An Online Learning View via a Projections' Path in the Sparse-land

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Joint work with
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Sparsity

Sparse Modeling

- Sparse modeling has been a major focus of research effort over the last decade or so.
- Sparsity promoting regularization of cost functions copes with:
  - Ill conditioning-overfitting when solving inverse problems; Learning from data is an instance of inverse problems.
  - **Promote** zeros when the underlying models have many near-to-zero values.

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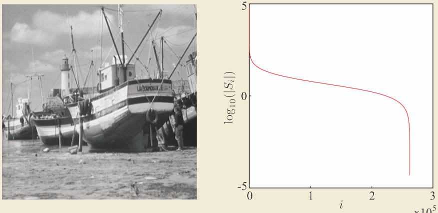
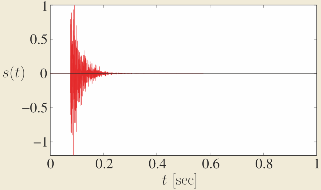
The need for sparse Models: Two examples

- Compression



(a) (b)

- Echo Cancelation



Sparse Modeling

The Generic Model

OUTPUT=INPUT $\times$ SPARSE MODEL+NOISE

Sparse Modeling

The Regression Model

- A generic model that covers a large class of problems (Filtering, Prediction)

$$y_n = \boldsymbol{u}_n^T \boldsymbol{a}_* + v_n$$

  - $\boldsymbol{a}_* \in \mathbb{R}^L$, is the unknown vector.
  - $\boldsymbol{u}_n \in \mathbb{R}^L$, is the incoming signal (sensing vectors).
  - $y_n \in \mathbb{R}$, is the observed signal (measurements).
  - v_n is the additive noise process.

- $\boldsymbol{a}_*$ is assumed to be sparse. That is, only a few, $K << L$, of its components are nonzero

$$\boldsymbol{a}_* = [0, 0, \underbrace{\star}_{1}, 0, \ldots, 0, \underbrace{\star}_{2}, 0, 0, \ldots, 0, \underbrace{\star}_{K}, 0, \ldots, 0]^T$$

- In its simplest formulation the task comprises the estimation of $\boldsymbol{a}_*$, based on a set of measurements $(y_n, \boldsymbol{u}_n)$, $n = 1 \ldots N$.

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Sparse Modeling

Dictionary Learning

- This is a powerful tool in analysing signals in terms of overcomplete basis vectors.

$$\underbrace{[\boldsymbol{y}_1, \ldots, \boldsymbol{y}_N]}_{L \times N} = \underbrace{[\boldsymbol{u}_1, \ldots, \boldsymbol{u}_m]}_{L \times m} \underbrace{[\boldsymbol{a}_1, \ldots, \boldsymbol{a}_N]}_{m \times N}, \quad m > L$$

$$Y = UA$$

- $\boldsymbol{y}_n, \in \mathbb{R}^L \ n = 1, 2, \ldots, N$, are the observation vectors.
- $\boldsymbol{u}_i \in \mathbb{R}^L, \ i = 1, 2, \ldots, m$, are the unknown atoms of the dictionary.
- $\boldsymbol{a}_n \in \mathbb{R}^m, \ n = 1, 2, \ldots, N$, are the vectors of the unknown weights, corresponding in the respective expansion of the nth input vector:
$$\boldsymbol{y}_n = \sum_{i=1}^{m} \boldsymbol{u}_i a_{ni}$$
- where, $\boldsymbol{a}_n, \ n = 1, 2, \ldots, N$, sparse vectors.

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Low Rank Matrix Factorization

- This task is at the heart of dimensionality reduction.

$$Y = UA = \sum_{i=1}^{r} \boldsymbol{u}_i \hat{\boldsymbol{a}}_i^T$$

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- $r < N$.
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Low Rank Matrix Factorization

- Matrix Completion is a special constrained version of low rank matrix factorization
- Y has missing elements and the lower rank matrix factorization is constrained to provide the non-missing elements at the respective positions

$$\hat{Y} = \begin{bmatrix} * & * & * & * & * & * \\ * & * & * & * & * & * \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ * & * & * & * & * & * \end{bmatrix}$$

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- Robust PCA is another special constrained version of low rank matrix factorization.

$$Y = L + V$$

L is a low rank matrix and V is a sparse matrix. The latter models OUTLIER NOISE. Being outlier is sparse.

- The goal of the task is to obtain estimates $\tilde{L}$ and $\tilde{V}$ by imposing sparsity on the singular values of Y as well as on the elements of V, constrained so that $Y = \tilde{L} + \tilde{V}$.

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Robust Regression

- Robust Regression is an old problem, with a major impact coming from the works of Huber. The revival of interest is due to a new look via sparsity-aware learning techniques. For example, the noise may comprise a few large values (outliers) on top of the Gaussian component. Since the large values are only a few, they can be treated via sparse modeling arguments.

Sparse Regression Modeling

There are two paths that lead to the "truth", e.g, obtain an estimate $\hat{\boldsymbol{a}}$ of the unknown $\boldsymbol{a}_*$.

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Batch Learning Problem

Linear Regression Model $\quad y_n = \boldsymbol{u}_n^T \boldsymbol{a}_* + v_n$

- $\boldsymbol{U} := [\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_N]^T \in \mathbb{R}^{N \times L}$
- $\boldsymbol{y} := [y_1, y_2, \ldots, y_N]^T \in \mathbb{R}^N$, and $\boldsymbol{v} := [v_1, v_2, \ldots, v_N]^T \in \mathbb{R}^N$.

Batch Formulation: $\quad \boldsymbol{y} = \boldsymbol{U}\boldsymbol{a}_* + \boldsymbol{v}$

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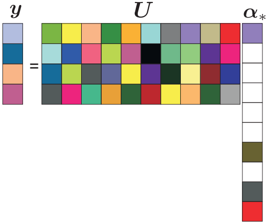

Estimating the unknown

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Batch vs Online Learning

Batch formulation: $\boldsymbol{y} = \boldsymbol{U}\boldsymbol{a}_* + \boldsymbol{v}$

Online Formulation: $y_n = \boldsymbol{u}_n^T \boldsymbol{a}_* + v_n,$

obtain an estimate, $\boldsymbol{a}_n$, after $(y_n, \boldsymbol{u}_n)$ has been received

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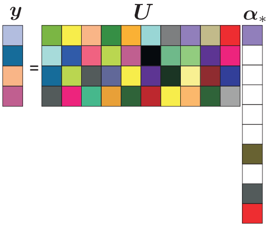

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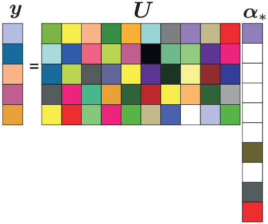

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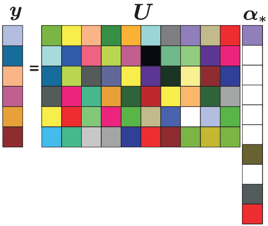

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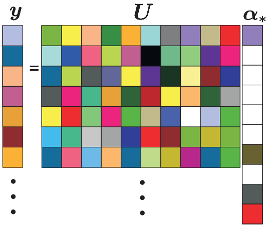

Sparse Vs Online Learning

Sparsity-promoting Batch algorithm (Compressed Sensing)

- Are mobilized after a finite number of data, $(\boldsymbol{u}_n, y_n)_{n=0}^{N-1}$, is collected.
- For any new datum, the estimation of $\boldsymbol{a}_*$, is repeated from scratch.
- Computational complexity might become prohibitive.
- Excessive storage demands.
- It is a "mature" research field with a diverse number of techniques and applications.

Sparsity-promoting Online algorithms

- Infinite number of data.
- For any new datum, the estimate of $\boldsymbol{a}_*$ is updated dynamically.
- Cases of time-varying $\boldsymbol{a}_*$ are "naturally" handled.
- Low complexity is required for streaming applications.
- Fast convergence / Tracking.
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Sparsity-Promoting Methods

ℓ_0-norm constrained minimization

- ℓ_0 (pseudo) norm minimization: NP-hard nonconvex task.
- $\hat{\boldsymbol{a}} : \min_{\boldsymbol{a}\in\mathbb{R}^l} \|\boldsymbol{a}\|_0, \quad \text{s.t.} \quad \|\boldsymbol{y} - U\boldsymbol{a}\|_2^2 \leq \epsilon$
- The above is carried out via greedy-type algorithmic arguments.

Constrained Least Squares Estimation: Three equivalent formulations

- $\hat{\boldsymbol{a}} := \arg\min_{\boldsymbol{a}\in\mathbb{R}^l} \left\{\|\boldsymbol{y} - U\boldsymbol{a}\|_2^2 + \lambda\|\boldsymbol{a}\|_1\right\}$
- $\hat{\boldsymbol{a}} : \min_{\boldsymbol{a}\in\mathbb{R}^l} \|\boldsymbol{y} - U\boldsymbol{a}\|_2^2, \quad \text{s.t.} \quad \|\boldsymbol{a}\|_1 \leq \rho$
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- Why ℓ_1 norm: It is the "closest" to ℓ_0 "norm" (number of nonzero elements) that retains its convex nature.

Sparsity-Promoting Methods

ℓ_0-norm constrained minimization

- ℓ_0 (pseudo) norm minimization: NP-hard nonconvex task.
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Constrained Least Squares Estimation: Three equivalent formulations

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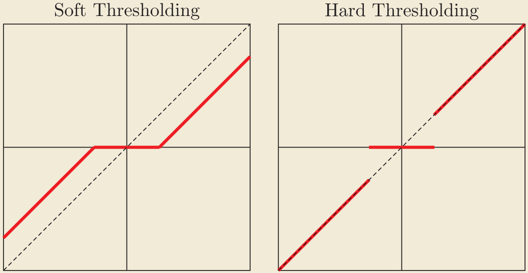
Sparsity-Promoting Methods

Hard and Soft thresholding

- The ℓ_1 norm is associated with a soft thresholding operation on the respective coefficients. This is a continuous function operation, but it adds bias even for the large values. On the other hand, hard thresholding is a discontinuous one.



Batch Penalized Least-Squares Estimator

Penalized Least-Squares - General Case

$$\min_{\boldsymbol{a} \in \mathbb{R}^L} \left\{ \frac{1}{2} \|\boldsymbol{y} - \boldsymbol{U}\boldsymbol{a}\|_2^2 + \lambda \sum_{i=1}^{L} p(|a_i|) \right\}$$

- $p(\cdot)$, sparsity-promoting penalty function,
- λ, regularization parameter.

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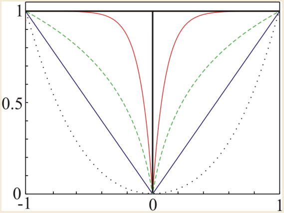
Examples: Penalty functions

- $p(|a_i|) := |a_i|^\gamma, \ \forall a_i \in \mathbb{R}$
- $p(|a_i|) = \lambda\left(1 - e^{-\beta|a_i|}\right)$
- $p(|a_i|) := \frac{\lambda}{\log(\gamma+1)}\log(\gamma|a_i|+1), \ \forall a_i \in \mathbb{R}$



Online Sparsity-Promoting Methods

Penalized Recursive LS

$$\min_{\boldsymbol{a}\in\mathbb{R}^L} \left\{ \frac{1}{2}\sum_{n=1}^{N} \beta^{N-n} e_n^2 + \lambda \sum_{i=1}^{L} p(|a_i|) \right\},$$

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$$\boldsymbol{r}_N := \sum_{n=1}^{N} \beta^{N-n} y_n \boldsymbol{u}_n, \; \boldsymbol{R}_N := \sum_{n=1}^{N} \beta^{N-n} \boldsymbol{u}_n \boldsymbol{u}_n^T$$

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- Complexity $\mathcal{O}(L^2)$
- Regularization parameter needs fine tuning

- [Angelosante, Bazerque and Giannakis, 2010]
- [Eksioglu and Tanc, 2011]

Online Sparsity-Promoting Methods

Penalized stochastic gradient descent: LMS type

$$\min_{\boldsymbol{a}\in\mathbb{R}^L} \left\{ \frac{1}{2} e_n^2 + \lambda \sum_{i=1}^{L} p(|a_i|) \right\}$$

$$\boldsymbol{a}_{n+1} = \boldsymbol{a}_n + \mu e_n(\boldsymbol{a}) \boldsymbol{u}_n - \mu\lambda \boldsymbol{f}(\boldsymbol{a}_n)$$

$$\boldsymbol{f}(\boldsymbol{a}_n) = \left[\frac{\partial p(|a_{n,1}|)}{\partial a_{n,1}}, \frac{\partial p(|a_{n,2}|)}{\partial a_{n,2}}, \ldots, \frac{\partial p(|a_{n,L}|)}{\partial a_{n,L}} \right]^T$$

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The Set-Theoretic Estimation Approach

The main concept

A descendent of POCS

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Projection onto a Closed Convex Set

Let C be a closed convex set in $\mathbb{R}^L$. Then, for each $\boldsymbol{a} \in \mathbb{R}^L$ there exists a unique $\boldsymbol{a}_* \in C$ such that

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Metric Projection is the mapping
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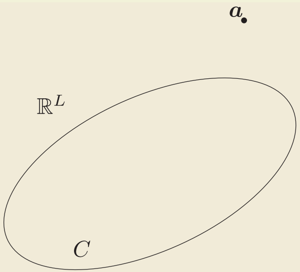

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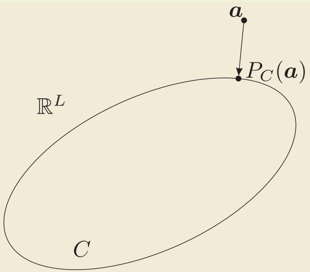

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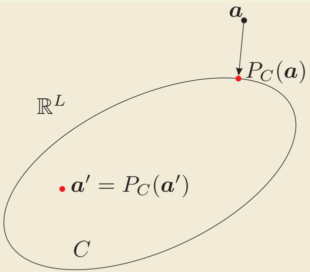

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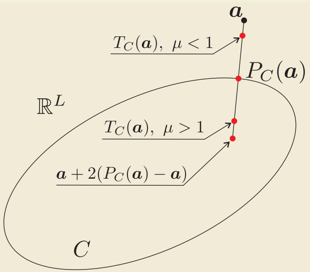
Relaxed Projection Mapping

The relaxed Projection is the mapping
$T_C(\boldsymbol{a}) := \boldsymbol{a} + \mu(P_C(\boldsymbol{a}) - \boldsymbol{a})$,
$\mu \in (0, 2), \forall \boldsymbol{a} \in \mathbb{R}^L$.



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Given a finite number of closed convex sets $C_1, \ldots, C_q$, with $\bigcap_{i=1}^q C_i \neq \emptyset$, let their associated projection mappings be $P_{C_1}, \ldots, P_{C_q}$. For any $\boldsymbol{a} \in \mathbb{R}^L$, define the sequence of projections:

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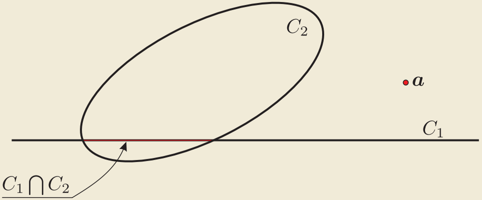

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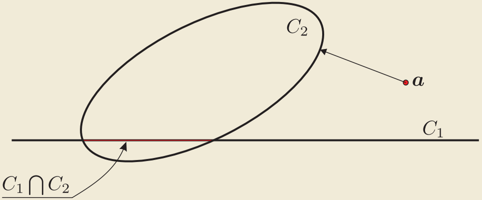

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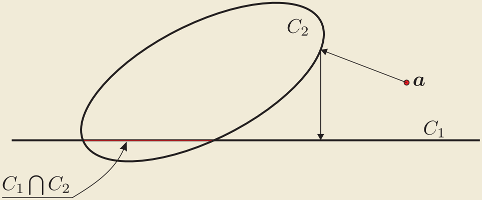

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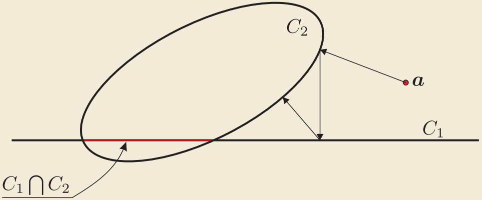

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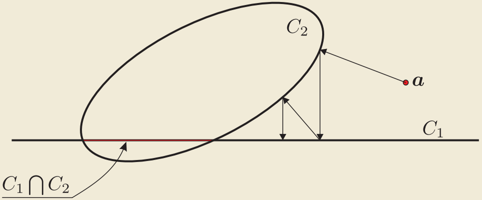

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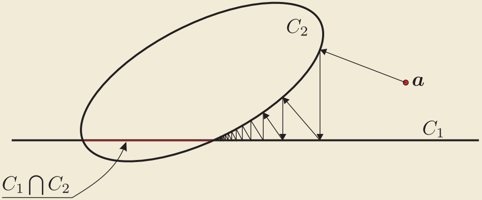

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Convex Combination of Projection Mappings [Pierra '84]

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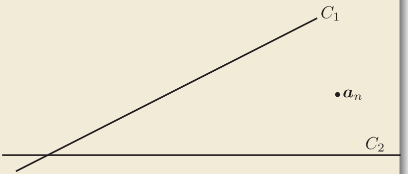

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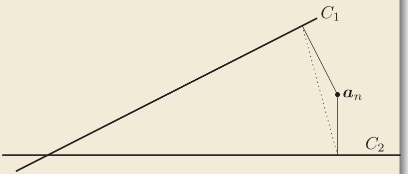

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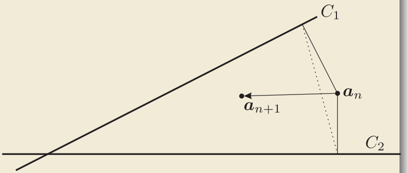

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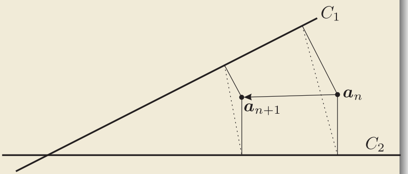
converges weakly to a point $\boldsymbol{a}_*$ in $\bigcap_{i=1}^q C_i$, where $\mu_n \in (\epsilon, \mathcal{M}_n)$, for $\epsilon \in (0,1)$, and $\mathcal{M}_n := \frac{\sum_{i=1}^q w_i \|P_{C_i}(\boldsymbol{a}_n) - \boldsymbol{a}_n\|^2}{\|\sum_{i=1}^q w_i P_{C_i}(\boldsymbol{a}_n) - \boldsymbol{a}_n\|^2}$.



The Set-Theoretic Estimation Approach

Convex Combination of Projection Mappings [Pierra '84]

Given a finite number of closed convex sets $C_1, \ldots, C_q$, with $\bigcap_{i=1}^{q} C_i \neq \emptyset$, let their associated projection mappings be $P_{C_1}, \ldots, P_{C_q}$. Let also a set of positive constants $w_1, \ldots, w_q$ such that $\sum_{i=1}^{q} w_i = 1$. Then for any $\boldsymbol{a}_0$, the sequence

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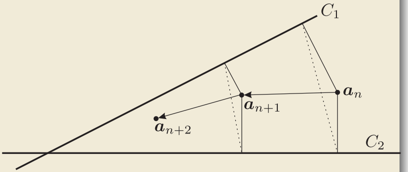

Set-Theoretic Estimation: The Online Case Approach

Constructing the Convex Sets

For each received set of measurements (training pairs) $(\boldsymbol{u}_n, y_n)$, construct a hyperslab:

$S_n[\epsilon] :=$
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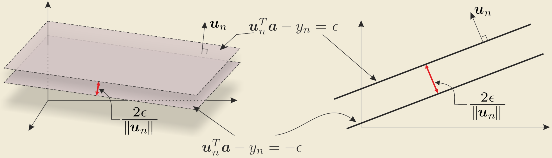

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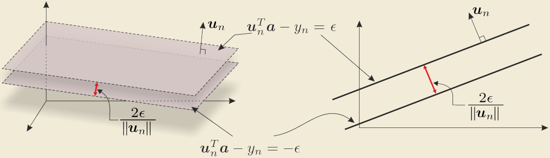

Solution

Find a point in the intersection of all the hyperslabs

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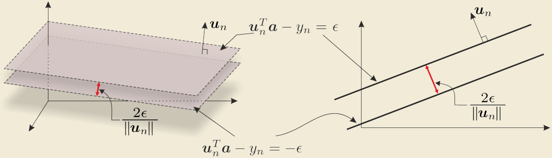

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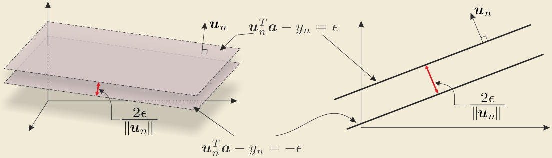

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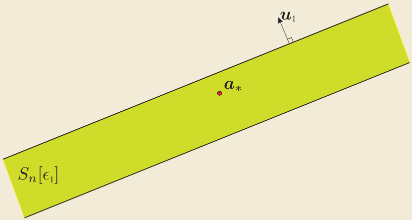

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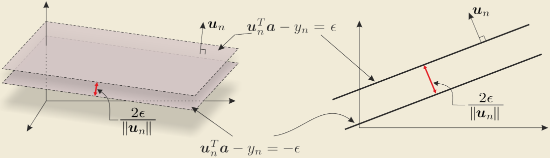

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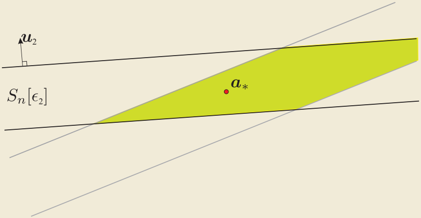

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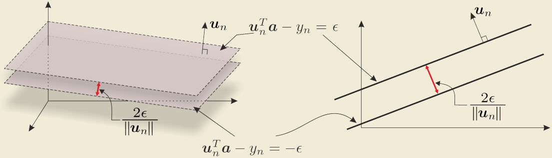

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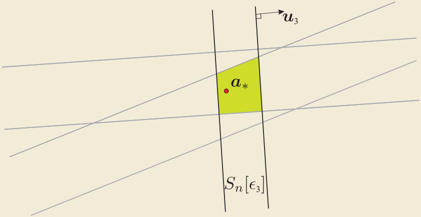

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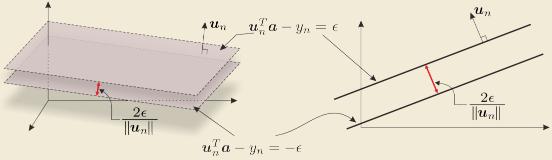

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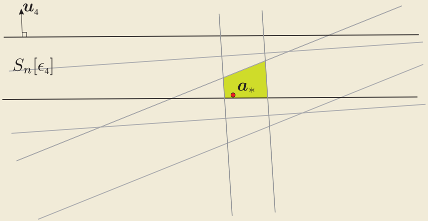

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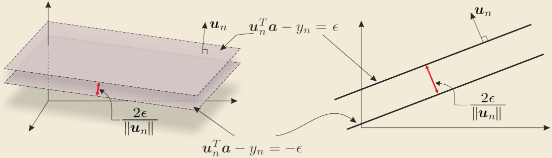

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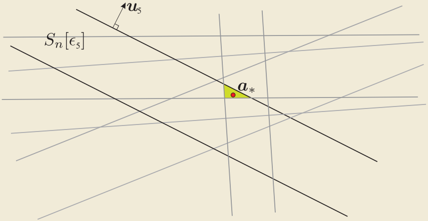

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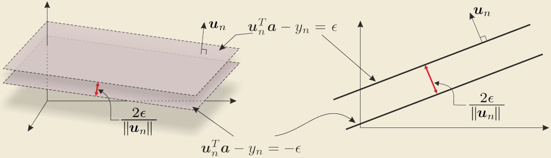


Solution

[Yamada 2001], [Yamada, Slavakis, Yamada 2002], [Yamada, Ogura 2004], [Slavakis, Yamada Ogura 2006].

[Chouvardas, Slavakis, Theodoridis, Yamada, 2013]: Under the assumption of Bounded noise it converges with probability 1 arbitrarily close to the true model.

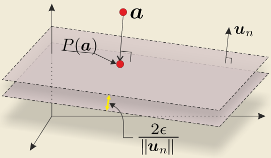
Adaptive Projection Subgradient Method (APSM)

The Algorithm

$$\boldsymbol{a}_{n+1} := \boldsymbol{a}_n + \mu_n \left(\sum_{i=n-q+1}^{n} \omega_i^{(n)} \left(P_{S_n[\epsilon]}(\boldsymbol{a}_n) - \boldsymbol{a}_n \right) \right)$$

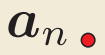
Projection onto Hyperslab

$$P_{S_n[\epsilon]}(\boldsymbol{a}) = \boldsymbol{a} + \begin{cases} \frac{y_n - \epsilon - \boldsymbol{u}_n^T \boldsymbol{a}}{\|\boldsymbol{u}_n\|^2} \boldsymbol{u}_n, & \text{if } y_n - \epsilon > \boldsymbol{u}_n^T \boldsymbol{a} \\ 0, & \text{if } |\boldsymbol{u}_n^T \boldsymbol{a} - y_n| \leq \epsilon \\ \frac{y_n + \epsilon - \boldsymbol{u}_n^T \boldsymbol{a}}{\|\boldsymbol{u}_n\|^2} \boldsymbol{u}_n, & \text{if } y_n + \epsilon < \boldsymbol{u}_n^T \boldsymbol{a} \end{cases}$$



Adaptive Projection Subgradient Method (APSM)

Geometric illustration example



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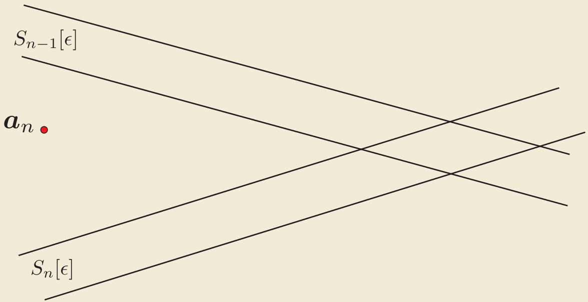

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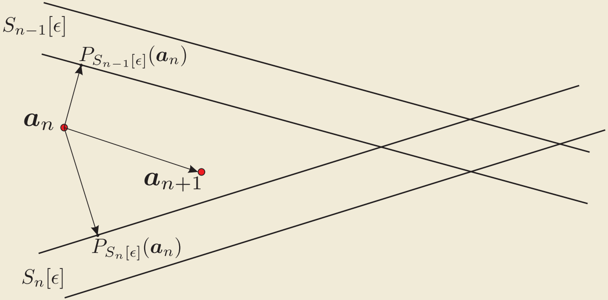

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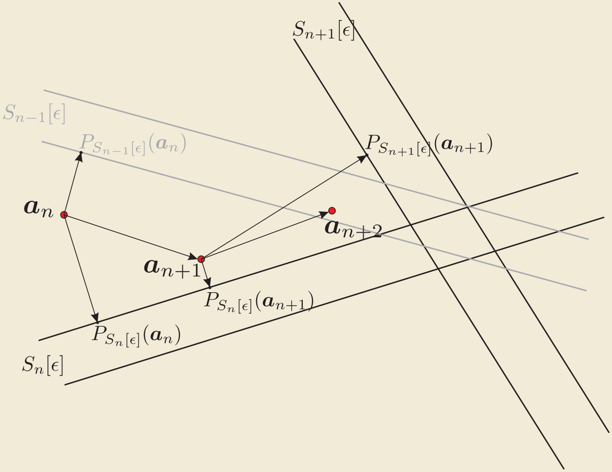

APSM under the ℓ_1 ball constraint

The ℓ_1-ball case

- Given $(\boldsymbol{u}_n, y_n)$, $n = 0, 1, 2, \ldots$, find $\boldsymbol{a}$ such that

$$\left|\boldsymbol{a}^T \boldsymbol{u}_n - y_n\right| \leq \epsilon, \quad n = 0, 1, 2, \ldots$$
$$\|\boldsymbol{a}\|_1 \leq \delta.$$

- The recursion:

$$\boldsymbol{a}_{n+1} := P_{B_{\ell_1}[\delta]}\left(\boldsymbol{a}_n + \mu_n\left(\sum_{j=n-q+1}^{n} \omega_j^{(n)} P_{S_j[\epsilon]}(\boldsymbol{a}_n) - \boldsymbol{a}_n\right)\right),$$

converges to

$$\boldsymbol{a}_* \in B_{\ell_1}[\delta] \cap \left(\bigcap_{n \geq n_0} S_n[\epsilon]\right).$$

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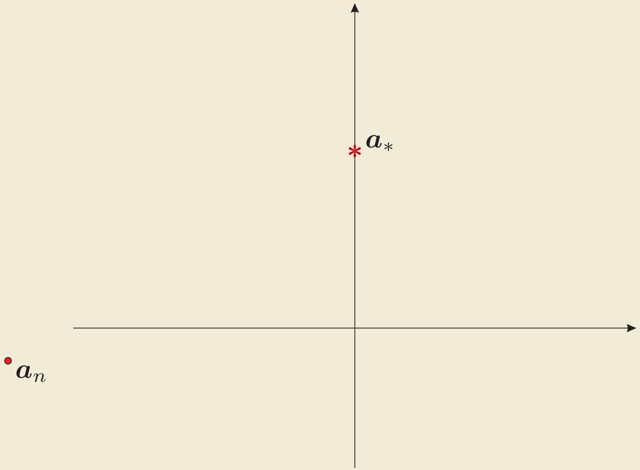

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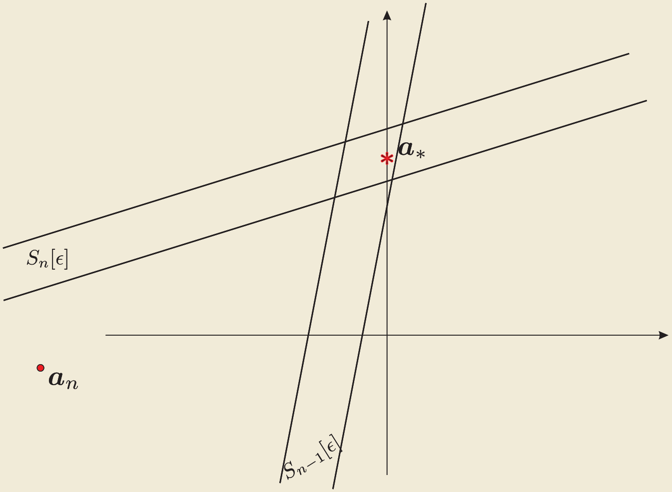

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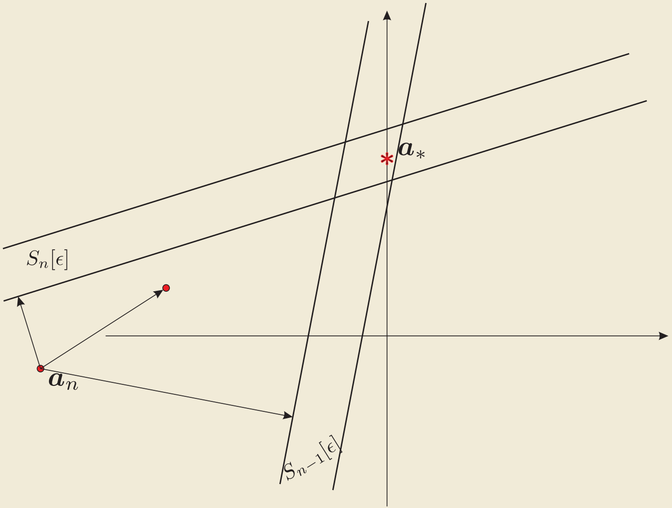

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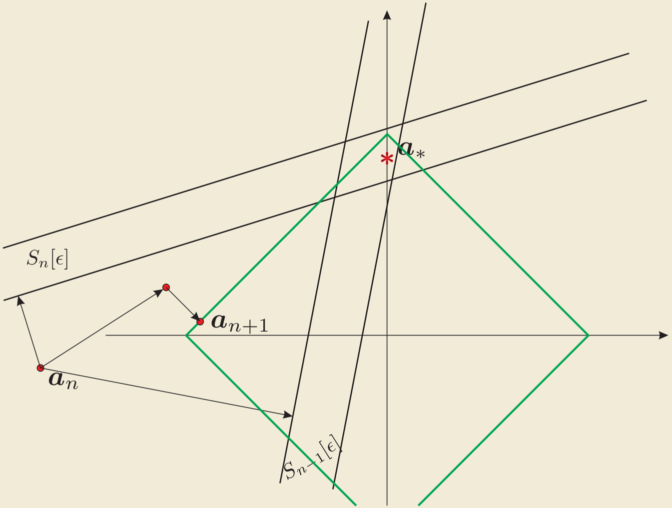

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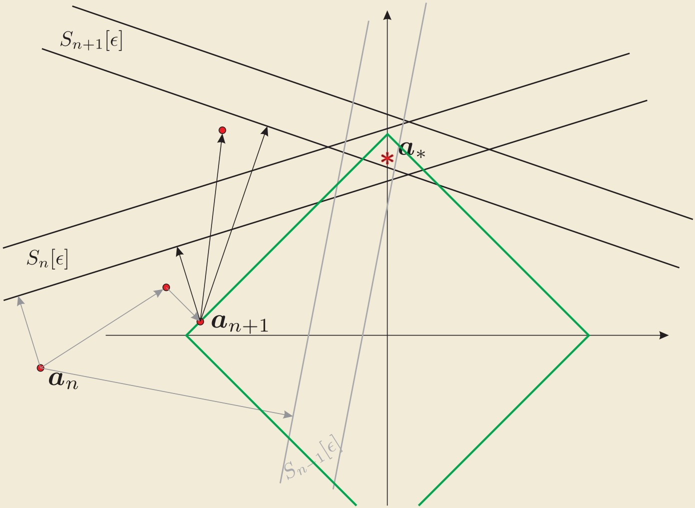

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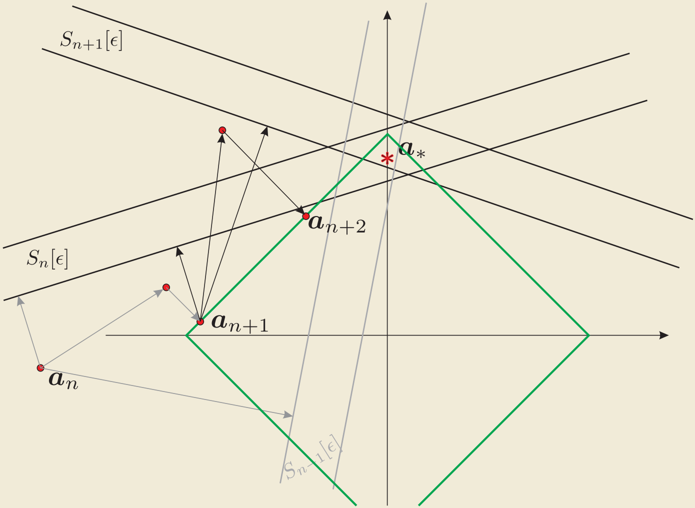

APSM under the weighted ℓ_1 ball constraint

The weighted ℓ_1-ball case:

- Convergence can be significantly speeded up if ℓ_1-ball, is replaced by the weighted ℓ_1 ball.
- Definition:

$$\|\boldsymbol{a}\|_{1,w} := \sum_{i=1}^{L} w_i |a_i|.$$

- Time-adaptive weighted norm:

$$w_{n,i} := \frac{1}{|a_{n,i}| + \epsilon'_n}.$$

- A time varying constraint case.
- The recursion:

$$\boldsymbol{a}_{n+1} := P_{B_{\ell_1}[\boldsymbol{w}_n,\delta]} \left(\boldsymbol{a}_n + \mu_n \left(\sum_{j=n-q+1}^{n} \omega_j^{(n)} P_{S_j[\epsilon]}(\boldsymbol{a}_n) - \boldsymbol{a}_n \right) \right).$$

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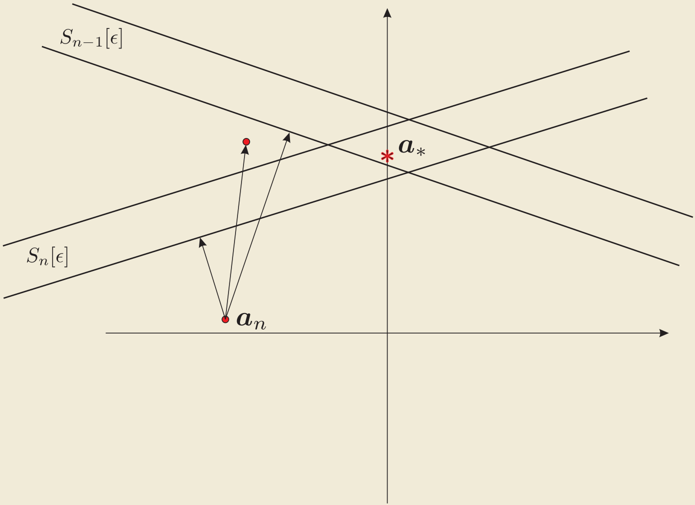
APSM under the weighted ℓ_1 ball constraint

Geometric illustration example



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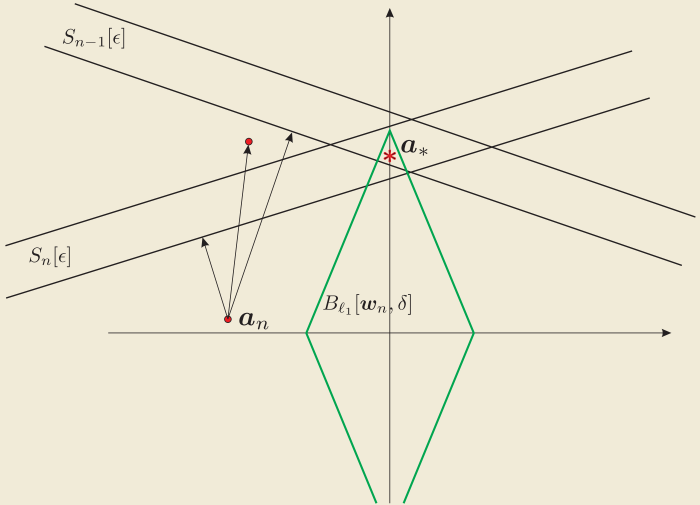

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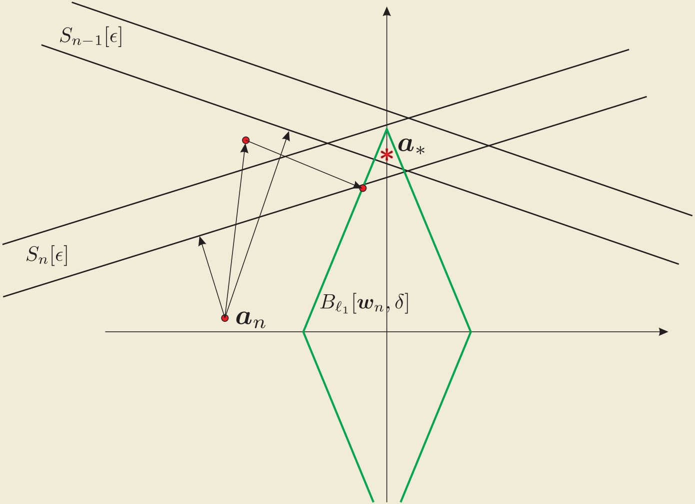

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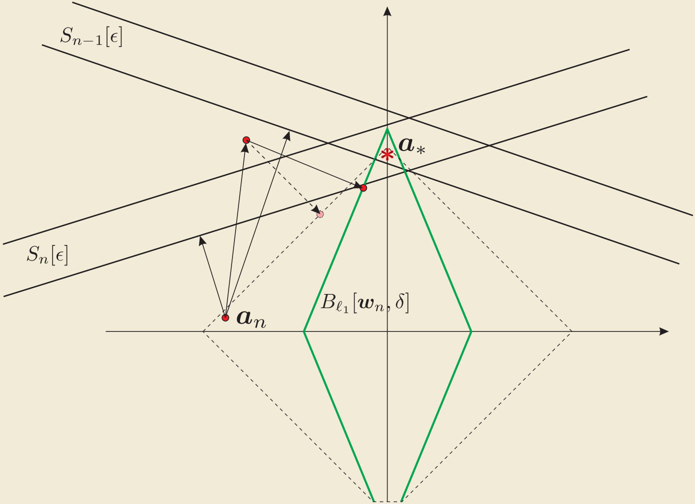


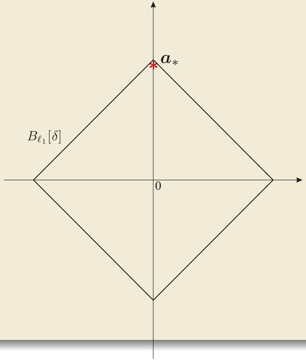
APSM under the weighted ℓ_1 ball constraint

Convergence of the Scheme

- Does this scheme converge?

  Note that our constraint, i.e., the weighted ℓ_1-ball is a time-varying constraint.

  Remark: This case was not covered by the existing theory.



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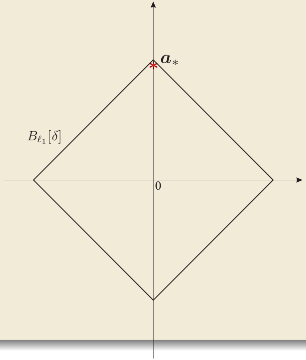

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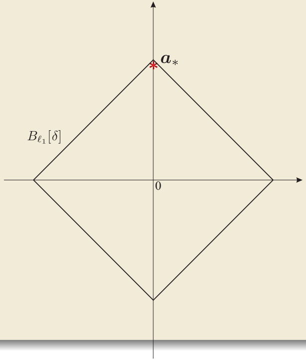

APSM under the weighted ℓ_1 ball constraint

Convergence of the Scheme

- Does this scheme converge?
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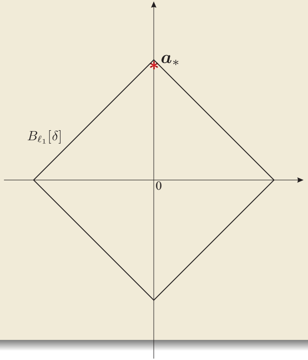

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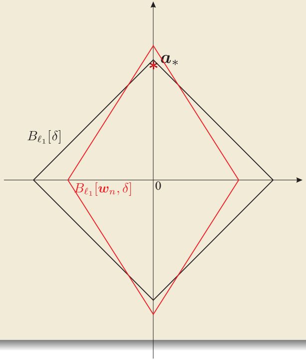

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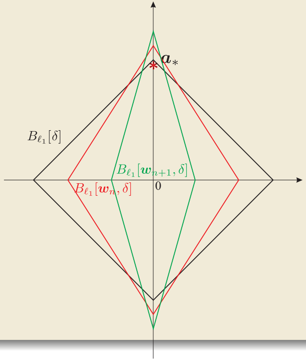
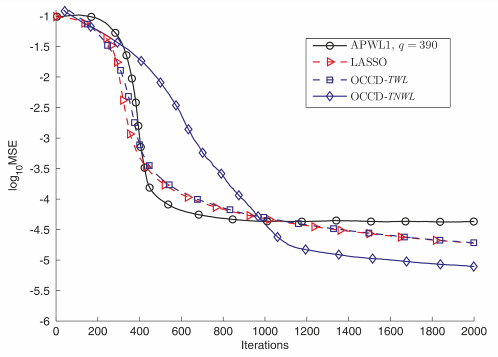


Simulation Examples

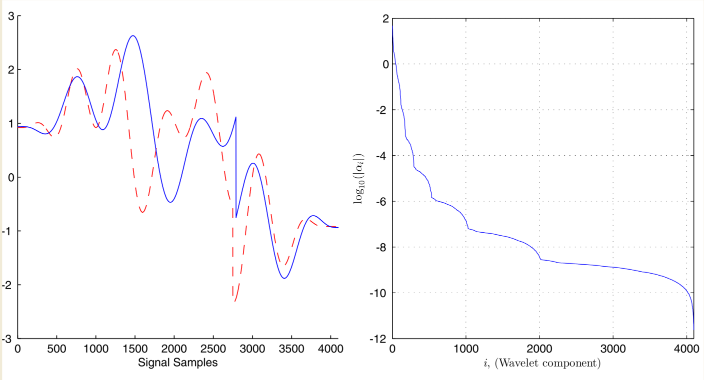
Example: Time-invariant signal sparse in wavelet domain



$L := 1024$, $\|\boldsymbol{a}_*\|_0 := 100$ wavelet coefficients. The radius of the ℓ_1-ball is set to $\delta := 101$.

Simulation Examples

Example: Time varying signal compressible in wavelet domain



$L := 4096.$

The sum of two chirp signals.

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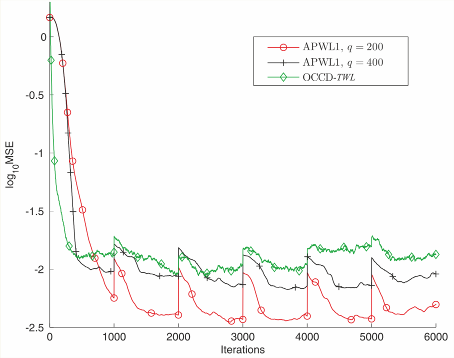

$L := 4096$. The radius of the ℓ_1-ball is set to $\delta := 40$.

Movies of the OCCD, and the APWL1sub.

Generalized Thresholding Rules

Thresholding rules associated with non-convex penalty functions

- Penalized LS thresholding operators:

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- PLSTO basically defines a mapping

$$\tilde{\boldsymbol{a}} \mapsto \min_{\boldsymbol{a}} \frac{1}{2}(\tilde{\boldsymbol{a}} - \boldsymbol{a})^2 + \lambda p(|\boldsymbol{a}|)$$

  which corresponds to a Shrinkage operator.

Generalized Thresholding Rules

Examples: Penalty functions

- $p(|a|) := |a|^\gamma, \ \forall a \in \mathbb{R}$
- $p(|a|) = \lambda \left(1 - e^{-\beta|a|}\right)$
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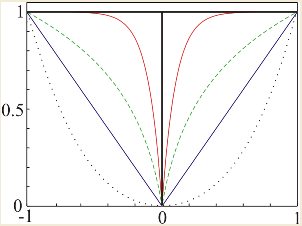

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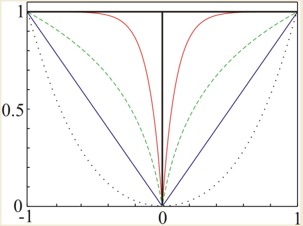
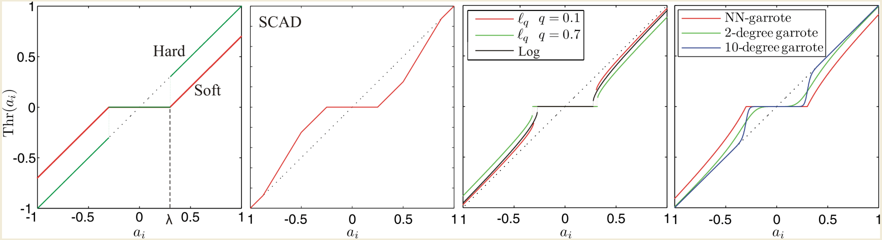


Examples: Penalized Least-Squares Thresholding Operators



Generalized Thresholding Rules

Generalized Thresholding (GT) operator: Definition:

For any $\boldsymbol{a} \in \mathbb{R}^L$, $\boldsymbol{z} := T_{\mathsf{GT}}^{(K)}(\boldsymbol{a})$ is obtained coordinate-wise:

$$\forall l \in \overline{1, L}, \quad z_l := \begin{cases} a_l, & \text{If, } a_l \text{ is one of the largest } K \text{ components,} \\ \mathsf{shr}(a_l), & \text{otherwise} \end{cases}$$

Shrinkage Function (Shr)

- $\tau \mathsf{shr}(\tau) \geq 0$, $\forall \tau \in \mathbb{R}$.
- shr acts as a *strict* shrinkage operator over all intervals which do not include 0.
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In words

- Choose the largest K components of the estimate.
- The rest are shrunk according to the shrinkage rule.

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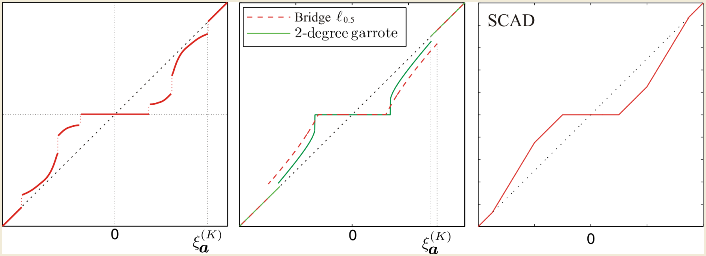
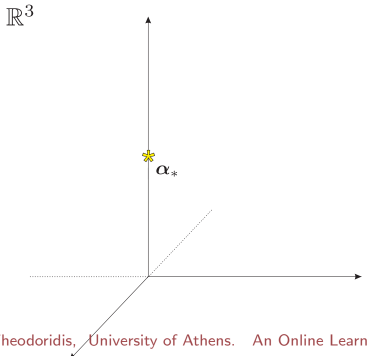


Adaptive Projection-Based Algorithm With Generalized Thresholding (APGT)

The Algorithm

$$\boldsymbol{a}_{n+1} := T_n \left(\boldsymbol{a}_n + \mu_n \left(\sum_{i=n-q+1}^{n} \omega_i^{(n)} \left(P(\boldsymbol{a}_n) - \boldsymbol{a}_n \right) \right) \right)$$

- Each piece of a-priori information, is also represented by a set



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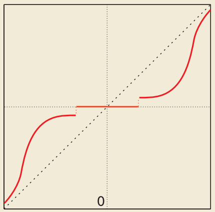

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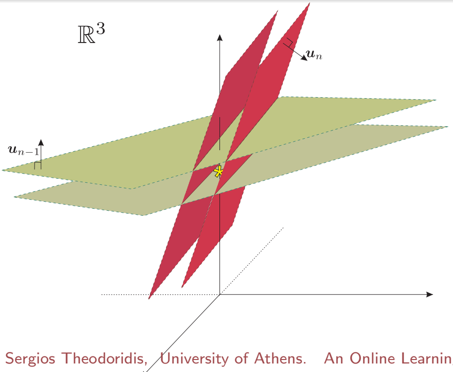

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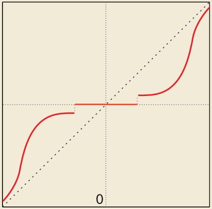

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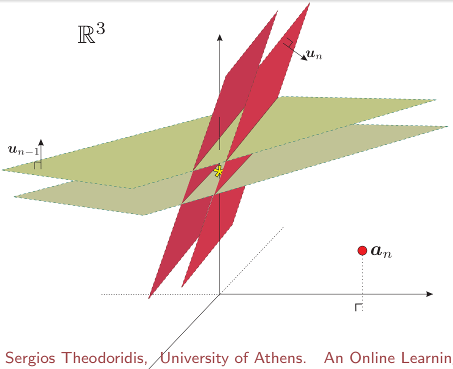

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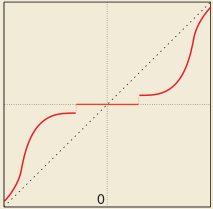

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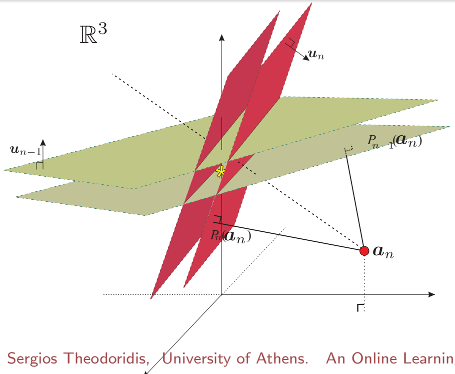

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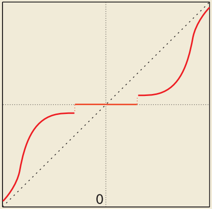

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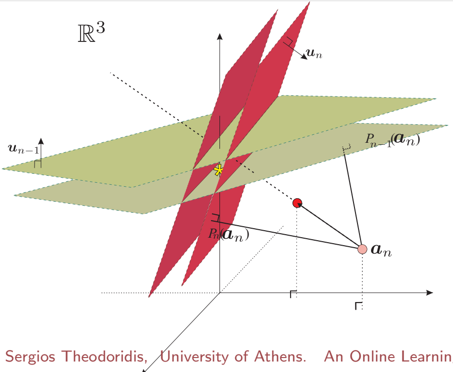

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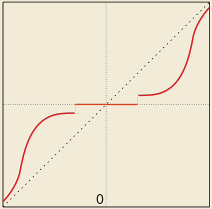

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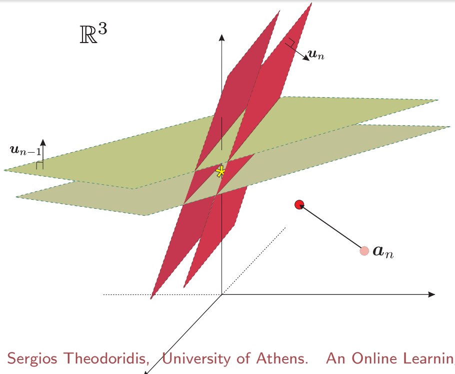

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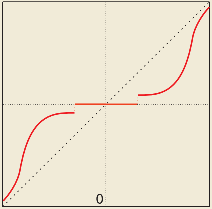

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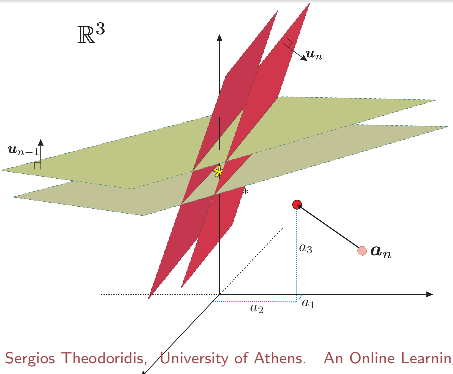

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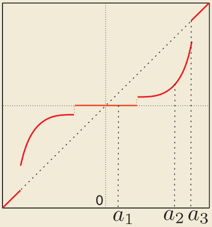

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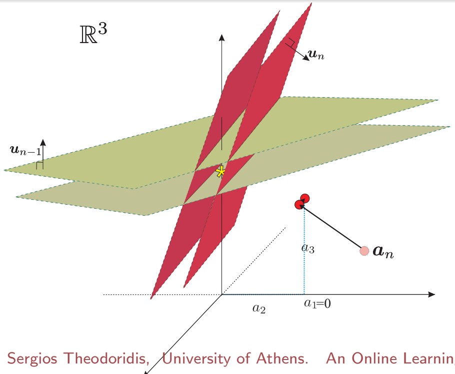

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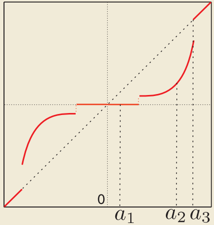

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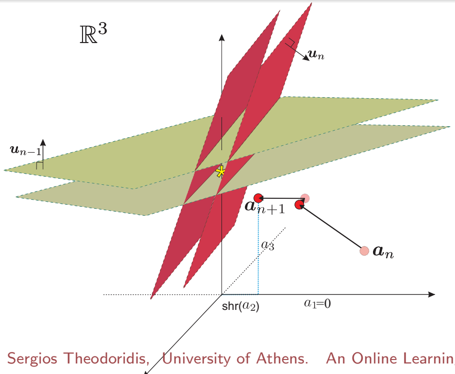

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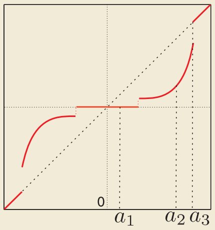

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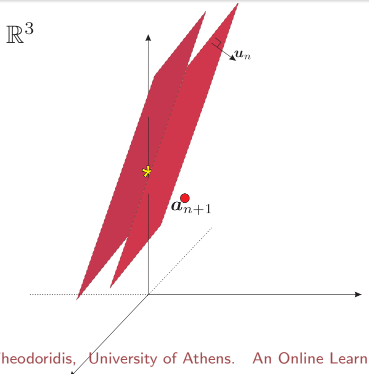

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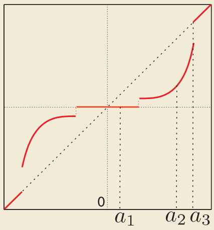

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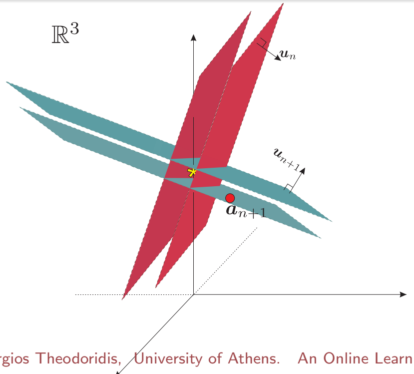

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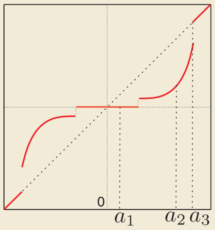

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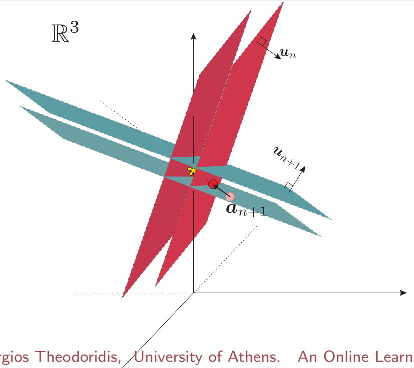

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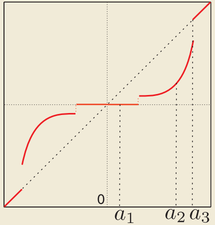

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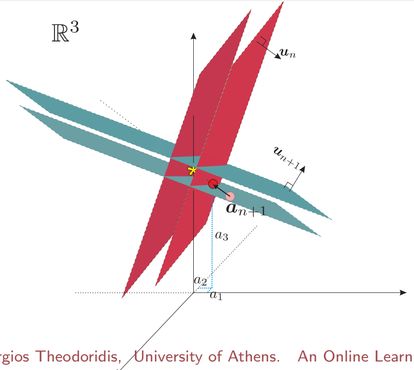

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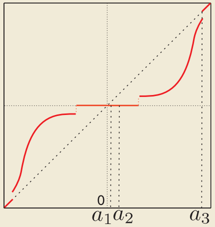

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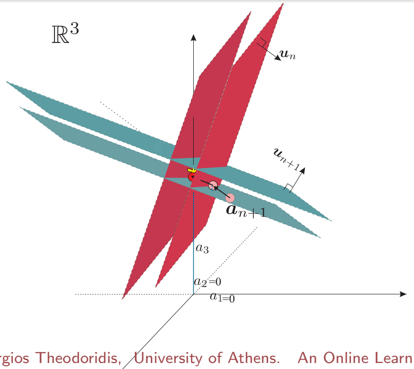

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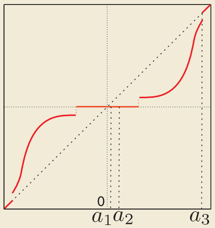

Adaptive Projection-Based Algorithm With GT (APGT)

Convergence of APGT

- Partially Quasi-nonexpansive Mapping.
  $\forall \boldsymbol{x} \in \mathbb{R}^L, \exists Y_{\boldsymbol{x}} \subset \mathrm{Fix}(T) : \forall \boldsymbol{y} \in Y_{\boldsymbol{x}},$
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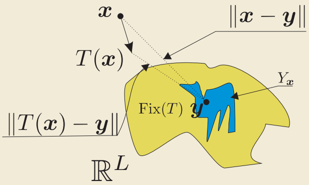

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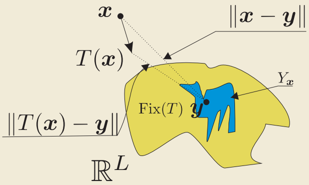
- The fixed point set of GT is a union of subspaces (non-convex).



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  $\forall \boldsymbol{x} \in \mathbb{R}^L, \exists Y_{\boldsymbol{x}} \subset \mathrm{Fix}(T) : \forall \boldsymbol{y} \in Y_{\boldsymbol{x}},$
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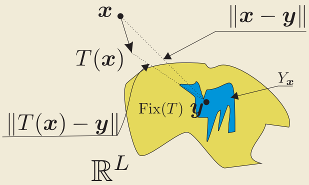
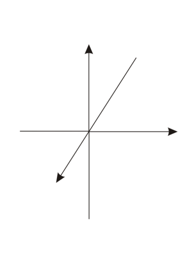
- The fixed point set of GT is a union of subspaces (non-convex).



Examples: Union of Subspaces for $s = 2$



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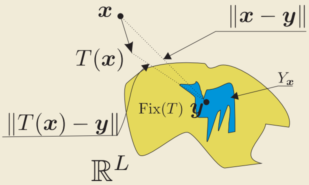

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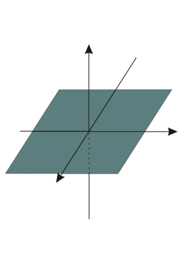

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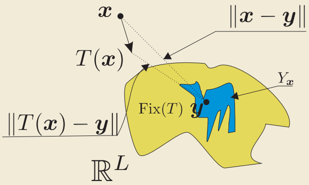

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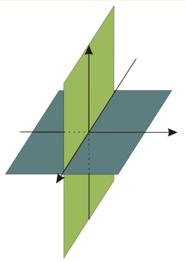

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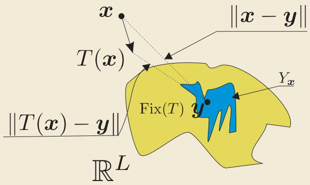

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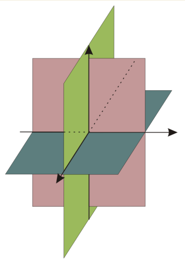

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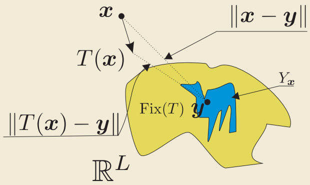

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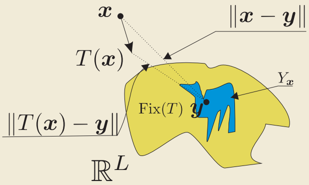

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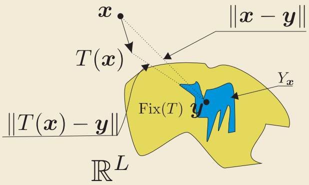

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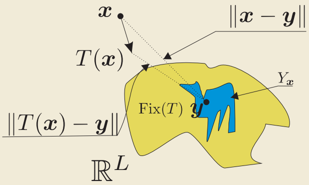

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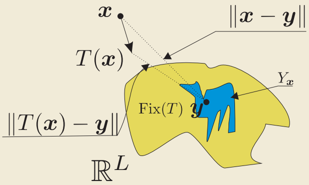

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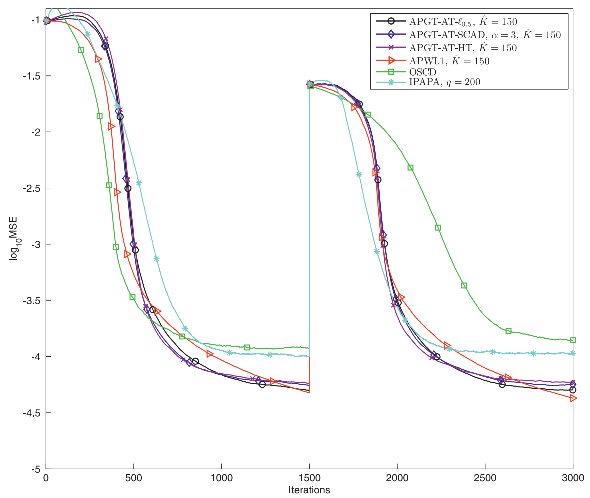
  - the solution is located arbitrarily close to an intersection of an infinite number of hyperslabs.

Example: Time-varying case exhibiting an abrupt change

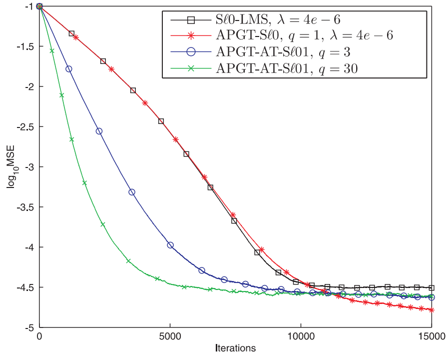


APGT: $\mathcal{O}(qL + qK)$
OSCD: $\mathcal{O}(L^2)$
IPAPA: $\mathcal{O}(q^3)$

$L := 1024$, $s = 100$ (up to $n = 1500$, and $s = 110$ afterwards)

Simulation Examples

Example: Sparse system identification with colored input



$L := 600$, $s = 60$, AR input (cond $\simeq 100$) .

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