconds containing 1000 Diracs is about 105 seconds.

5. CONCLUSIONS AND FUTURE WORK

In this paper we have presented a fast sequential algorithm to retrieve infinite streams of Diracs in noiseless and noisy environments. In the noiseless case perfect reconstruction is achieved. In the noisy scenario we propose to retrieve groups of K Diracs sequentially and to retain only those Diracs whose locations have been consistently estimated in overlapping sliding windows.

We showed that the algorithm is able to process $10K$ samples in about 100 seconds and can retrieve with high accuracy 1000 Diracs even in very low SNR regimes.

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