Exact Support Recovery for Sparse Spikes Deconvolution

Gabriel Peyré

Joint work with
Vincent Duval & Quentin Denoyelle

DAUPHINE UNIVERSITÉ PARIS

cnrs

www.numerical-tours.com

Sparse Deconvolution

Neural spikes (1D)

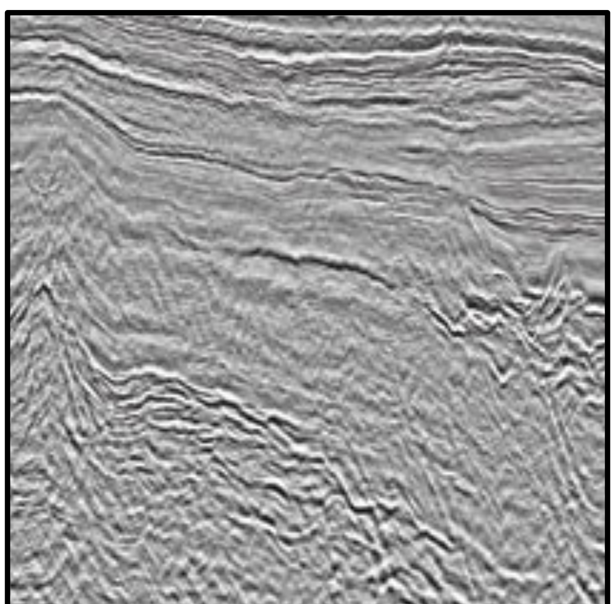
$$y = \varphi \star m_0 + w$$

m_0 is "sparse"

Sparse Deconvolution

Neural spikes (1D)

$$y = \varphi \star m_0 + w$$
m_0 is “sparse”



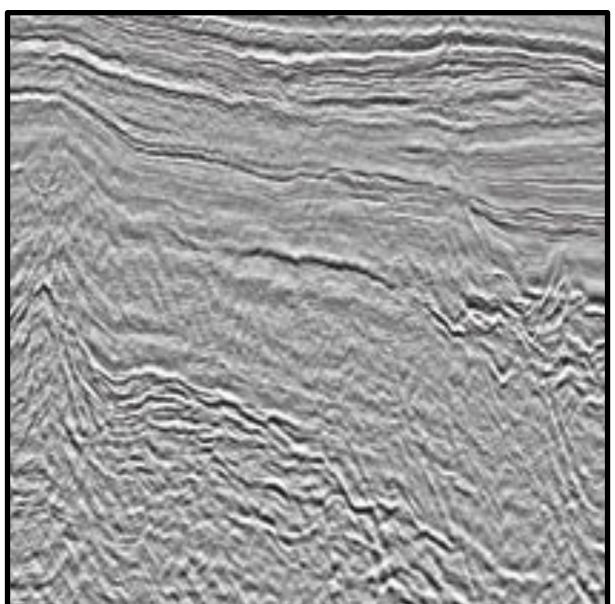
Seismic imaging (1.5D)

Sparse Deconvolution

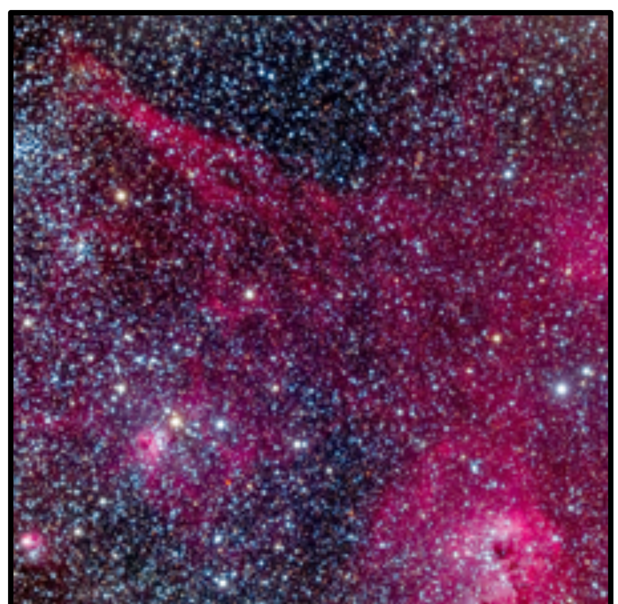
Neural spikes (1D)

$$y = \varphi \star m_0 + w$$

m_0 is "sparse"



Seismic imaging (1.5D)



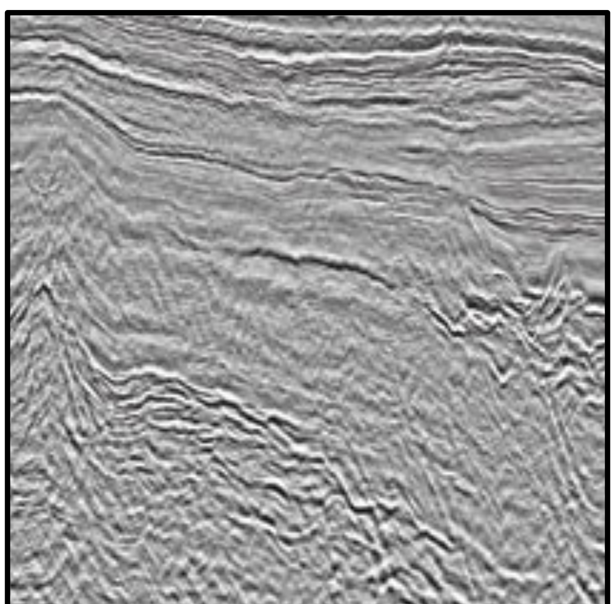
Astrophysics (2D)

Sparse Deconvolution

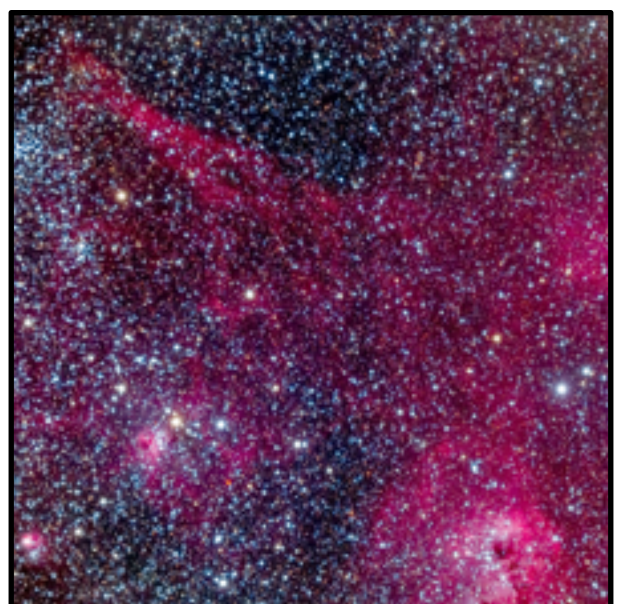
Neural spikes (1D)

$$y = \varphi \star m_0 + w$$

m_0 is "sparse"



Seismic imaging (1.5D)



Astrophysics (2D)

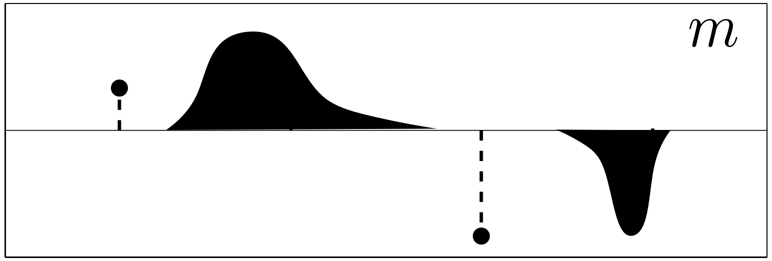
Presented results extend to nD problems

Overview

- Sparse Spikes Super-resolution
- Robust Support Recovery
- Asymptotic Positive Measure Recovery

Deconvolution of Measures

Radon measure m on $\mathbb{T} = (\mathbb{R}/\mathbb{Z})^d$.

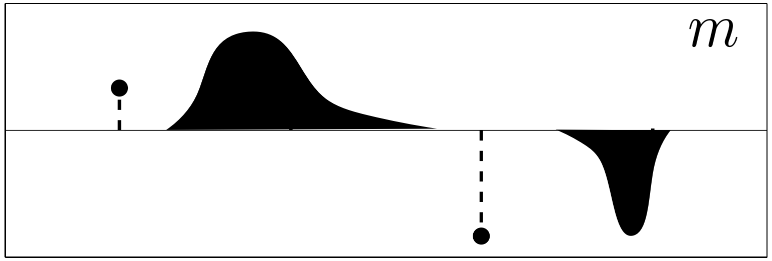


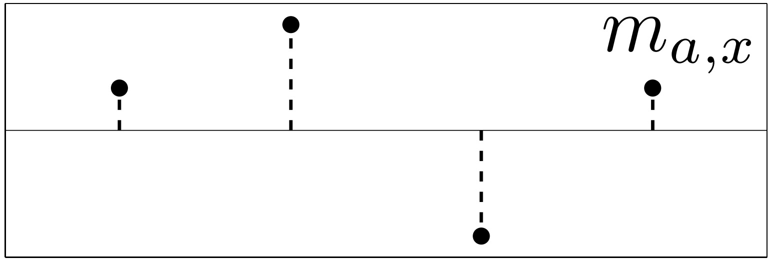
Deconvolution of Measures

Radon measure m on $\mathbb{T} = (\mathbb{R}/\mathbb{Z})^d$.

Discrete measure:

$$m_{a,x} = \sum_{i=1}^N a_i \delta_{x_i}, \ a \in \mathbb{R}^N, x \in \mathbb{T}^N$$



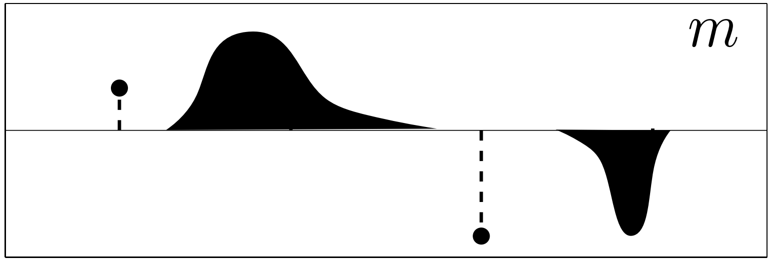


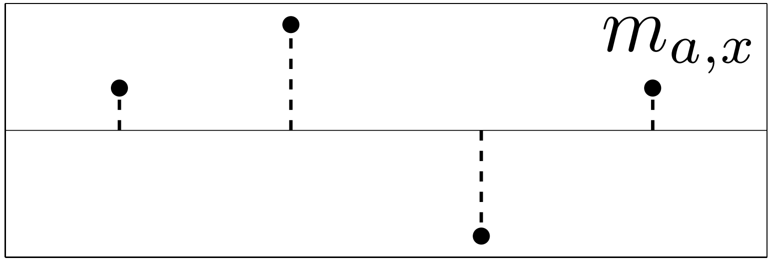
Deconvolution of Measures

Radon measure m on $\mathbb{T} = (\mathbb{R}/\mathbb{Z})^d$.

Discrete measure:

$$m_{a,x} = \sum_{i=1}^N a_i \delta_{x_i}, \; a \in \mathbb{R}^N, x \in \mathbb{T}^N$$





Linear measurements:

$$y = \Phi(m) + w \qquad \varphi \in C^2(\mathbb{T} \times \mathbb{T})$$

$$\Phi(m) = \int_{\mathbb{T}} \varphi(x, \cdot) \mathrm{d}m(x)$$

Deconvolution of Measures

Radon measure m on $\mathbb{T} = (\mathbb{R}/\mathbb{Z})^d$.

Discrete measure:

$$m_{a,x} = \sum_{i=1}^{N} a_i \delta_{x_i}, \; a \in \mathbb{R}^N, x \in \mathbb{T}^N$$

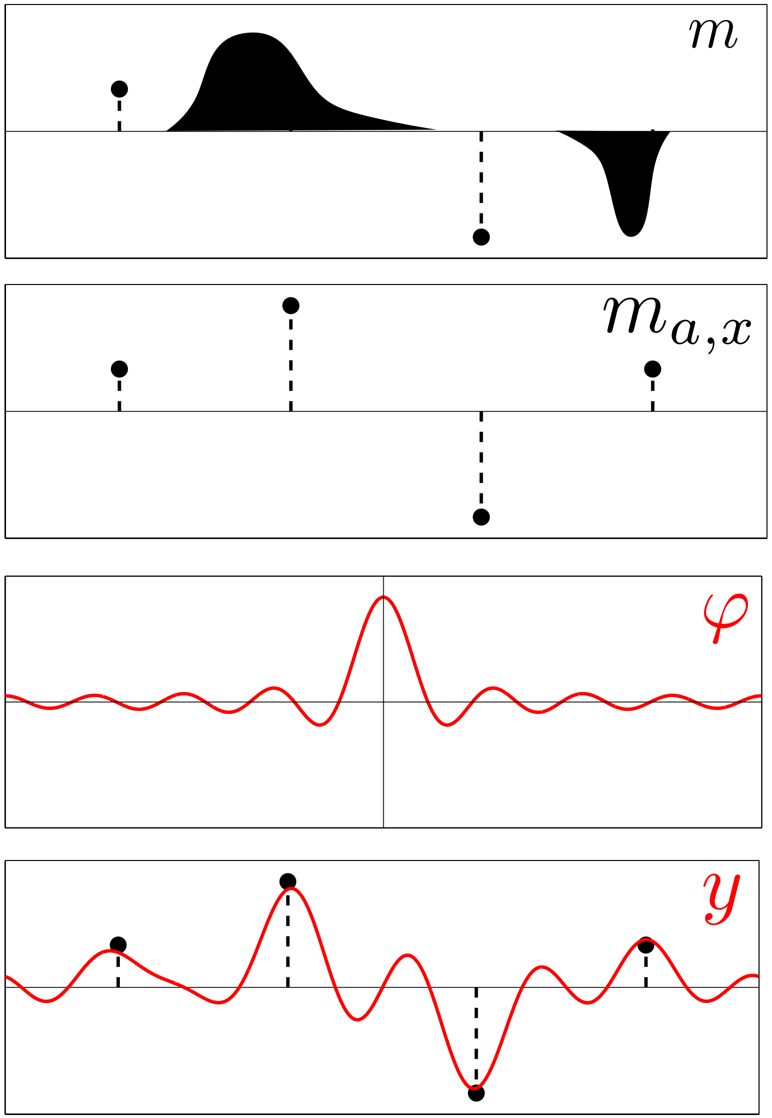
Linear measurements:

$$y = \Phi(m) + w \qquad \varphi \in C^2(\mathbb{T} \times \mathbb{T})$$

$$\Phi(m) = \int_{\mathbb{T}} \varphi(x, \cdot) \mathrm{d}m(x)$$

Example: 1-D ($d = 1$) convolution

$$\varphi(x, t) = \varphi(t - x)$$



Deconvolution of Measures

Radon measure m on $\mathbb{T} = (\mathbb{R}/\mathbb{Z})^d$.

Discrete measure:

$$m_{a,x} = \sum_{i=1}^N a_i \delta_{x_i}, \; a \in \mathbb{R}^N, x \in \mathbb{T}^N$$

Linear measurements:

$$y = \Phi(m) + w \qquad \varphi \in C^2(\mathbb{T} \times \mathbb{T})$$

$$\Phi(m) = \int_{\mathbb{T}} \varphi(x, \cdot) \mathrm{d}m(x)$$

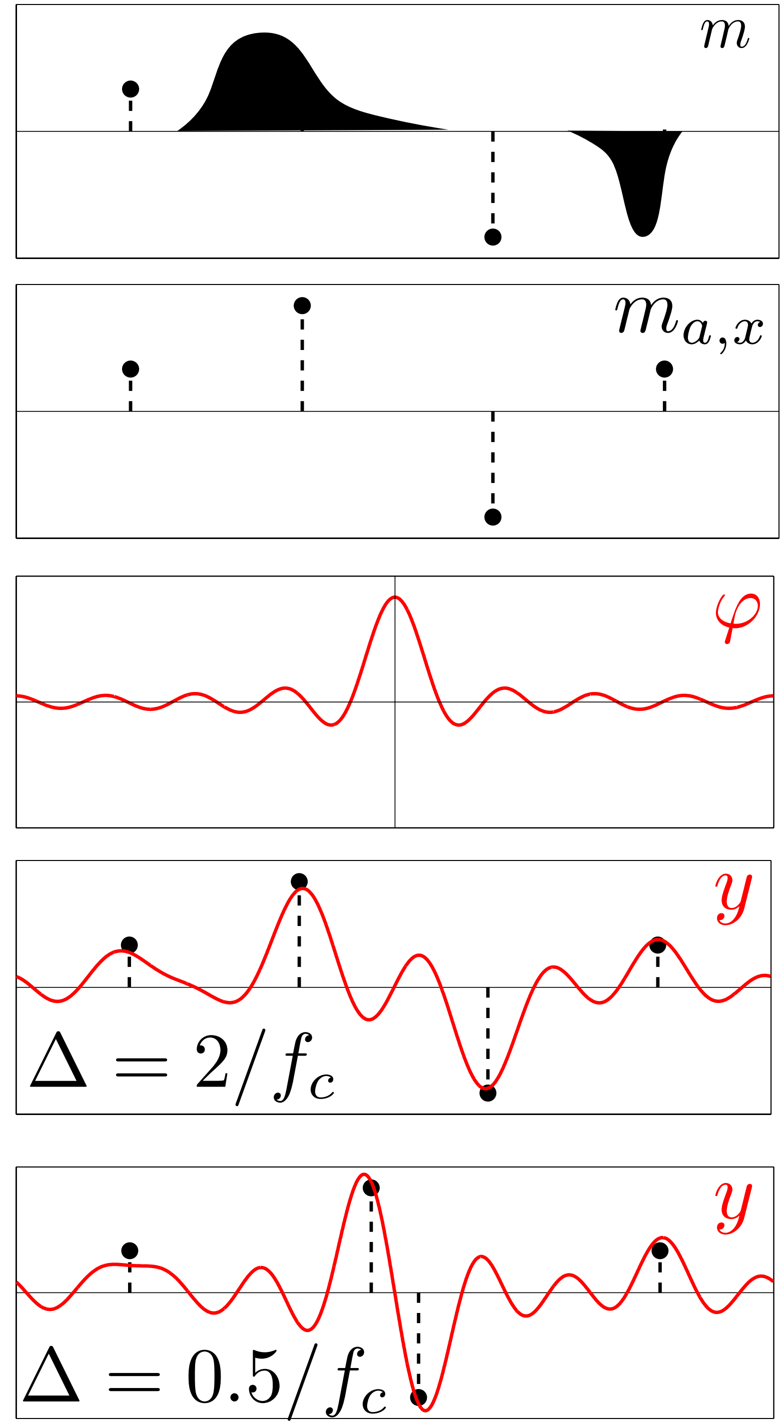
Example: 1-D ($d = 1$) convolution

$$\varphi(x,t) = \varphi(t-x)$$

Minimum separation:

$$\Delta = \min_{i \neq j} |x_i - x_j|$$

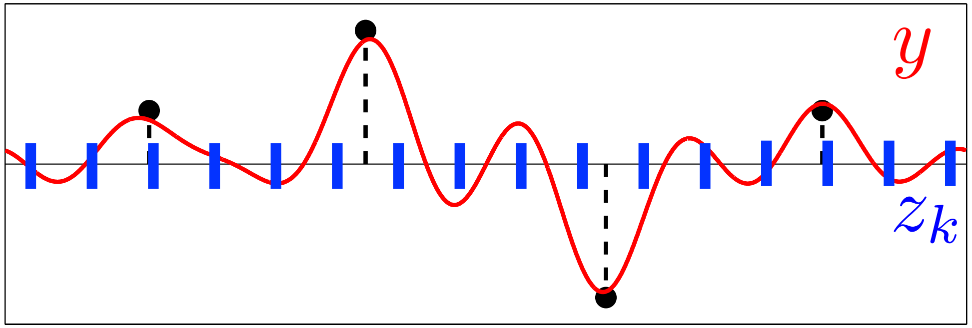
$\rightarrow$ Signal-dependent recovery criteria.



Sparse l^1 Deconvolution

Discrete ℓ^1 regularization:

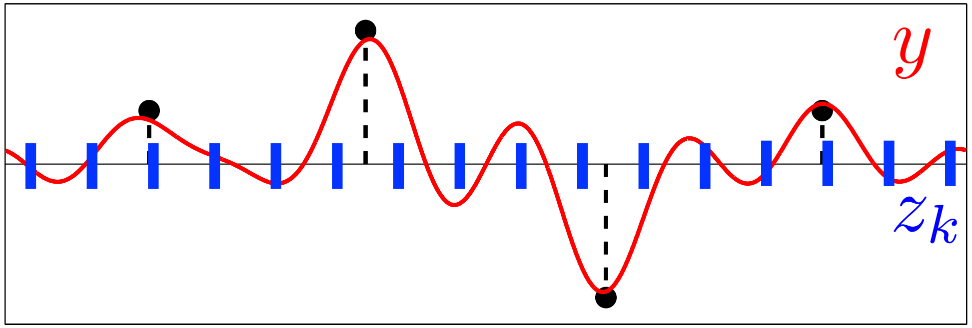
Computation grid $z = (z_k)_{k=1}^K$.



Sparse l^1 Deconvolution

Discrete ℓ^1 regularization:

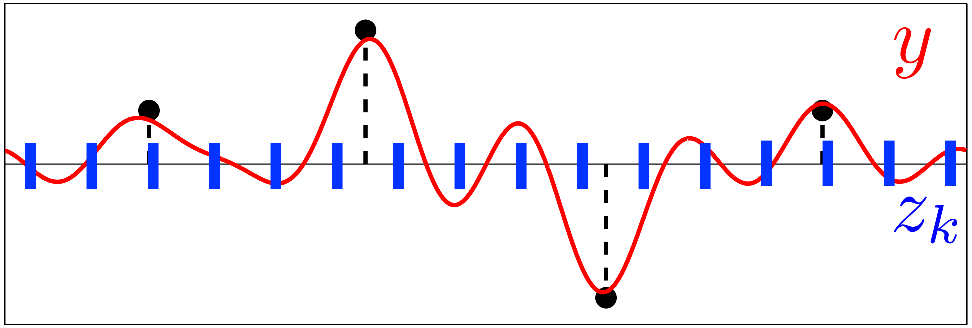
Computation grid $z = (z_k)_{k=1}^K$.



Basis-pursuit / Lasso: $\min_{a \in \mathbb{R}^N} \frac{1}{2}\|y - \bar{\Phi} a\|^2 + \lambda \|a\|_1$

$$\bar{\Phi} : a \in \mathbb{R}^K \mapsto \Phi(m_{a,z}) = \sum_k a_k \varphi(z_k, \cdot) \in \mathrm{Im}(\Phi)$$

Sparse l^1 Deconvolution



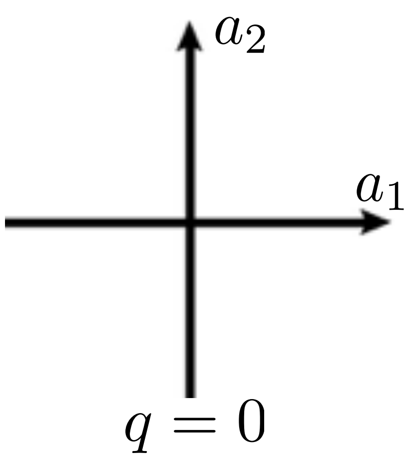
Discrete ℓ^1 regularization:

Computation grid $z = (z_k)_{k=1}^K$.

Basis-pursuit / Lasso: $$\min_{a \in \mathbb{R}^N} \frac{1}{2}\|y - \bar{\Phi} a\|^2 + \lambda \|a\|_1$$

$$\bar{\Phi} : a \in \mathbb{R}^K \mapsto \Phi(m_{a,z}) = \sum_k a_k \varphi(z_k, \cdot) \in \mathrm{Im}(\Phi)$$

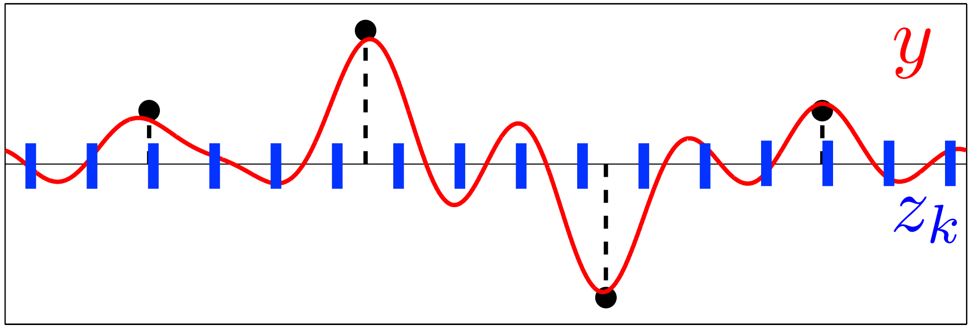
Why ℓ^1 *?* "ℓ^0 ball"



Sparse l^1 Deconvolution

Discrete ℓ^1 regularization:

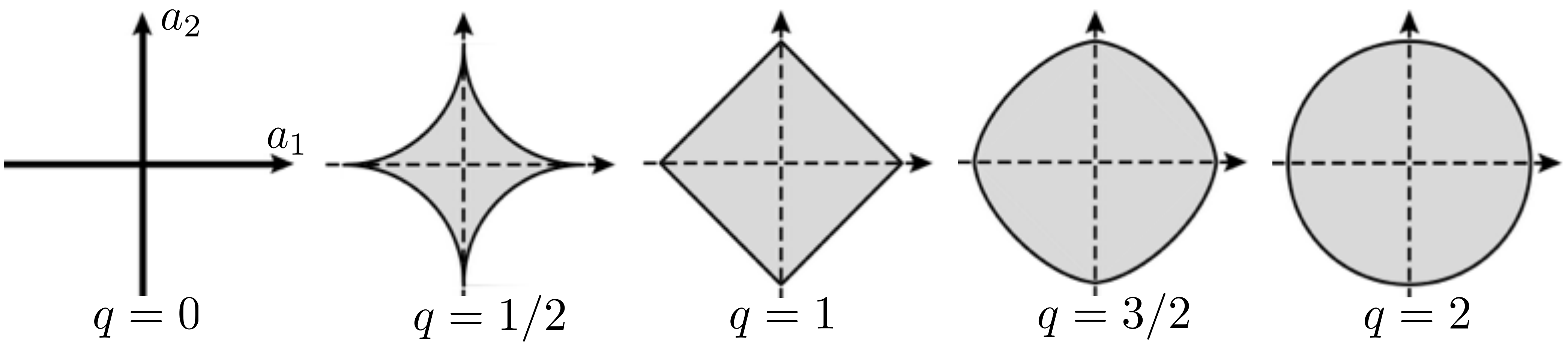
Computation grid $z = (z_k)_{k=1}^K$.



Basis-pursuit / Lasso: $\min_{a \in \mathbb{R}^N} \frac{1}{2}\|y - \bar{\Phi} a\|^2 + \lambda \|a\|_1$

$$\bar{\Phi} : a \in \mathbb{R}^K \mapsto \Phi(m_{a,z}) = \sum_k a_k \varphi(z_k, \cdot) \in \mathrm{Im}(\Phi)$$

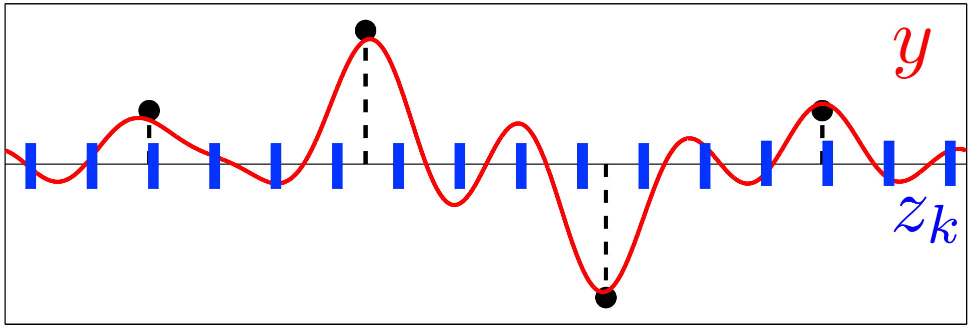
Why ℓ^1*?* “ℓ^0 ball” $\longrightarrow$ ℓ^q ball $\{a \in \mathbb{R}^K \; ; \; \sum_k |a_k|^q \leqslant 1\}$



Sparse l^1 Deconvolution

Discrete ℓ^1 regularization:

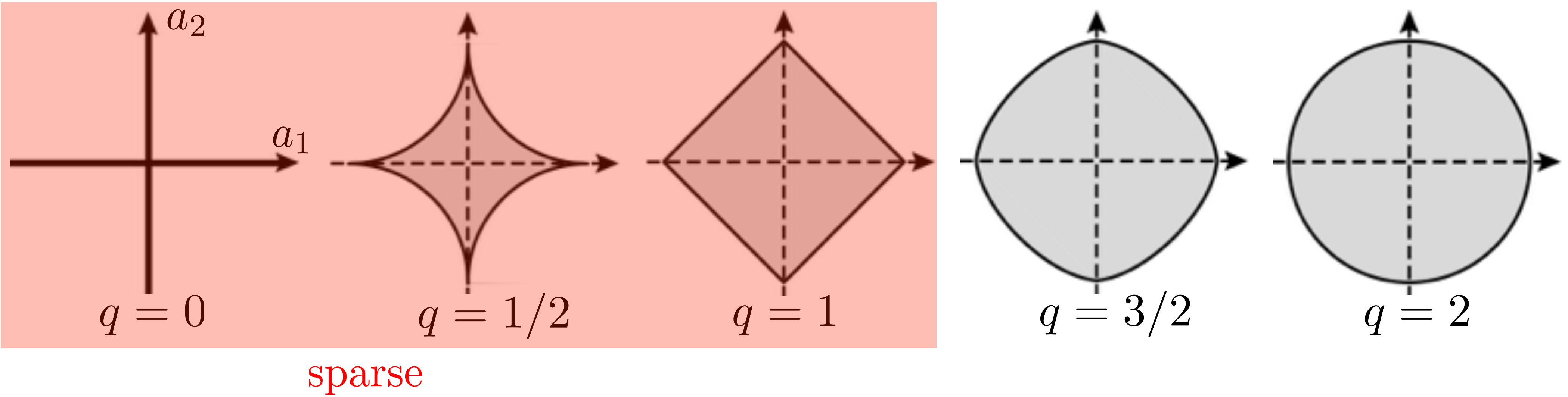
Computation grid $z = (z_k)_{k=1}^K$.



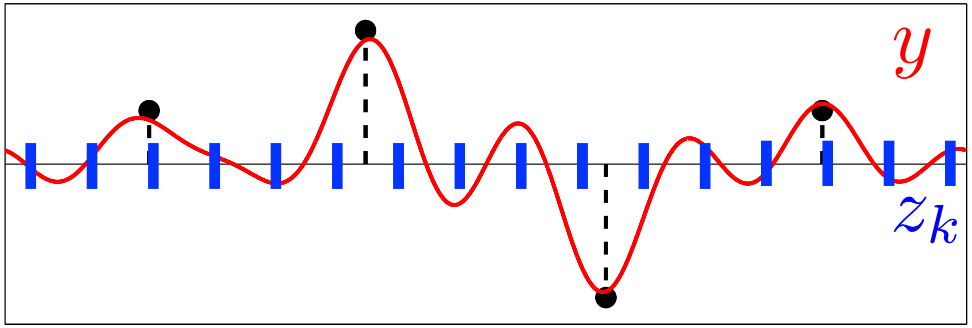
Basis-pursuit / Lasso: $\min_{a \in \mathbb{R}^N} \frac{1}{2}\|y - \bar{\Phi}a\|^2 + \lambda \|a\|_1$

$$\bar{\Phi} : a \in \mathbb{R}^K \mapsto \Phi(m_{a,z}) = \sum_k a_k \varphi(z_k, \cdot) \in \mathrm{Im}(\Phi)$$

Why ℓ^1 ? "ℓ^0 ball" $\longrightarrow$ ℓ^q ball $\{a \in \mathbb{R}^K \; ; \; \sum_k |a_k|^q \leqslant 1\}$



Sparse l¹ Deconvolution



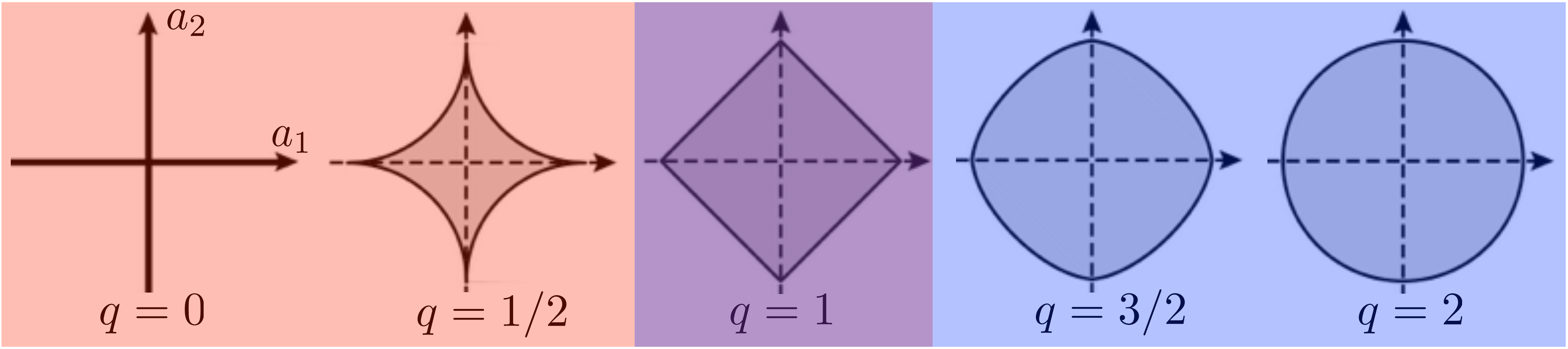
Discrete ℓ^1 regularization:

Computation grid $z = (z_k)_{k=1}^K$.

Basis-pursuit / Lasso: $$\min_{a \in \mathbb{R}^N} \frac{1}{2}\|y - \bar{\Phi} a\|^2 + \lambda \|a\|_1$$

$$\bar{\Phi} : a \in \mathbb{R}^K \mapsto \Phi(m_{a,z}) = \sum_k a_k \varphi(z_k, \cdot) \in \mathrm{Im}(\Phi)$$

Why ℓ^1 *?* "ℓ^0 ball" $\longrightarrow$ ℓ^q ball $\{a \in \mathbb{R}^K \,;\, \sum_k |a_k|^q \leqslant 1\}$



sparse convex

Grid-free Sparse Recovery

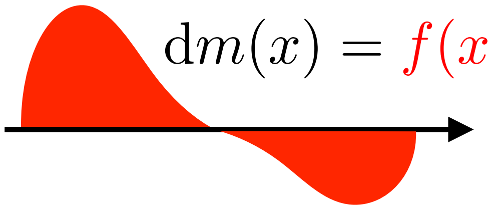
Grid-free regularization: total variation of measures:

$$|m|(\mathbb{T}) = \sup \left\{ \int \eta \mathrm{d}m \ : \ \eta \in C(\mathbb{T}), \|\eta\|_\infty \leqslant 1 \right\}$$

Grid-free Sparse Recovery

Grid-free regularization: total variation of measures:

$$|m|(\mathbb{T}) = \sup \left\{ \int \eta \mathrm{d}m \; : \; \eta \in C(\mathbb{T}), \|\eta\|_\infty \leqslant 1 \right\}$$



$\mathrm{d}m(x) = f(x)\mathrm{d}x$

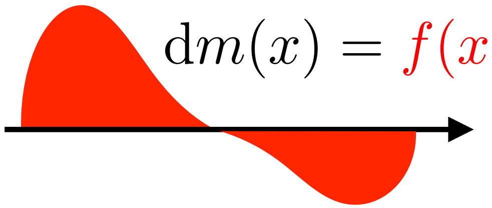
$|m|(\mathbb{T}) = \int |f| = \|f\|_{L^1}$

Grid-free Sparse Recovery

Grid-free regularization: total variation of measures:

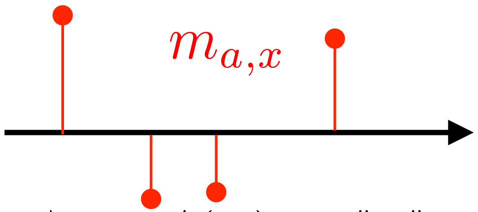
$$|m|(\mathbb{T}) = \sup \left\{ \int \eta \mathrm{d}m \ : \ \eta \in C(\mathbb{T}), \|\eta\|_\infty \leqslant 1 \right\}$$

$\mathrm{d}m(x) = f(x)\mathrm{d}x$



$|m|(\mathbb{T}) = \int |f| = \|f\|_{L^1}$

$m_{a,x}$

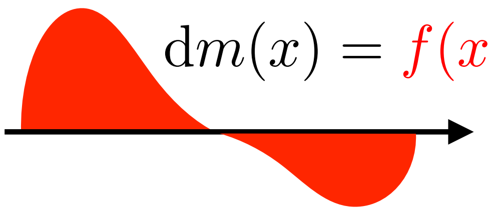


$|m_{a,x}|(\mathbb{T}) = \|a\|_{\ell^1}$

Grid-free Sparse Recovery

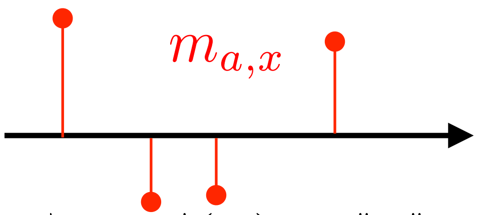
Grid-free regularization: total variation of measures:

$$|m|(\mathbb{T}) = \sup \left\{ \int \eta \mathrm{d}m \;:\; \eta \in C(\mathbb{T}), \|\eta\|_\infty \leqslant 1 \right\}$$



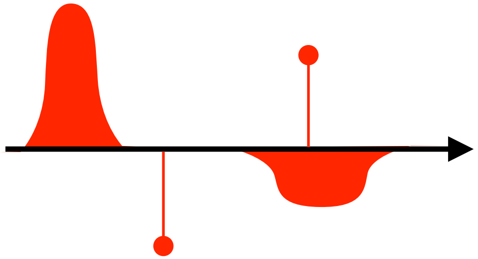
$\mathrm{d}m(x) = f(x)\mathrm{d}x$

$|m|(\mathbb{T}) = \int |f| = \|f\|_{L^1}$



$m_{a,x}$

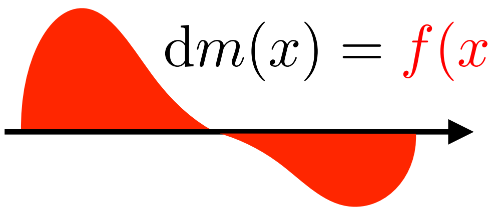
$|m_{a,x}|(\mathbb{T}) = \|a\|_{\ell^1}$



Grid-free Sparse Recovery

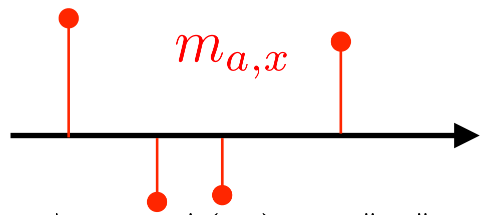
Grid-free regularization: total variation of measures:

$$|m|(\mathbb{T}) = \sup\left\{ \int \eta \mathrm{d}m \;:\; \eta \in C(\mathbb{T}), \|\eta\|_\infty \leqslant 1 \right\}$$



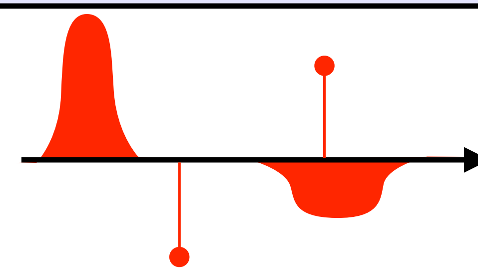
$\mathrm{d}m(x) = f(x)\mathrm{d}x$

$|m|(\mathbb{T}) = \int |f| = \|f\|_{L^1}$



$m_{a,x}$

$|m_{a,x}|(\mathbb{T}) = \|a\|_{\ell^1}$

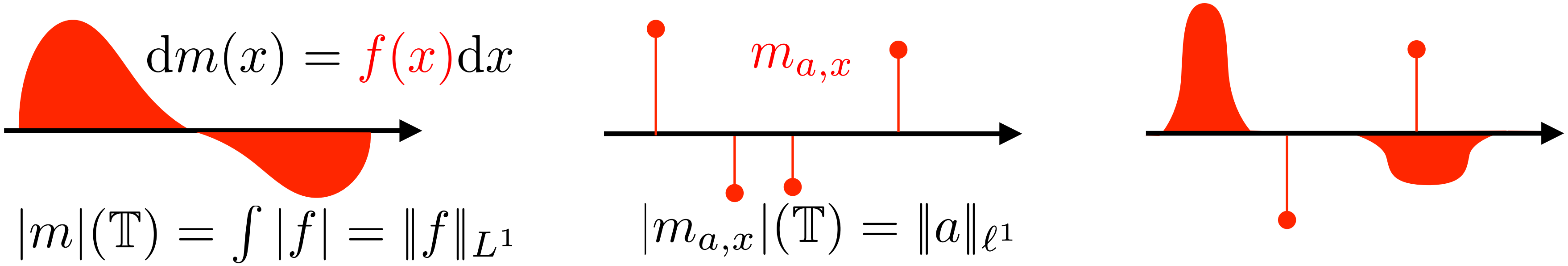


Sparse recovery:
$$\min_m \frac{1}{2}\|\Phi(m) - y\|^2 + \lambda |m|(\mathbb{T}) \qquad (\mathcal{P}_\lambda(y))$$

Grid-free Sparse Recovery

Grid-free regularization: total variation of measures:

$$|m|(\mathbb{T}) = \sup\left\{\int \eta \mathrm{d}m \ : \ \eta \in C(\mathbb{T}), \|\eta\|_\infty \leqslant 1\right\}$$



Sparse recovery:

$$\min_m \frac{1}{2}\|\Phi(m) - y\|^2 + \lambda |m|(\mathbb{T}) \qquad (\mathcal{P}_\lambda(y))$$

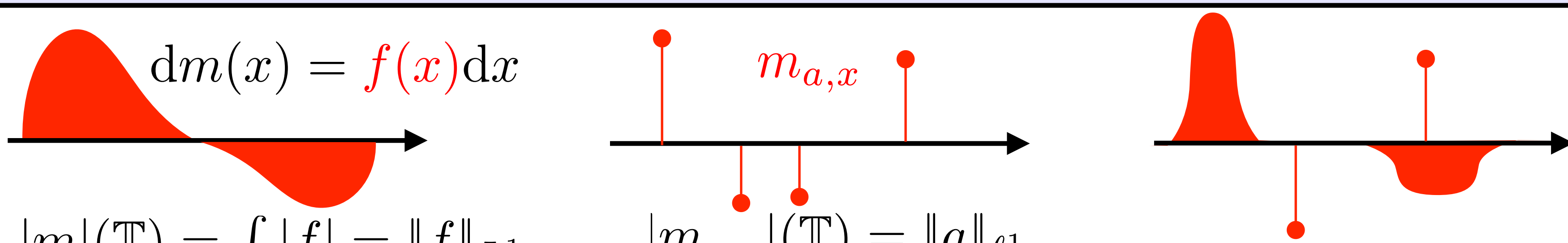
$\lambda \to 0^+$

$$\min_m \{|m|(\mathbb{T}) \ ; \ \Phi m = y\} \qquad (\mathcal{P}_0(y))$$

Grid-free Sparse Recovery

Grid-free regularization: total variation of measures:

$$|m|(\mathbb{T}) = \sup \left\{ \int \eta \mathrm{d}m \;:\; \eta \in C(\mathbb{T}), \|\eta\|_\infty \leqslant 1 \right\}$$



Sparse recovery:

$$\min_m \frac{1}{2}\|\Phi(m) - y\|^2 + \lambda |m|(\mathbb{T}) \qquad (\mathcal{P}_\lambda(y))$$

$\lambda \to 0^+$

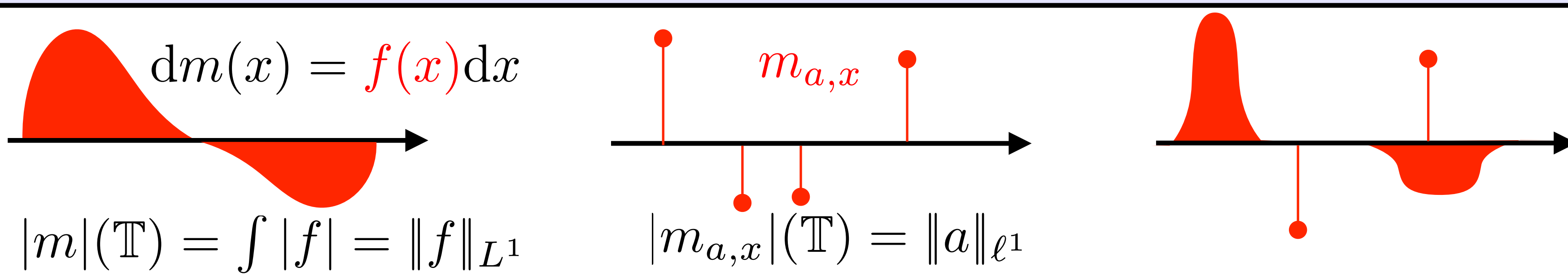
$$\min_m \; \{|m|(\mathbb{T}) \;;\; \Phi m = y\} \qquad (\mathcal{P}_0(y))$$

Proposition: If $\dim(\mathrm{Im}(\Phi)) < +\infty$, $\exists (a, x) \in \mathbb{R}^N \times \mathbb{T}^N$ with $N \leqslant \dim(\mathrm{Im}(\Phi))$ such that $m_{a,x}$ is a solution to $\mathcal{P}_\lambda(y)$.

Grid-free Sparse Recovery

Grid-free regularization: total variation of measures:

$$|m|(\mathbb{T}) = \sup \left\{ \int \eta \mathrm{d}m \ : \ \eta \in C(\mathbb{T}), \|\eta\|_\infty \leqslant 1 \right\}$$



Sparse recovery:

$$\min_m \frac{1}{2}\|\Phi(m) - y\|^2 + \lambda |m|(\mathbb{T}) \qquad (\mathcal{P}_\lambda(y))$$

$\lambda \to 0^+$

$$\min_m \ \{|m|(\mathbb{T}) \ ; \ \Phi m = y\} \qquad (\mathcal{P}_0(y))$$

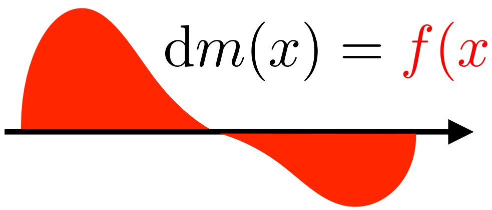
Proposition: If $\dim(\mathrm{Im}(\Phi)) < +\infty$, $\exists (a, x) \in \mathbb{R}^N \times \mathbb{T}^N$ with $N \leqslant \dim(\mathrm{Im}(\Phi))$ such that $m_{a,x}$ is a solution to $\mathcal{P}_\lambda(y)$.

$\longrightarrow$ Algorithms: [Bredies, Pikkarainen, 2010] (proximal-based)
[Candès, Fernandez-G. 2012] (root finding)

Grid-free Sparse Recovery

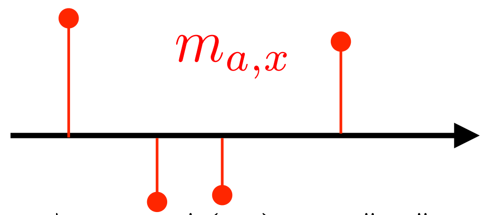
Grid-free regularization: total variation of measures:

$$|m|(\mathbb{T}) = \sup \left\{ \int \eta \mathrm{d}m \ : \ \eta \in C(\mathbb{T}), \|\eta\|_\infty \leqslant 1 \right\}$$

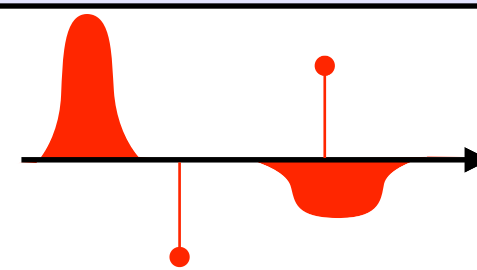


$\mathrm{d}m(x) = f(x)\mathrm{d}x$

$|m|(\mathbb{T}) = \int |f| = \|f\|_{L^1}$



$|m_{a,x}|(\mathbb{T}) = \|a\|_{\ell^1}$



Sparse recovery:

$$\min_m \frac{1}{2}\|\Phi(m) - y\|^2 + \lambda |m|(\mathbb{T}) \qquad (\mathcal{P}_\lambda(y))$$

$\lambda \to 0^+$

$$\min_m \ \{|m|(\mathbb{T}) \ ; \ \Phi m = y\} \qquad (\mathcal{P}_0(y))$$

Proposition: If $\dim(\mathrm{Im}(\Phi)) < +\infty$, $\exists (a, x) \in \mathbb{R}^N \times \mathbb{T}^N$ with $N \leqslant \dim(\mathrm{Im}(\Phi))$ such that $m_{a,x}$ is a solution to $\mathcal{P}_\lambda(y)$.

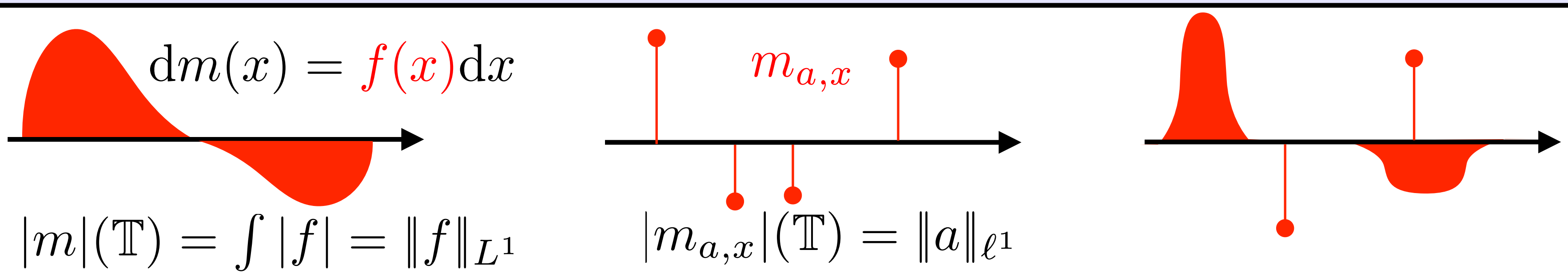
$\longrightarrow$ Algorithms: [Bredies, Pikkarainen, 2010] (proximal-based)
[Candès, Fernandez-G. 2012] (root finding)

Competitors: Prony's methods (MUSIC, ESPRIT, FRI).

Grid-free Sparse Recovery

Grid-free regularization: total variation of measures:

$$|m|(\mathbb{T}) = \sup\left\{ \int \eta \mathrm{d}m \;:\; \eta \in C(\mathbb{T}), \|\eta\|_\infty \leqslant 1 \right\}$$



Sparse recovery:

$$\min_m \frac{1}{2}\|\Phi(m) - y\|^2 + \lambda |m|(\mathbb{T}) \qquad (\mathcal{P}_\lambda(y))$$

$\lambda \to 0^+$

$$\min_m \{|m|(\mathbb{T}) \;;\; \Phi m = y\} \qquad (\mathcal{P}_0(y))$$

Proposition: If $\dim(\mathrm{Im}(\Phi)) < +\infty$, $\exists (a, x) \in \mathbb{R}^N \times \mathbb{T}^N$ with $N \leqslant \dim(\mathrm{Im}(\Phi))$ such that $m_{a,x}$ is a solution to $\mathcal{P}_\lambda(y)$.

$\longrightarrow$ Algorithms: [Bredies, Pikkarainen, 2010] (proximal-based)
[Candès, Fernandez-G. 2012] (root finding)

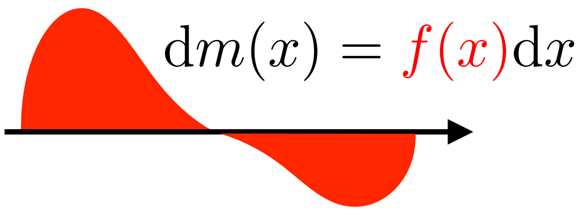
Competitors: Prony's methods (MUSIC, ESPRIT, FRI).

"+": always works when $w = 0$, less sensitive to sign.

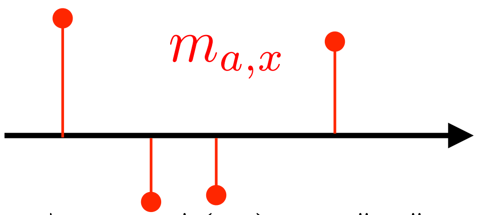
Grid-free Sparse Recovery

Grid-free regularization: total variation of measures:

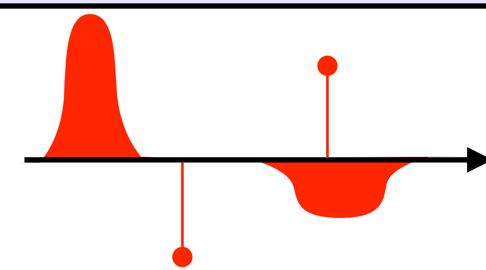
$$|m|(\mathbb{T}) = \sup \left\{ \int \eta \mathrm{d}m \;\; : \;\; \eta \in C(\mathbb{T}), \|\eta\|_\infty \leqslant 1 \right\}$$



$|m|(\mathbb{T}) = \int |f| = \|f\|_{L^1}$



$|m_{a,x}|(\mathbb{T}) = \|a\|_{\ell^1}$



Sparse recovery:

$$\min_m \frac{1}{2}\|\Phi(m) - y\|^2 + \lambda |m|(\mathbb{T}) \qquad (\mathcal{P}_\lambda(y))$$

$\lambda \to 0^+$

$$\min_m \; \{|m|(\mathbb{T}) \; ; \; \Phi m = y\} \qquad (\mathcal{P}_0(y))$$

Proposition: If $\dim(\mathrm{Im}(\Phi)) < +\infty$, $\exists (a, x) \in \mathbb{R}^N \times \mathbb{T}^N$ with $N \leqslant \dim(\mathrm{Im}(\Phi))$ such that $m_{a,x}$ is a solution to $\mathcal{P}_\lambda(y)$.

$\longrightarrow$ Algorithms: [Bredies, Pikkarainen, 2010] (proximal-based)
[Candès, Fernandez-G. 2012] (root finding)

Competitors: Prony's methods (MUSIC, ESPRIT, FRI).

"+": always works when $w = 0$, less sensitive to sign.

"-": only for convolution operator, $\varphi(x, t) = \varphi(x - t)$

Overview

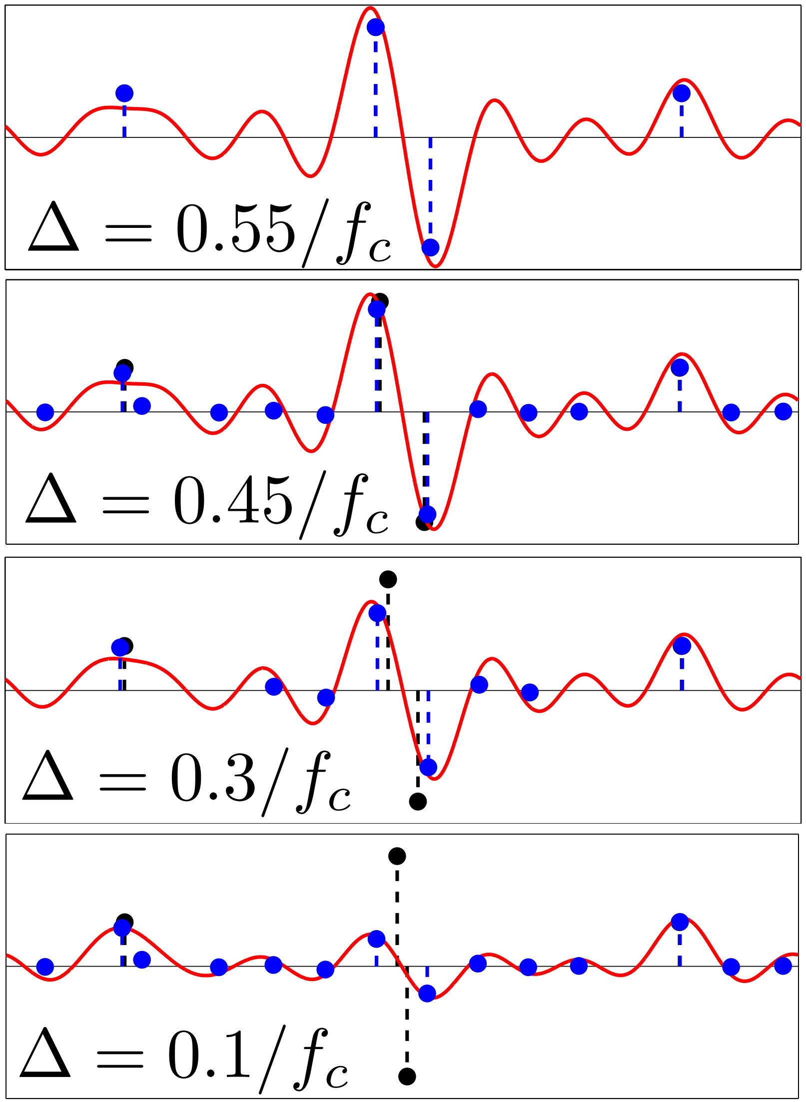
- Sparse Spikes Super-resolution
- **Robust Support Recovery**
- Asymptotic Positive Measure Recovery

Robustness and Support-stability

$$\min_m \{|m|(\mathbb{T}) \;;\; \Phi m = y\} \quad (\mathcal{P}_0(y))$$

Low-pass filter $\text{supp}(\hat{\varphi}) = [-f_c, f_c]$.

When is m_0 solution of $\mathcal{P}_0(\Phi m_0)$?



Robustness and Support-stability

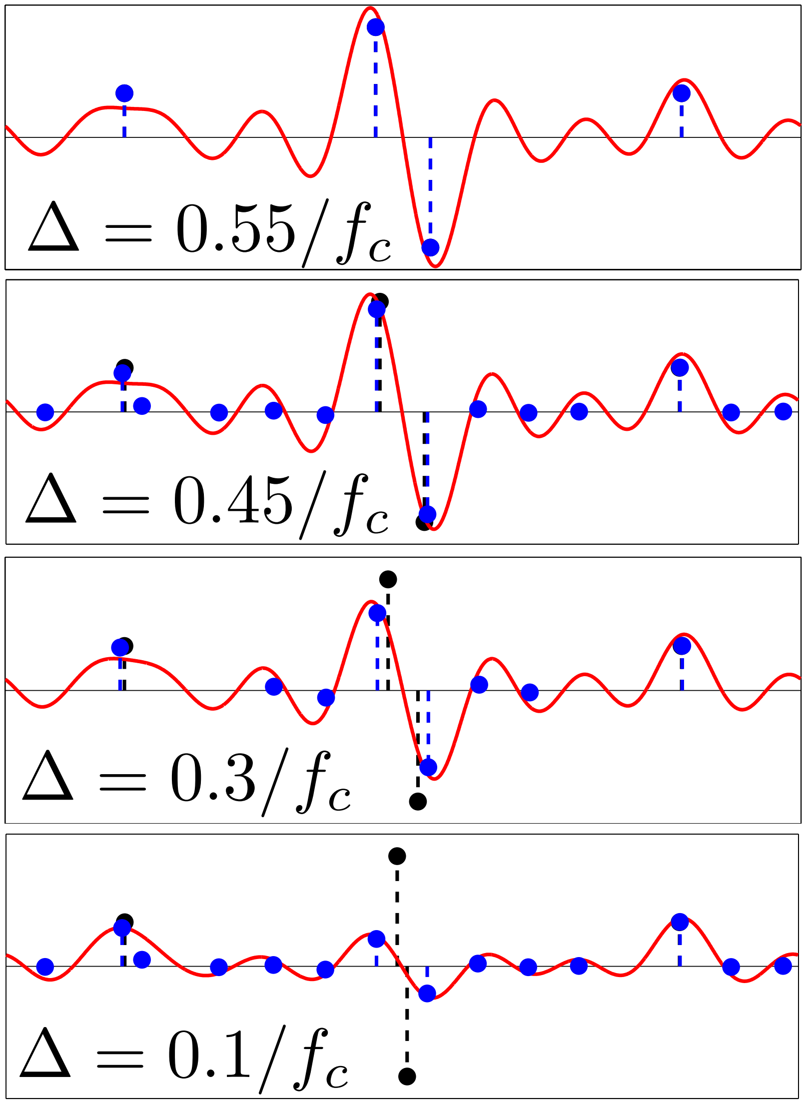
$$\min_m \; \{|m|(\mathbb{T}) \;;\; \Phi m = y\} \qquad (\mathcal{P}_0(y))$$

Low-pass filter $\text{supp}(\hat{\varphi}) = [-f_c, f_c]$.

When is m_0 solution of $\mathcal{P}_0(\Phi m_0)$?

Theorem: [Candès, Fernandez G.]

$$\Delta > \frac{1.26}{f_c} \Rightarrow m_0 \text{ solves } \mathcal{P}_0(\Phi m_0).$$



Robustness and Support-stability

$$\min_m \ \{|m|(\mathbb{T}) \ ; \ \Phi m = y\} \qquad (\mathcal{P}_0(y))$$

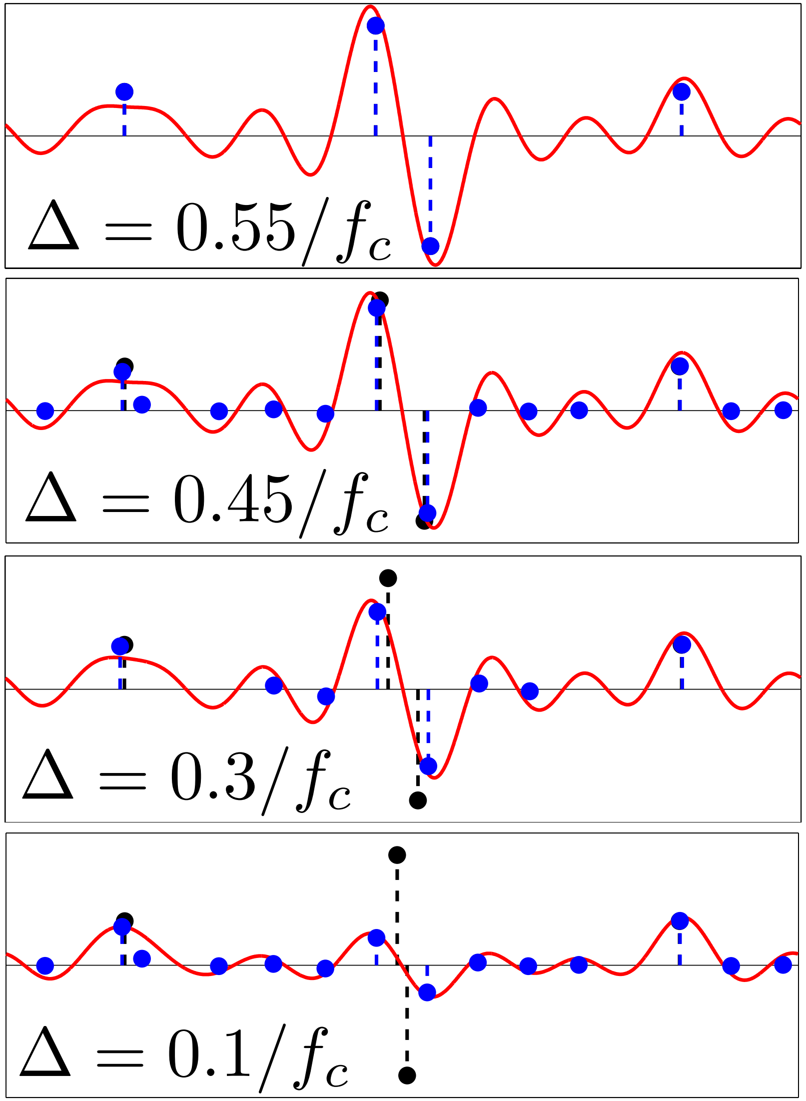
Low-pass filter $\text{supp}(\hat{\varphi}) = [-f_c, f_c]$.

When is m_0 solution of $\mathcal{P}_0(\Phi m_0)$?

Theorem: [Candès, Fernandez G.]

$$\Delta > \frac{1.26}{f_c} \Rightarrow m_0 \text{ solves } \mathcal{P}_0(\Phi m_0).$$

How close to m_0
are solutions of $\mathcal{P}_\lambda(\Phi m_0 + w)$?



Robustness and Support-stability

$$\min_m \{|m|(\mathbb{T}) \; ; \; \Phi m = y\} \quad (\mathcal{P}_0(y))$$

Low-pass filter $\text{supp}(\hat{\varphi}) = [-f_c, f_c]$.

When is m_0 solution of $\mathcal{P}_0(\Phi m_0)$?

Theorem: [Candès, Fernandez G.]

$$\Delta > \frac{1.26}{f_c} \Rightarrow m_0 \text{ solves } \mathcal{P}_0(\Phi m_0).$$

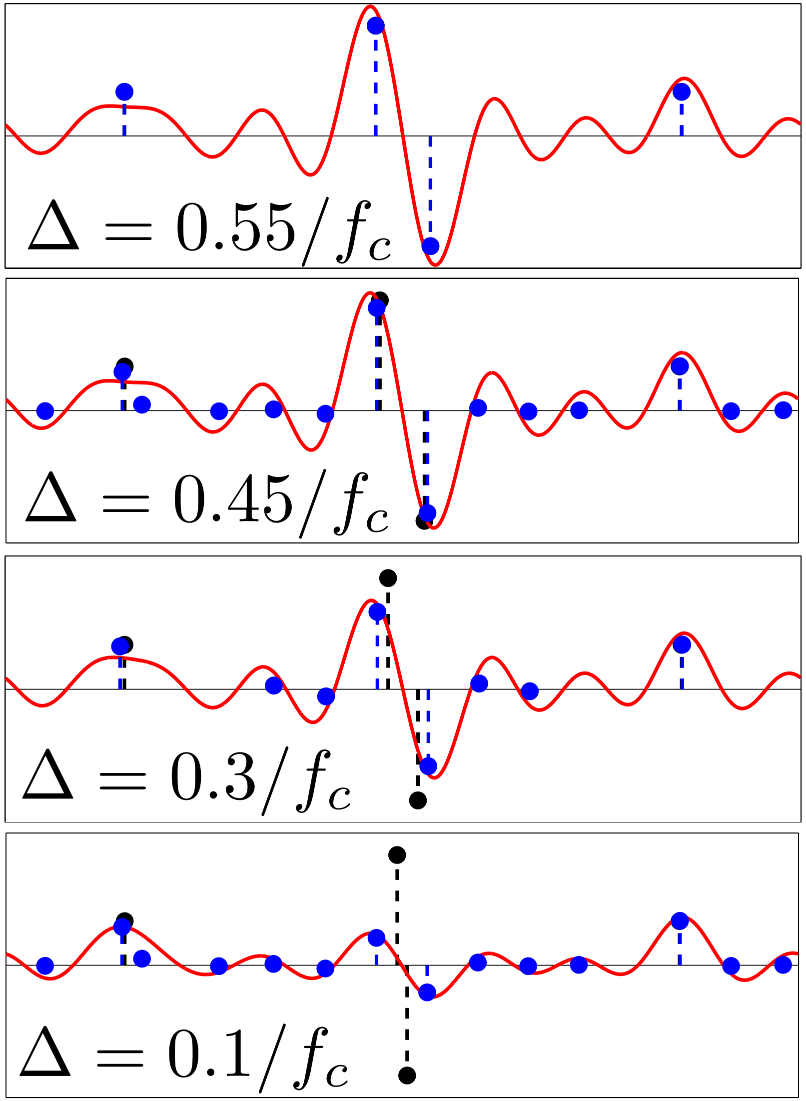
How close to m_0 are solutions of $\mathcal{P}_\lambda(\Phi m_0 + w)$?

Weighted L^2 error:

$\longrightarrow$ [Candès, Fernandez-G. 2012]

Support localization:

$\longrightarrow$ [Fernandez-G.][de Castro 2012]



Robustness and Support-stability

$$\min_m \ \{|m|(\mathbb{T}) \ ; \ \Phi m = y\} \qquad (\mathcal{P}_0(y))$$

Low-pass filter $\text{supp}(\hat{\varphi}) = [-f_c, f_c]$.

When is m_0 solution of $\mathcal{P}_0(\Phi m_0)$?

Theorem: [Candès, Fernandez G.]

$$\Delta > \frac{1.26}{f_c} \Rightarrow m_0 \text{ solves } \mathcal{P}_0(\Phi m_0).$$

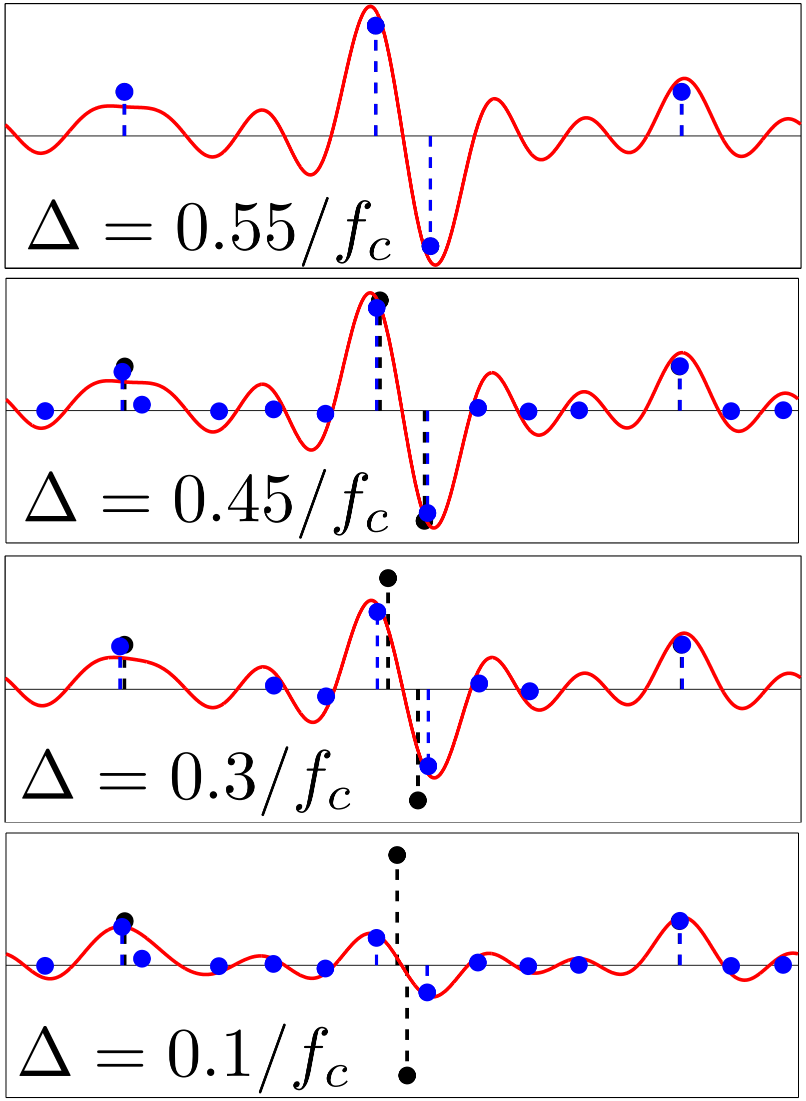
How close to m_0 are solutions of $\mathcal{P}_\lambda(\Phi m_0 + w)$?

Weighted L^2 error:

$\longrightarrow$ [Candès, Fernandez-G. 2012]

Support localization:

$\longrightarrow$ [Fernandez-G.][de Castro 2012]



Open problems: Exact support recovery? General kernels?

From Primal to Dual

$$\min_m |m|(\mathbb{T}) + \frac{1}{2\lambda}\|\Phi m - y\|^2 \quad \mathcal{P}_\lambda(y)$$

From Primal to Dual

$$\min_{m} |m|(\mathbb{T}) + \frac{1}{2\lambda}\|\Phi m - y\|^2 \quad \mathcal{P}_\lambda(y)$$

$$= \min_{m} \left[\sup_{\|\eta\|_\infty \leqslant 1} -\langle \eta, m \rangle + \frac{1}{2\lambda}\|\Phi m - y\|^2 \right]$$

From Primal to Dual

$$\min_m |m|(\mathbb{T}) + \frac{1}{2\lambda}\|\Phi m - y\|^2 \quad \mathcal{P}_\lambda(y)$$

$$m \in \partial \iota_{\|.\|_\infty \leqslant 1}(\eta)$$

$$= \min_m \left[\sup_{\|\eta\|_\infty \leqslant 1} -\langle \eta,\, m\rangle + \frac{1}{2\lambda}\|\Phi m - y\|^2 \right]$$

From Primal to Dual

$$\min_m |m|(\mathbb{T}) + \frac{1}{2\lambda}\|\Phi m - y\|^2 \quad \mathcal{P}_\lambda(y)$$

$$m \in \partial \iota_{\|.\|_\infty \leqslant 1}(\eta)$$

$$= \min_m \left[\sup_{\|\eta\|_\infty \leqslant 1} -\langle \eta,\, m\rangle + \frac{1}{2\lambda}\|\Phi m - y\|^2 \right]$$

$$= \sup_{\|\eta\|_\infty \leqslant 1} \left[\min_m - \langle \eta,\, m\rangle + \frac{1}{2\lambda}\|\Phi m - y\|^2 \right]$$

From Primal to Dual

$$\min_m |m|(\mathbb{T}) + \frac{1}{2\lambda}\|\Phi m - y\|^2 \qquad \mathcal{P}_\lambda(y)$$

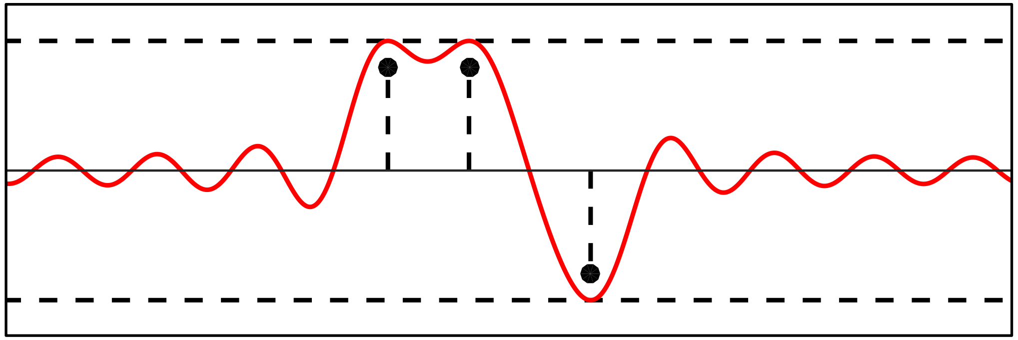
$$= \min_m \left[\sup_{\|\eta\|_\infty \leqslant 1} -\langle \eta, m\rangle + \frac{1}{2\lambda}\|\Phi m - y\|^2 \right]$$

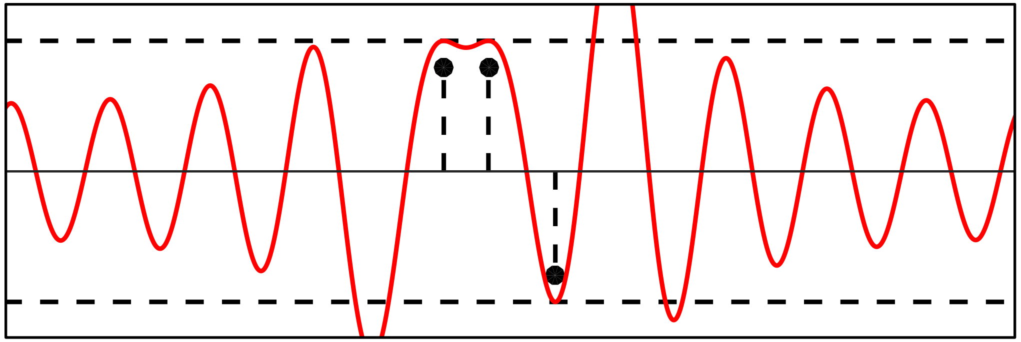
$$= \sup_{\|\eta\|_\infty \leqslant 1} \left[\min_m -\langle \eta, m\rangle + \frac{1}{2\lambda}\|\Phi m - y\|^2 \right]$$

$$m \in \partial \iota_{\|\cdot\|_\infty \leqslant 1}(\eta)$$

$$\eta = \Phi^* \frac{\Phi m - y}{\lambda} = \Phi^* p$$

Ideal low-pass filter: $\rightarrow \eta = \Phi^* p$ trigonometric polynomial.





From Primal to Dual

$$\min_m |m|(\mathbb{T}) + \frac{1}{2\lambda}\|\Phi m - y\|^2 \quad \mathcal{P}_\lambda(y)$$

$$= \min_m \left[\sup_{\|\eta\|_\infty \leqslant 1} -\langle \eta,\, m\rangle + \frac{1}{2\lambda}\|\Phi m - y\|^2 \right]$$

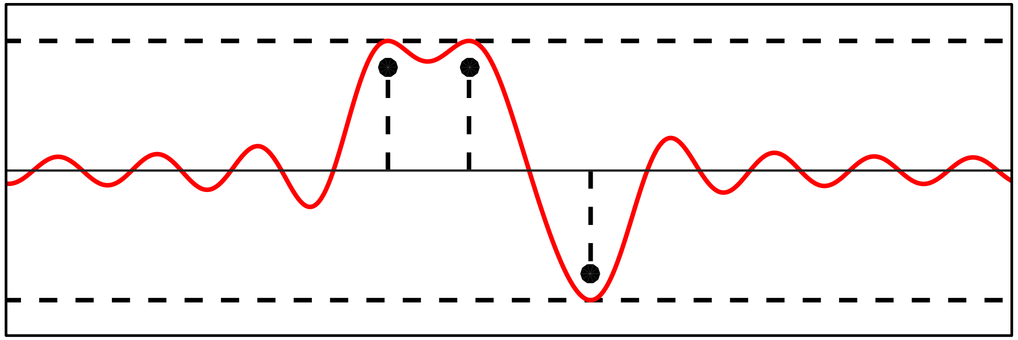
$$= \sup_{\|\eta\|_\infty \leqslant 1} \left[\min_m -\langle \eta,\, m\rangle + \frac{1}{2\lambda}\|\Phi m - y\|^2 \right]$$

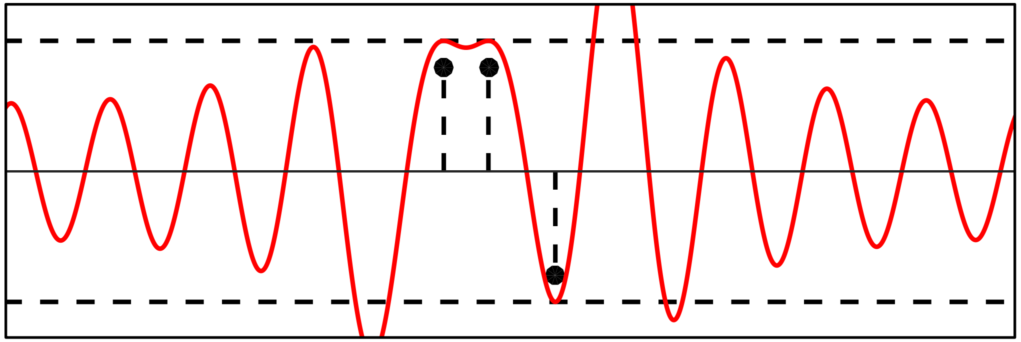
$$= \sup_{\|\Phi^* p\|_\infty \leqslant 1} -\langle p,\, y\rangle - \frac{\lambda}{2}\|p\|^2 \quad \mathcal{D}_\lambda(y)$$

$$m \in \partial \iota_{\|\cdot\|_\infty \leqslant 1}(\eta)$$

$$\eta = \Phi^* \frac{\Phi m - y}{\lambda} = \Phi^* p$$

Ideal low-pass filter: $\rightarrow \eta = \Phi^* p$ trigonometric polynomial.





From Primal to Dual

$$\min_m |m|(\mathbb{T}) + \frac{1}{2\lambda}\|\Phi m - y\|^2 \qquad \mathcal{P}_\lambda(y)$$

$$= \min_m \left[\sup_{\|\eta\|_\infty \leqslant 1} -\langle \eta,\, m\rangle + \frac{1}{2\lambda}\|\Phi m - y\|^2 \right]$$

$$= \sup_{\|\eta\|_\infty \leqslant 1} \left[\min_m - \langle \eta,\, m\rangle + \frac{1}{2\lambda}\|\Phi m - y\|^2 \right]$$

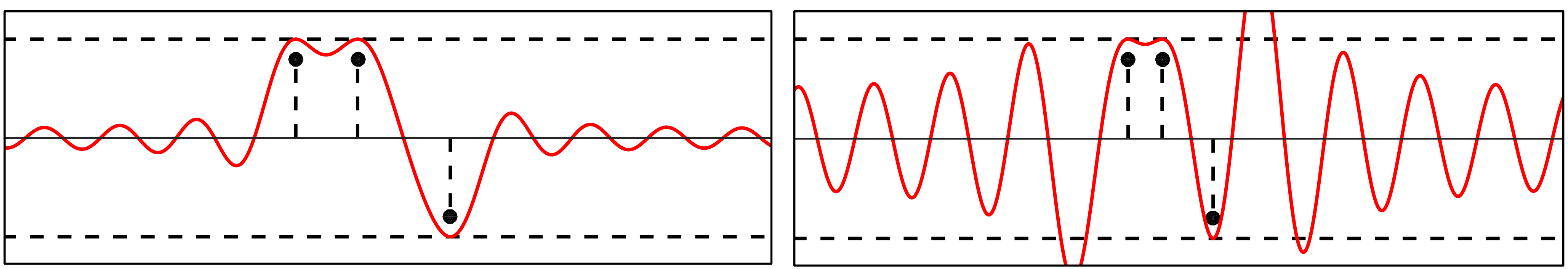
$$= \sup_{\|\Phi^* p\|_\infty \leqslant 1} -\langle p,\, y\rangle - \frac{\lambda}{2}\|p\|^2 \qquad \mathcal{D}_\lambda(y)$$

$$m \in \partial \iota_{\|\cdot\|_\infty \leqslant 1}(\eta)$$

$$\eta = \Phi^* \frac{\Phi m - y}{\lambda} = \Phi^* p$$

$$\Leftrightarrow \quad \eta \in \partial |m|(\mathbb{T})$$

Ideal low-pass filter: $\rightarrow \eta = \Phi^* p$ trigonometric polynomial.



From Primal to Dual

$$\min_m |m|(\mathbb{T}) + \frac{1}{2\lambda}\|\Phi m - y\|^2 \qquad \mathcal{P}_\lambda(y)$$

$$= \min_m \left[\sup_{\|\eta\|_\infty \leqslant 1} -\langle \eta,\, m\rangle + \frac{1}{2\lambda}\|\Phi m - y\|^2 \right]$$

$$= \sup_{\|\eta\|_\infty \leqslant 1} \left[\min_m - \langle \eta,\, m\rangle + \frac{1}{2\lambda}\|\Phi m - y\|^2 \right]$$

$$= \sup_{\|\Phi^* p\|_\infty \leqslant 1} -\langle p,\, y\rangle - \frac{\lambda}{2}\|p\|^2 \qquad \mathcal{D}_\lambda(y)$$

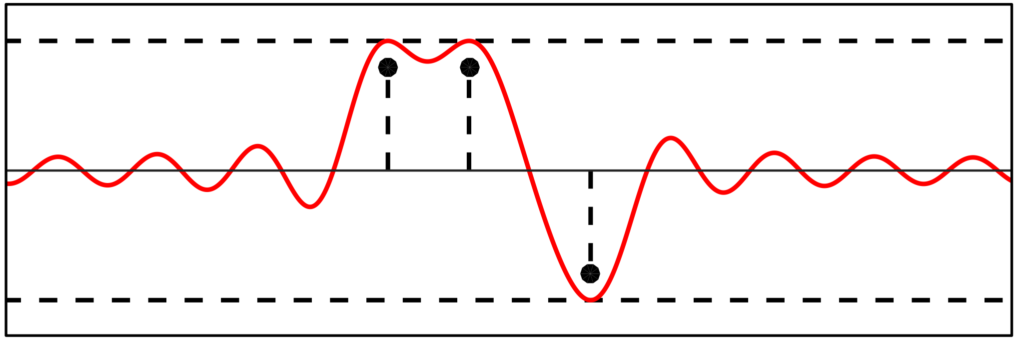
$$m \in \partial \iota_{\|\cdot\|_\infty \leqslant 1}(\eta)$$

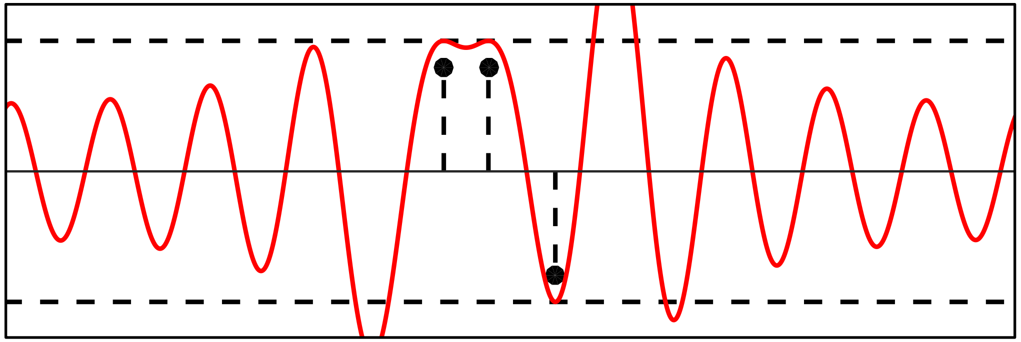
$$\eta = \Phi^* \frac{\Phi m - y}{\lambda} = \Phi^* p$$

$$\Leftrightarrow \quad \eta \in \partial |m|(\mathbb{T})$$

$$\partial |m_{a,x}|(\mathbb{T}) = \left\{ \eta \;;\; \begin{array}{c} \forall\, t, |\eta(t)| \leqslant 1 \\ \forall\, i, \eta(x_i) = \mathrm{sign}(a_i) \end{array} \right\}$$

Ideal low-pass filter: $\rightarrow \eta = \Phi^* p$ trigonometric polynomial.





From Primal to Dual

$$\min_m |m|(\mathbb{T}) + \frac{1}{2\lambda}\|\Phi m - y\|^2 \quad \mathcal{P}_\lambda(y)$$

$$= \min_m \left[\sup_{\|\eta\|_\infty \leqslant 1} -\langle \eta,\, m\rangle + \frac{1}{2\lambda}\|\Phi m - y\|^2 \right]$$

$$= \sup_{\|\eta\|_\infty \leqslant 1} \left[\min_m - \langle \eta,\, m\rangle + \frac{1}{2\lambda}\|\Phi m - y\|^2 \right]$$

$$= \sup_{\|\Phi^* p\|_\infty \leqslant 1} -\langle p,\, y\rangle - \frac{\lambda}{2}\|p\|^2 \quad \mathcal{D}_\lambda(y)$$

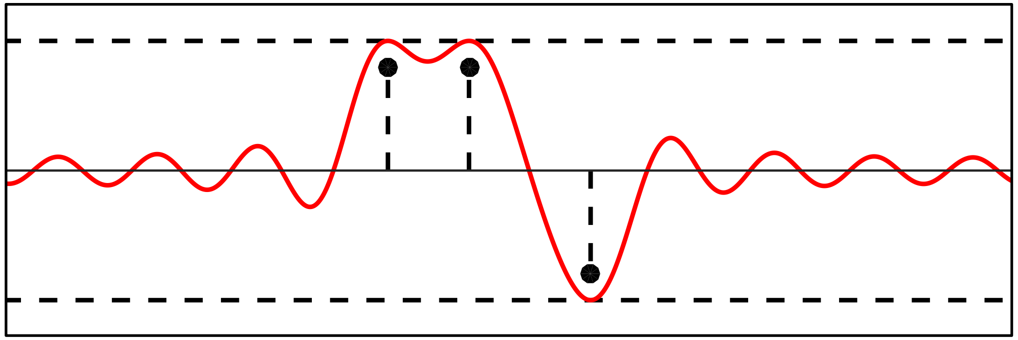
$$m \in \partial\iota_{\|\cdot\|_\infty \leqslant 1}(\eta)$$

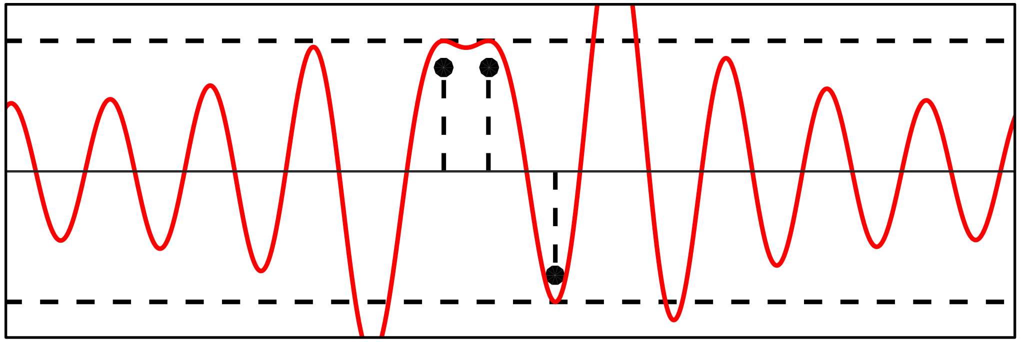
$$\eta = \Phi^* \frac{\Phi m - y}{\lambda} = \Phi^* p$$

$$\Leftrightarrow \quad \eta \in \partial|m|(\mathbb{T})$$

$$\partial|m_{a,x}|(\mathbb{T}) = \left\{ \eta \,;\, \begin{array}{c} \forall\, t,\, |\eta(t)| \leqslant 1 \\ \forall\, i,\, \eta(x_i) = \text{sign}(a_i) \end{array} \right\}$$

Ideal low-pass filter:
- $\rightarrow$ $\eta = \Phi^* p$ trigonometric polynomial.
- $\rightarrow$ Interpolates spikes location and sign.





From Primal to Dual

$$\min_m |m|(\mathbb{T}) + \frac{1}{2\lambda}\|\Phi m - y\|^2 \quad \mathcal{P}_\lambda(y)$$

$$= \min_m \left[\sup_{\|\eta\|_\infty \leqslant 1} -\langle \eta, m\rangle + \frac{1}{2\lambda}\|\Phi m - y\|^2 \right]$$

$$= \sup_{\|\eta\|_\infty \leqslant 1} \left[\min_m -\langle \eta, m\rangle + \frac{1}{2\lambda}\|\Phi m - y\|^2 \right]$$

$$= \sup_{\|\Phi^* p\|_\infty \leqslant 1} -\langle p, y\rangle - \frac{\lambda}{2}\|p\|^2 \quad \mathcal{D}_\lambda(y)$$

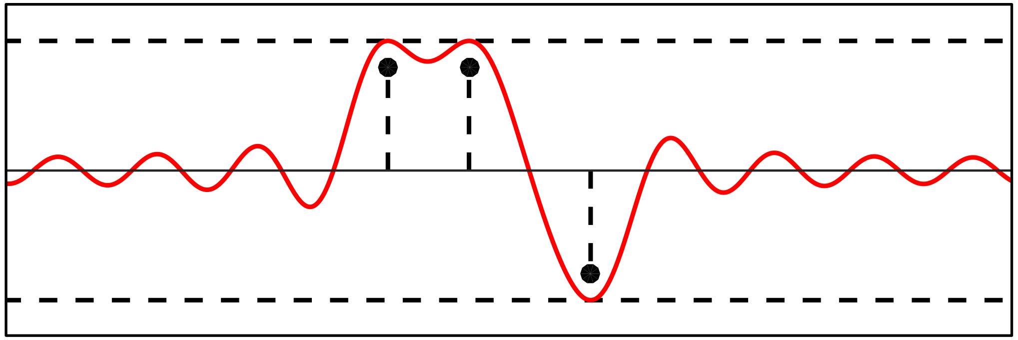
$$m \in \partial \iota_{\|\cdot\|_\infty \leqslant 1}(\eta)$$

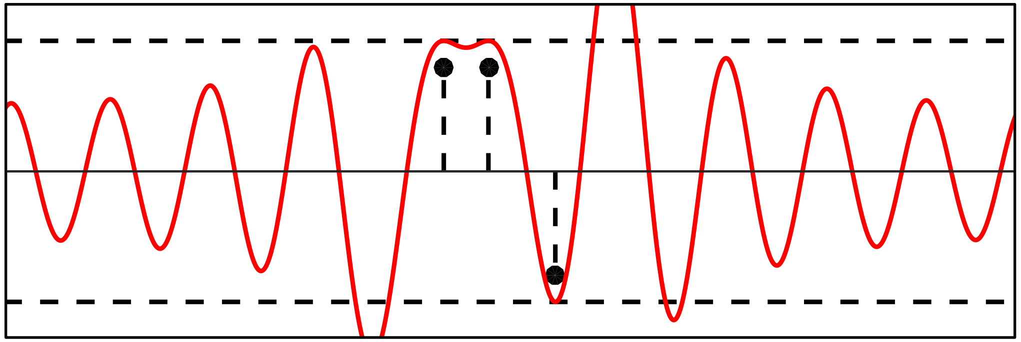
$$\eta = \Phi^* \frac{\Phi m - y}{\lambda} = \Phi^* p$$

$$\Leftrightarrow \quad \eta \in \partial |m|(\mathbb{T})$$

$$\partial |m_{a,x}|(\mathbb{T}) = \left\{ \eta \;;\; \begin{array}{l} \forall t, |\eta(t)| \leqslant 1 \\ \forall i, \eta(x_i) = \operatorname{sign}(a_i) \end{array} \right\}$$

Ideal low-pass filter:
- $\rightarrow$ $\eta = \Phi^* p$ trigonometric polynomial.
- $\rightarrow$ Interpolates spikes location and sign.
- $\rightarrow$ $|\eta(t)|^2 = 1$: polynomial equation of $\operatorname{supp}(m)$.





Asymptotic Dual and Certificate

$\mathcal{P}_\lambda(y)$

$$\min_m |m|(\mathbb{T}) + \frac{1}{2\lambda}\|\Phi m - y\|^2$$

$$p_\lambda \overset{\text{def.}}{=} \underset{\|\Phi^* p\|_\infty \leqslant 1}{\operatorname{argmax}} \langle p,\, y\rangle - \frac{\lambda}{2}\|p\|^2$$

$\mathcal{D}_\lambda(y)$

Asymptotic Dual and Certificate

$\mathcal{P}_\lambda(y)$

$$\min_{m} |m|(\mathbb{T}) + \frac{1}{2\lambda}\|\Phi m - y\|^2$$

$\mathcal{D}_\lambda(y)$

$$p_\lambda \overset{\text{def.}}{=} \underset{\|\Phi^* p\|_\infty \leqslant 1}{\text{argmax}} \; \langle p,\, y\rangle - \frac{\lambda}{2}\|p\|^2$$

$\lambda \to 0^+$

$\mathcal{P}_0(y)$

$$m_0 \in \underset{\Phi m = y}{\text{argmin}} \, |m|(\mathbb{T})$$

Asymptotic Dual and Certificate

$\mathcal{P}_\lambda(y)$

$$\min_m |m|(\mathbb{T}) + \frac{1}{2\lambda}\|\Phi m - y\|^2$$

$$p_\lambda \overset{\text{def.}}{=} \underset{\|\Phi^* p\|_\infty \leqslant 1}{\text{argmax}} \langle p,\, y\rangle - \frac{\lambda}{2}\|p\|^2$$

$\mathcal{D}_\lambda(y)$

$\lambda \to 0^+$

$\mathcal{P}_0(y)$

$$m_0 \in \underset{\Phi m = y}{\text{argmin}} |m|(\mathbb{T})$$

$$\mathcal{D}_0(y) = \underset{\|\Phi^* p\|_\infty \leqslant 1}{\text{argmax}} \langle p,\, y\rangle$$

$\mathcal{D}_0(y)$

Asymptotic Dual and Certificate

$\mathcal{P}_\lambda(y)$: $$\min_m |m|(\mathbb{T}) + \frac{1}{2\lambda}\|\Phi m - y\|^2$$

$\mathcal{D}_\lambda(y)$: $$p_\lambda \stackrel{\text{def.}}{=} \underset{\|\Phi^* p\|_\infty \leqslant 1}{\text{argmax}} \; \langle p,\, y\rangle - \frac{\lambda}{2}\|p\|^2$$

$\lambda \to 0^+$

$\mathcal{P}_0(y)$: $$m_0 \in \underset{\Phi m = y}{\text{argmin}} |m|(\mathbb{T})$$

$\mathcal{D}_0(y)$: $$\mathcal{D}_0(y) = \underset{\|\Phi^* p\|_\infty \leqslant 1}{\text{argmax}} \; \langle p,\, y\rangle$$

$\lambda \to 0^+$

$$p_0 \stackrel{\text{def.}}{=} \underset{p \in \mathcal{D}_0(y)}{\text{argmax}} \; -\frac{1}{2}\|p\|^2$$

Asymptotic Dual and Certificate

$\mathcal{P}_\lambda(y)$
$$\min_m |m|(\mathbb{T}) + \frac{1}{2\lambda}\|\Phi m - y\|^2$$

$\mathcal{D}_\lambda(y)$
$$p_\lambda \overset{\text{def.}}{=} \underset{\|\Phi^* p\|_\infty \leqslant 1}{\text{argmax}} \; \langle p,\, y\rangle - \frac{\lambda}{2}\|p\|^2$$

$\lambda \to 0^+$

$\mathcal{P}_0(y)$
$$m_0 \in \underset{\Phi m = y}{\text{argmin}} |m|(\mathbb{T})$$

$\mathcal{D}_0(y)$
$$\mathcal{D}_0(y) = \underset{\|\Phi^* p\|_\infty \leqslant 1}{\text{argmax}} \; \langle p,\, y\rangle$$

$\lambda \to 0^+$

Lemma: $\mathcal{D}_0(y) = \{p \; ; \; \Phi^* p \in \partial|m_0|(\mathbb{T})\}$

$$p_0 \overset{\text{def.}}{=} \underset{p \in \mathcal{D}_0(y)}{\text{argmax}} \; - \frac{1}{2}\|p\|^2$$

Asymptotic Dual and Certificate

$\mathcal{P}_\lambda(y)$: $$\min_m |m|(\mathbb{T}) + \frac{1}{2\lambda}\|\Phi m - y\|^2$$

$\mathcal{D}_\lambda(y)$: $$p_\lambda \overset{\text{def.}}{=} \underset{\|\Phi^* p\|_\infty \leqslant 1}{\operatorname{argmax}} \langle p,\, y\rangle - \frac{\lambda}{2}\|p\|^2$$

$\lambda \to 0^+$

$\mathcal{P}_0(y)$: $$m_0 \in \underset{\Phi m = y}{\operatorname{argmin}} |m|(\mathbb{T})$$

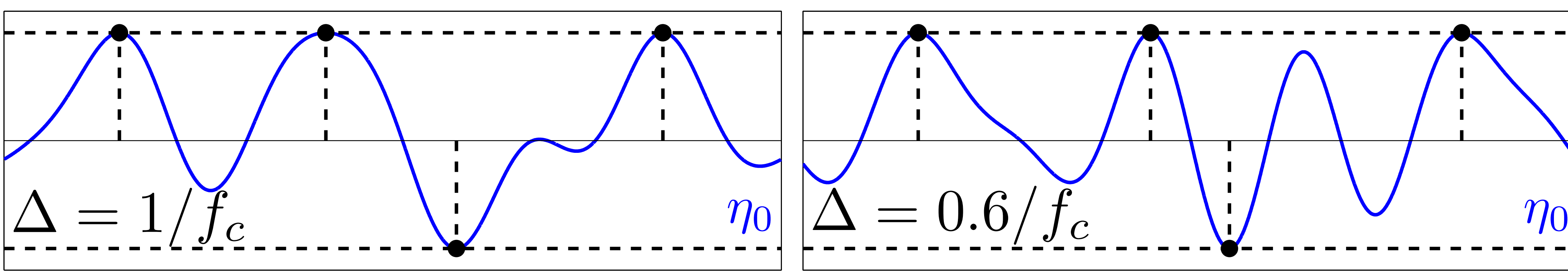
$\mathcal{D}_0(y)$: $$\mathcal{D}_0(y) = \underset{\|\Phi^* p\|_\infty \leqslant 1}{\operatorname{argmax}} \langle p,\, y\rangle$$

$\lambda \to 0^+$

Lemma: $\mathcal{D}_0(y) = \{p \,;\, \Phi^* p \in \partial |m_0|(\mathbb{T})\}$

$$p_0 \overset{\text{def.}}{=} \underset{p \in \mathcal{D}_0(y)}{\operatorname{argmax}} -\frac{1}{2}\|p\|^2$$

Definition: for any m_0 solution of $\mathcal{P}_0(y)$,
$$\eta_0 = \Phi^* p_0 = \underset{\eta = \Phi^* p \in \partial |m_0|(\mathbb{T})}{\operatorname{argmin}} \|p\|$$

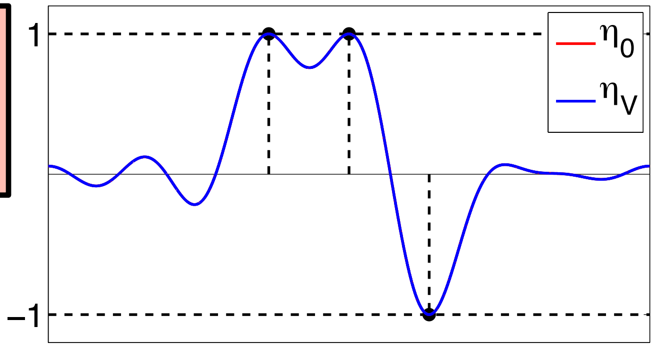


Vanishing Derivative Pre-certificate

Input measure: $m_0 = m_{a,x}$.

$$\eta_0 \overset{\text{def.}}{=} \underset{\eta = \Phi^* p}{\text{argmin}} \, \|p\| \text{ s.t. } \begin{cases} \forall\, i,\ \eta(x_i) = \text{sign}(a_i), \\ \|\eta\|_\infty \leqslant 1. \end{cases}$$

$$\exists \eta_0 \iff m_0 \text{ solves } \mathcal{P}_0(\Phi m_0)$$



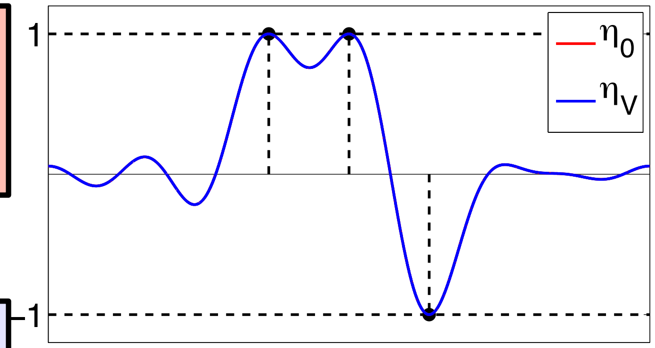
Vanishing Derivative Pre-certificate

Input measure: $m_0 = m_{a,x}$.

$$\eta_0 \overset{\text{def.}}{=} \underset{\eta=\Phi^* p}{\text{argmin}} \|p\| \text{ s.t. } \begin{cases} \forall\, i,\ \eta(x_i) = \text{sign}(a_i), \\ \|\eta\|_\infty \leqslant 1. \end{cases}$$

$$\exists \eta_0 \iff m_0 \text{ solves } \mathcal{P}_0(\Phi m_0)$$

$$\eta_V \overset{\text{def.}}{=} \underset{\eta=\Phi^* p}{\text{argmin}} \|p\| \text{ s.t. } \begin{cases} \forall\, i,\ \eta(x_i) = \text{sign}(a_i), \\ \forall\, i,\ \eta'(x_i) = 0. \end{cases}$$



$\eta_0 = \eta_V$

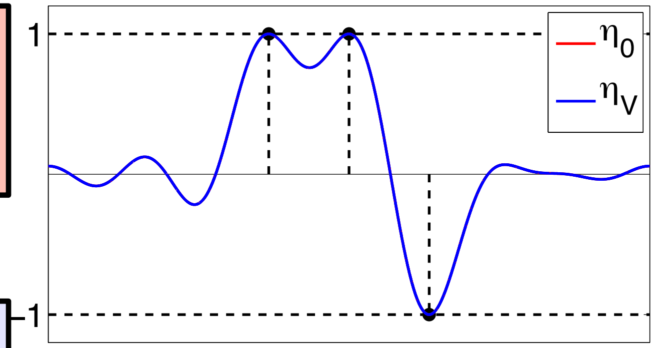
Vanishing Derivative Pre-certificate

Input measure: $m_0 = m_{a,x}$.

$$\eta_0 \overset{\text{def.}}{=} \underset{\eta=\Phi^* p}{\text{argmin}} \|p\| \text{ s.t. } \begin{cases} \forall\, i,\ \eta(x_i) = \text{sign}(a_i), \\ \|\eta\|_\infty \leqslant 1. \end{cases}$$

$$\exists \eta_0 \iff m_0 \text{ solves } \mathcal{P}_0(\Phi m_0)$$

$$\eta_V \overset{\text{def.}}{=} \underset{\eta=\Phi^* p}{\text{argmin}} \|p\| \text{ s.t. } \begin{cases} \forall\, i,\ \eta(x_i) = \text{sign}(a_i), \\ \forall\, i,\ \eta'(x_i) = 0. \end{cases}$$



$\eta_0 = \eta_V$

Proposition: $\eta_V = \Phi^* A_x^+(\text{sign}(a); 0)$

where $A_x(b) = \sum_i b_i^1 \varphi(x_i, \cdot) + b_i^2 \varphi'(x_i, \cdot)$

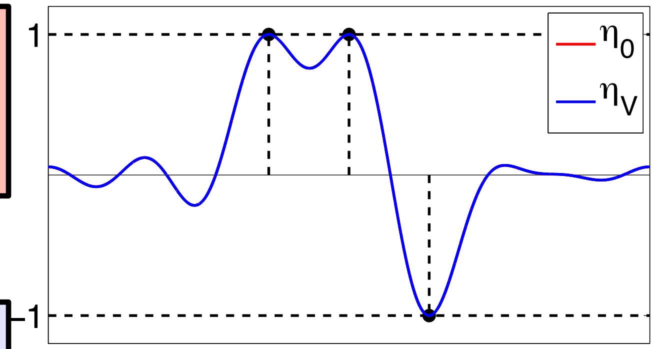
Vanishing Derivative Pre-certificate

Input measure: $m_0 = m_{a,x}$.

$$\eta_0 \overset{\text{def.}}{=} \underset{\eta=\Phi^* p}{\text{argmin}} \|p\| \text{ s.t. } \begin{cases} \forall\, i,\ \eta(x_i) = \text{sign}(a_i), \\ \|\eta\|_\infty \leqslant 1. \end{cases}$$

$$\exists \eta_0 \iff m_0 \text{ solves } \mathcal{P}_0(\Phi m_0)$$

$$\eta_V \overset{\text{def.}}{=} \underset{\eta=\Phi^* p}{\text{argmin}} \|p\| \text{ s.t. } \begin{cases} \forall\, i,\ \eta(x_i) = \text{sign}(a_i), \\ \forall\, i,\ \eta'(x_i) = 0. \end{cases}$$



$\eta_0 = \eta_V$

Proposition: $\eta_V = \Phi^* A_x^+(\text{sign}(a); 0)$

where $A_x(b) = \sum_i b_i^1 \varphi(x_i, \cdot) + b_i^2 \varphi'(x_i, \cdot)$

Non-degenerate certificate: $\eta \in \text{ND}(m_{a,x})$:

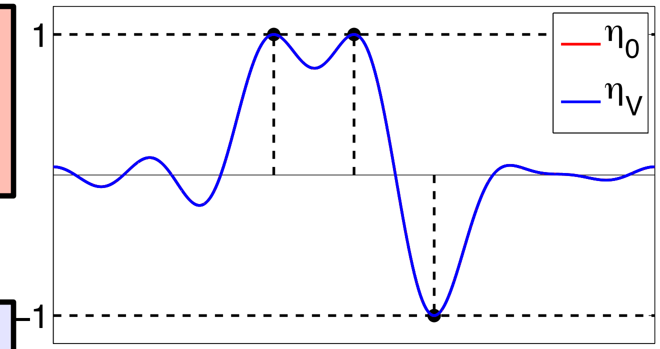
$\iff \quad \forall\, t \notin \{x_1, \ldots, x_N\}, |\eta(t)| < 1$ and $\forall\, i, \eta''(x_i) \neq 0$

Vanishing Derivative Pre-certificate

Input measure: $m_0 = m_{a,x}$.

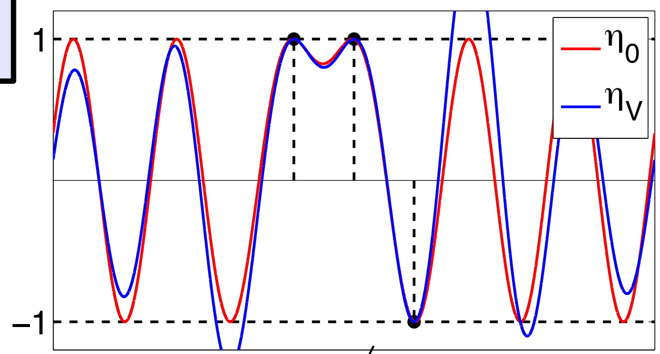
$$\eta_0 \overset{\text{def.}}{=} \underset{\eta=\Phi^* p}{\text{argmin}} \|p\| \text{ s.t. } \begin{cases} \forall\, i,\ \eta(x_i) = \text{sign}(a_i), \\ \|\eta\|_\infty \leqslant 1. \end{cases}$$

$$\exists \eta_0 \iff m_0 \text{ solves } \mathcal{P}_0(\Phi m_0)$$



$\eta_0 = \eta_V$

$$\eta_V \overset{\text{def.}}{=} \underset{\eta=\Phi^* p}{\text{argmin}} \|p\| \text{ s.t. } \begin{cases} \forall\, i,\ \eta(x_i) = \text{sign}(a_i), \\ \forall\, i,\ \eta'(x_i) = 0. \end{cases}$$



$\eta_0 \neq \eta_V$

Proposition: $\eta_V = \Phi^* A_x^+(\text{sign}(a); 0)$

where $A_x(b) = \sum_i b_i^1 \varphi(x_i, \cdot) + b_i^2 \varphi'(x_i, \cdot)$

Non-degenerate certificate: $\eta \in \text{ND}(m_{a,x})$:

$$\iff \forall\, t \notin \{x_1, \ldots, x_N\}, |\eta(t)| < 1 \text{ and } \forall\, i, \eta''(x_i) \neq 0$$

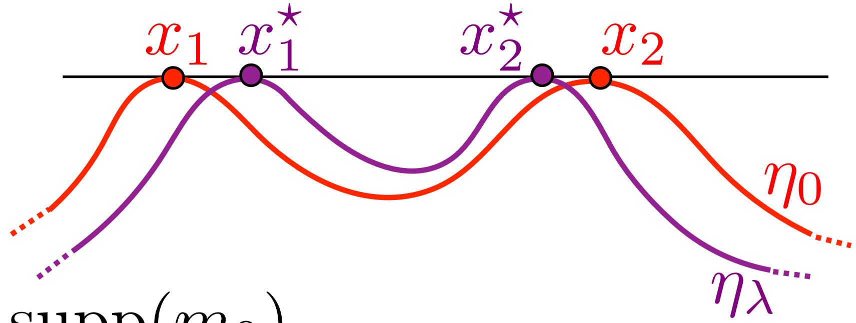
Theorem: $\eta_V \in \text{ND}(m_0) \implies \eta_V = \eta_0$

Support Stability Theorem

$$\eta_\lambda = \Phi^* p_\lambda \xrightarrow{\lambda \to 0} \eta_0 = \Phi^* p_0$$

$$\text{supp}(m_\lambda) \subset \{|\eta_\lambda| = 1\}$$

Support Stability Theorem



$$\eta_\lambda = \Phi^* p_\lambda \xrightarrow{\lambda \to 0} \eta_0 = \Phi^* p_0$$

$$\text{supp}(m_\lambda) \subset \{|\eta_\lambda| = 1\}$$

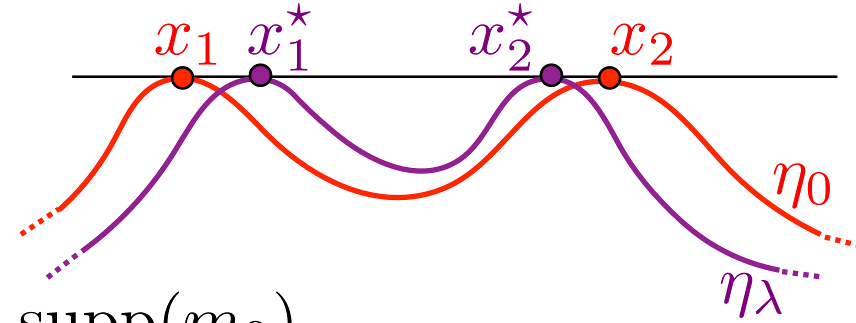
If $\eta_0 \in \text{ND}(m_0)$ then $\text{supp}(m_\lambda) \to \text{supp}(m_0)$

Support Stability Theorem

$$\eta_\lambda = \Phi^* p_\lambda \overset{\lambda \to 0}{\longrightarrow} \eta_0 = \Phi^* p_0$$

$$\text{supp}(m_\lambda) \subset \{|\eta_\lambda| = 1\}$$

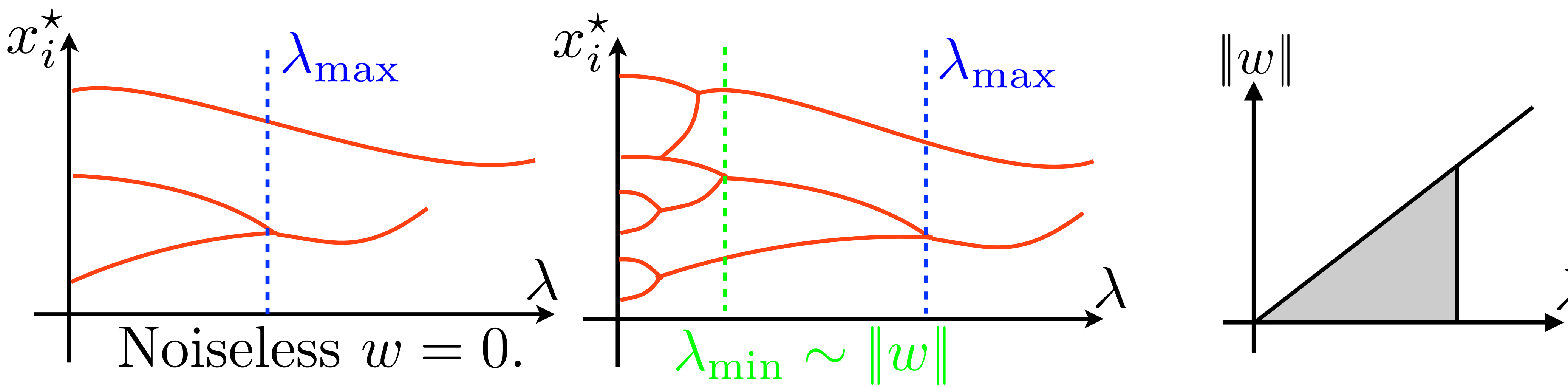
If $\eta_0 \in \text{ND}(m_0)$ then $\text{supp}(m_\lambda) \to \text{supp}(m_0)$



[Duval, Peyré 2014]

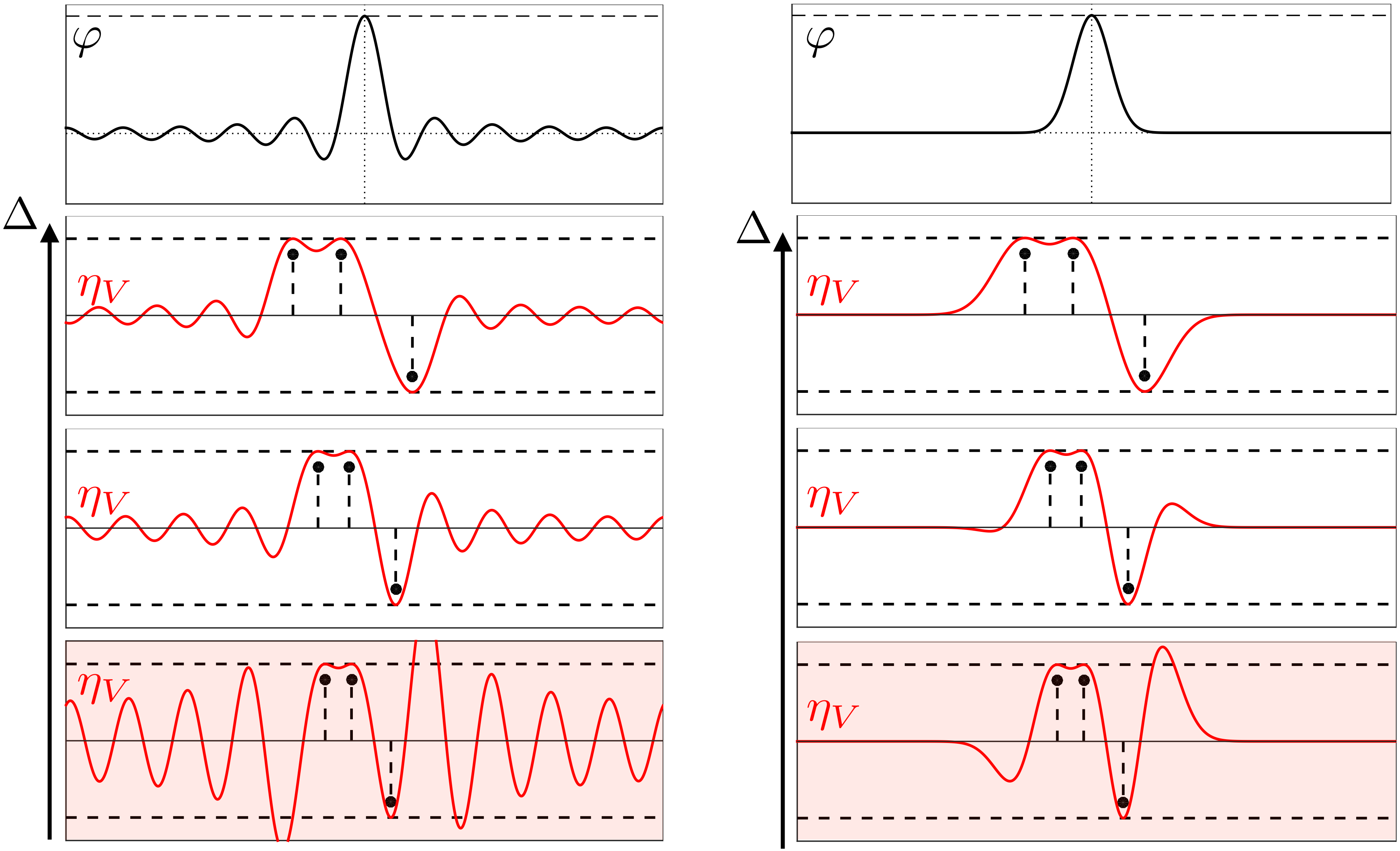
Theorem: If $\eta_V \in \text{ND}(m_0)$ for $m_0 = m_{a,x}$, then for $(\|w\|/\lambda, \lambda) = O(1)$, the solution of $\mathcal{P}_\lambda(y)$ for $y = \Phi(m_0) + w$ is

$$m_\lambda = \sum_{i=1}^N a_i^\star \delta_{x_i^\star} \quad \text{where} \quad \|(x,a) - (x^\star, a^\star)\| = O(\|w\|)$$



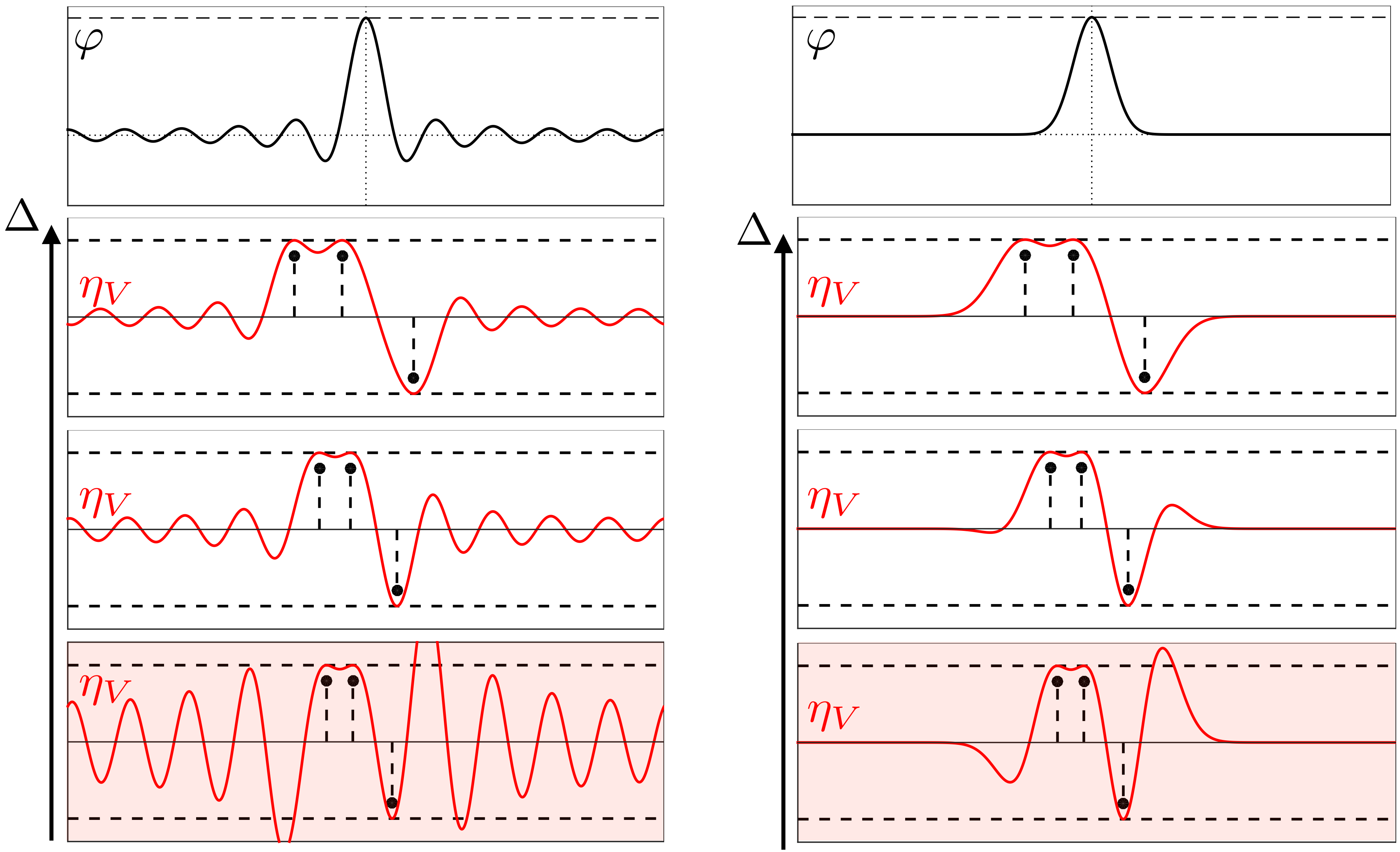
When is η_V Non-degenerate ?

Input measure: $m_0 = m_{a,\Delta x}$, $\Delta \to 0$



When is η_V Non-degenerate ?

Input measure: $m_0 = m_{a,\Delta x}, \quad \Delta \to 0$



Theorem: [Tang, Recht, 2013]

$\exists C, (\Delta > C\sigma) \implies (\eta_V \text{ is non degenerate})$

Valid for:

- $\varphi(x) = e^{-x^2/\sigma^2}$
- $\varphi(x) = (1 + (x/\sigma)^2)^{-1}$
- $\dots$

Overview

- Sparse Spikes Super-resolution
- Robust Support Recovery
- Asymptotic Positive Measure Recovery

Recovery of Positive Measures

Input measure: $m_0 = m_{a,x}$ where $a \in \mathbb{R}_+^N$.

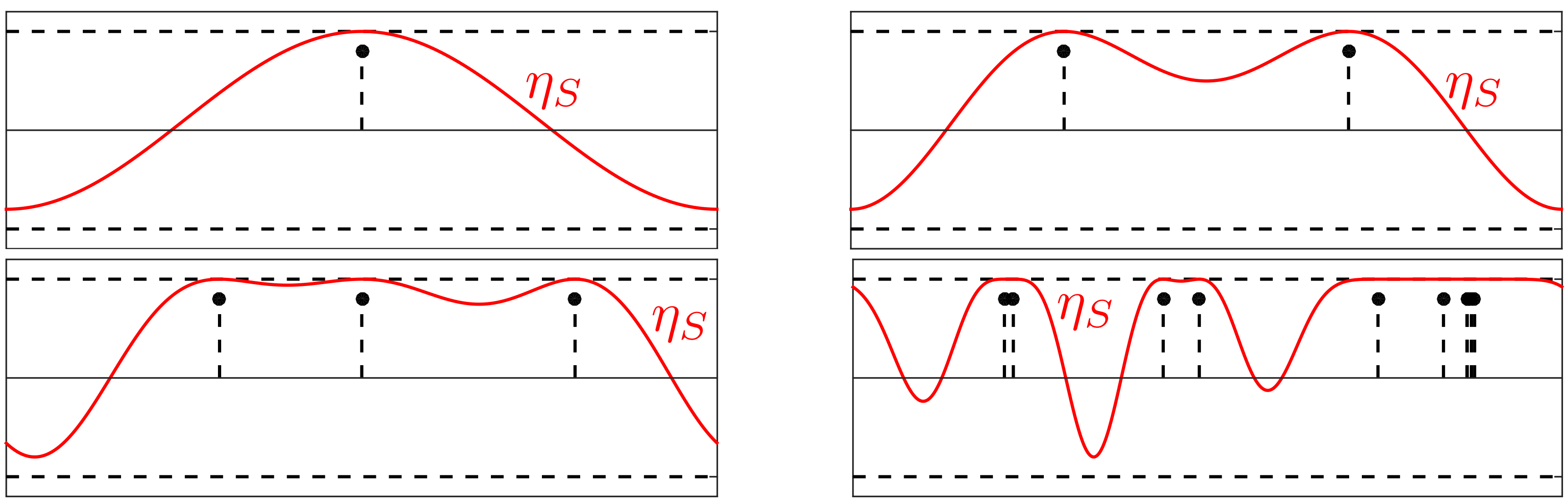
[de Castro et al. 2011]

Theorem: let $\Phi m = (\int e^{-2i\pi kt} \mathrm{d}m(t))_{k=-f_c}^{f_c}$ and

$$\eta_S(t) = 1 - \rho \prod_{i=1}^N \sin(\pi(t - x_i))^2$$

for $N \leqslant f_c$ and ρ small enough, $\eta_S \in \bar{\mathcal{D}}(m_0)$.

$\longrightarrow$ m_0 is recovered when there is no noise.



Recovery of Positive Measures

Input measure: $m_0 = m_{a,x}$ where $a \in \mathbb{R}_+^N$.

Theorem: let $\Phi m = (\int e^{-2i\pi k t} \mathrm{d}m(t))_{k=-f_c}^{f_c}$ and
$$\eta_S(t) = 1 - \rho \prod_{i=1}^N \sin(\pi(t - x_i))^2$$
for $N \leqslant f_c$ and ρ small enough, $\eta_S \in \bar{\mathcal{D}}(m_0)$.

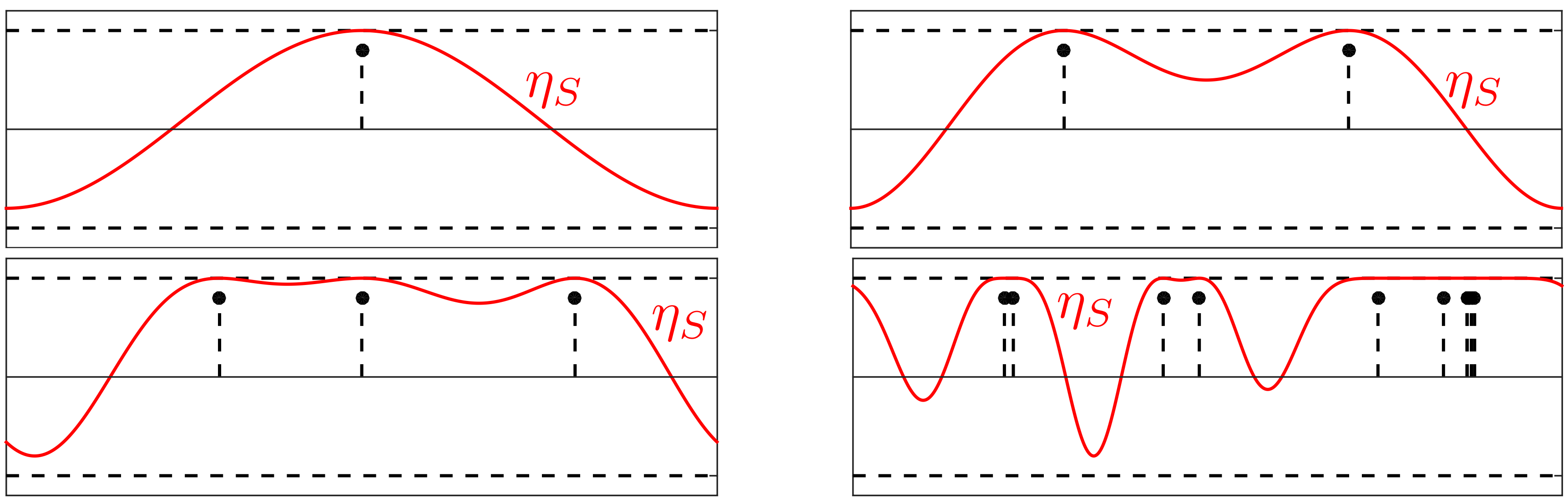
[de Castro et al. 2011]

$\longrightarrow$ m_0 is recovered when there is no noise.

$\longrightarrow$ behavior as $\forall\, i, x_i \to 0$?

[Morgenshtern, Candès, 2015] discrete ℓ^1 robustness.

[Demanet, Nguyen, 2015] discrete ℓ^0 robustness.



Recovery of Positive Measures

Input measure: $m_0 = m_{a,x}$ where $a \in \mathbb{R}_+^N$.

Theorem: let $\Phi m = (\int e^{-2i\pi k t} \mathrm{d}m(t))_{k=-f_c}^{f_c}$ and

$$\eta_S(t) = 1 - \rho \prod_{i=1}^N \sin(\pi(t - x_i))^2$$

for $N \leqslant f_c$ and ρ small enough, $\eta_S \in \bar{\mathcal{D}}(m_0)$.

[de Castro et al. 2011]

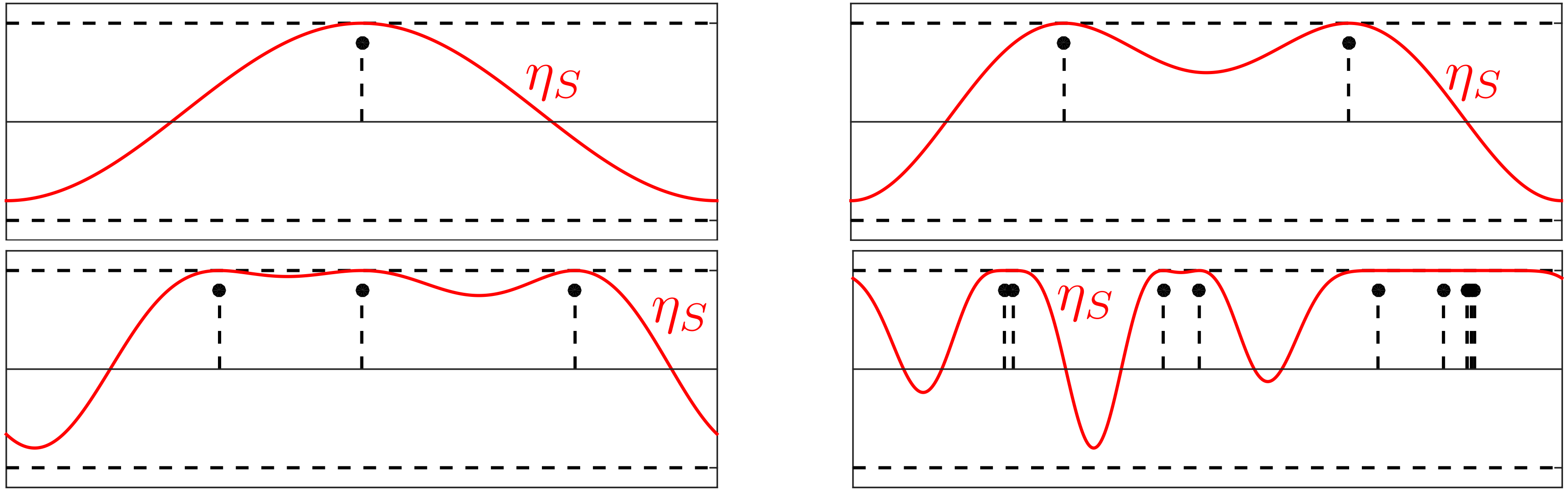
$\longrightarrow$ m_0 is recovered when there is no noise.

$\longrightarrow$ behavior as $\forall\, i, x_i \to 0$?

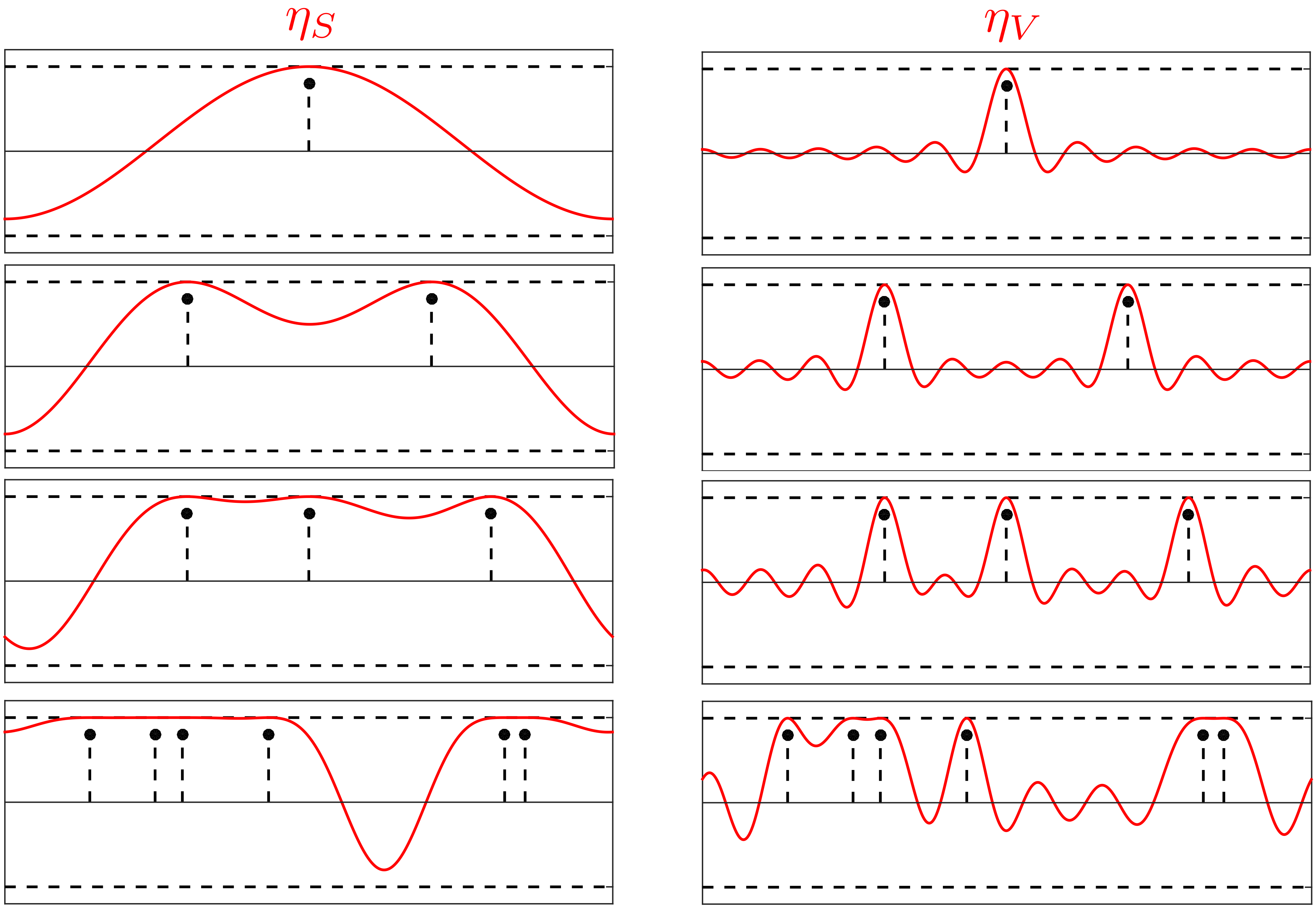
[Morgenshtern, Candès, 2015] discrete ℓ^1 robustness.

[Demanet, Nguyen, 2015] discrete ℓ^0 robustness.

$\longrightarrow$ noise robustness of support recovery ?

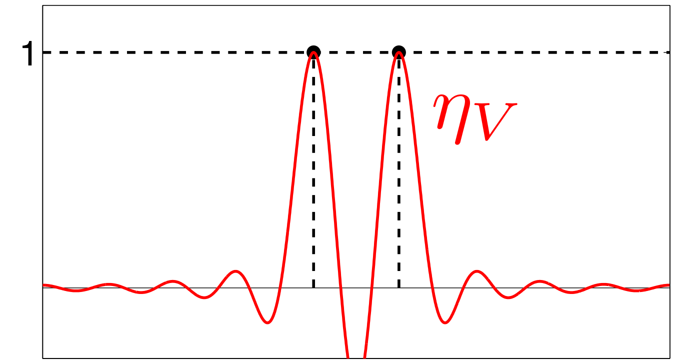


Comparison of Certificates



Asymptotic of Vanishing Certificate

$$m_0 = m_{a,\Delta x} \quad \text{where} \quad \Delta \to 0$$



Vanishing Derivative pre-certificate:

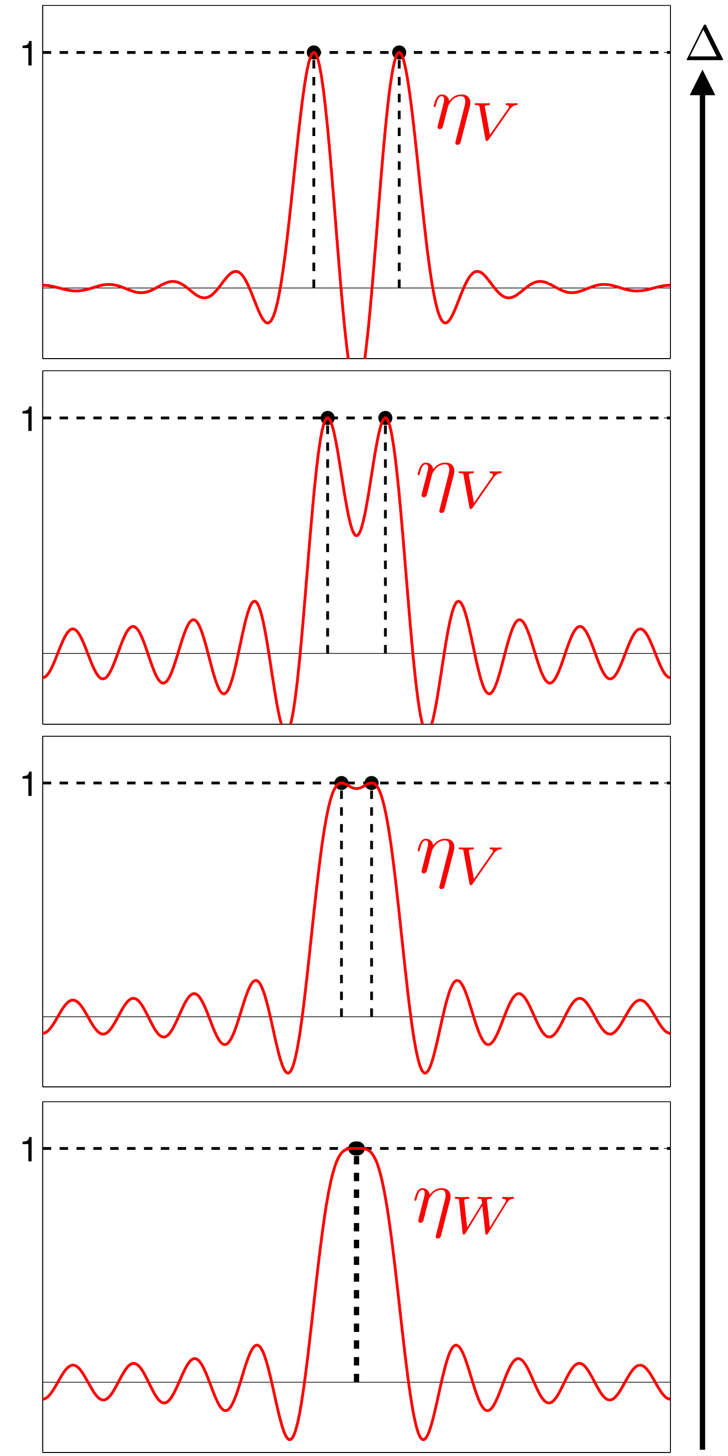
$$\eta_V \overset{\text{def.}}{=} \underset{\eta = \Phi^* p}{\text{argmin}} \, \|p\| \quad \text{s.t.} \quad \forall\, i, \begin{cases} \eta(\Delta x_i) = 1, \\ \eta'(\Delta x_i) = 0. \end{cases}$$

Asymptotic of Vanishing Certificate

$$m_0 = m_{a,\Delta x} \quad \text{where} \quad \Delta \to 0$$

Vanishing Derivative pre-certificate:

$$\eta_V \overset{\text{def.}}{=} \underset{\eta = \Phi^* p}{\operatorname{argmin}} \|p\| \quad \text{s.t.} \quad \forall\, i, \begin{cases} \eta(\Delta x_i) = 1, \\ \eta'(\Delta x_i) = 0. \end{cases}$$



Asymptotic of Vanishing Certificate

$m_0 = m_{a,\Delta x}$ where $\Delta \to 0$

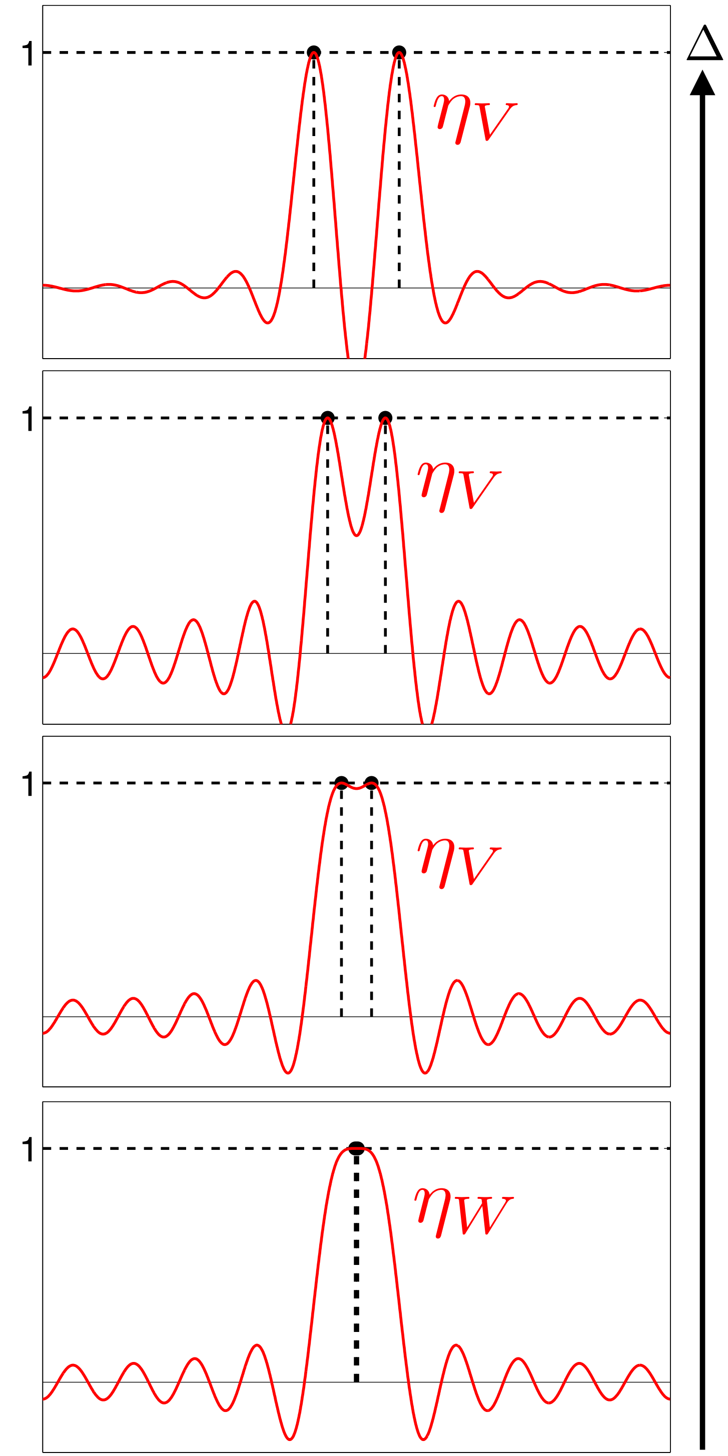
Vanishing Derivative pre-certificate:

$$\eta_V \overset{\text{def.}}{=} \underset{\eta = \Phi^* p}{\text{argmin}} \|p\| \quad \text{s.t.} \quad \forall\, i, \begin{cases} \eta(\Delta x_i) = 1, \\ \eta'(\Delta x_i) = 0. \end{cases}$$

$\Delta \to 0$

Asymptotic pre-certificate:

$$\eta_W \overset{\text{def.}}{=} \underset{\eta = \Phi^* p}{\text{argmin}} \|p\| \quad \text{s.t.} \quad \begin{cases} \eta(0) = 1, \\ \eta'(0) = \ldots = \eta^{(2N-1)}(0) = 0. \end{cases}$$

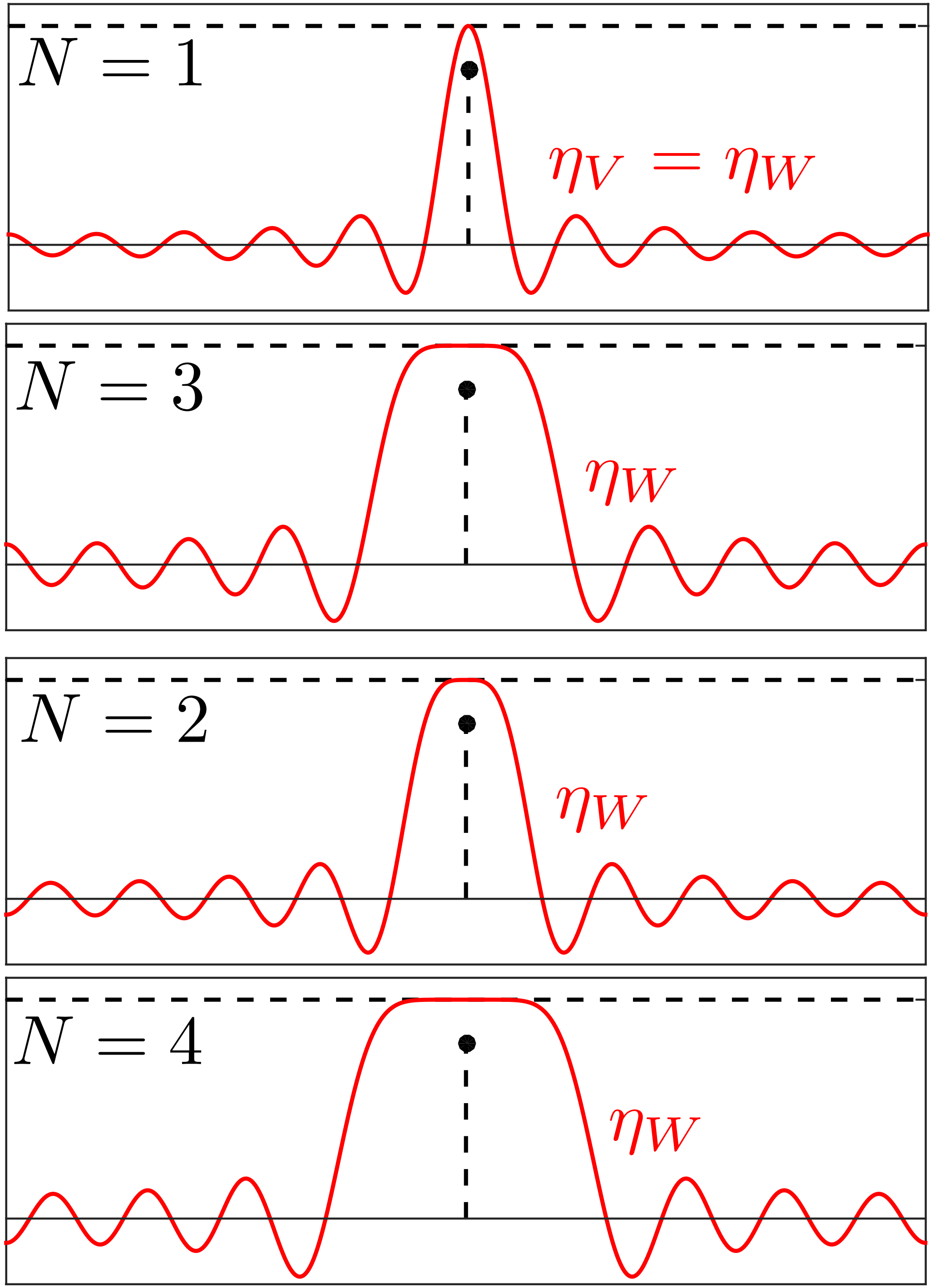


Asymptotic Certificate

$(2N-1)$-*Non degenerate:*

$$\eta_W \in \mathrm{ND}_N$$

$$\Longleftrightarrow \begin{cases} \forall\, t \neq 0, |\eta_W(t)| < 1 \\ {\eta_W}^{(2N)}(0) \neq 0 \end{cases}$$



Asymptotic Certificate

$(2N-1)$-Non degenerate:

$$\eta_W \in \mathrm{ND}_N$$

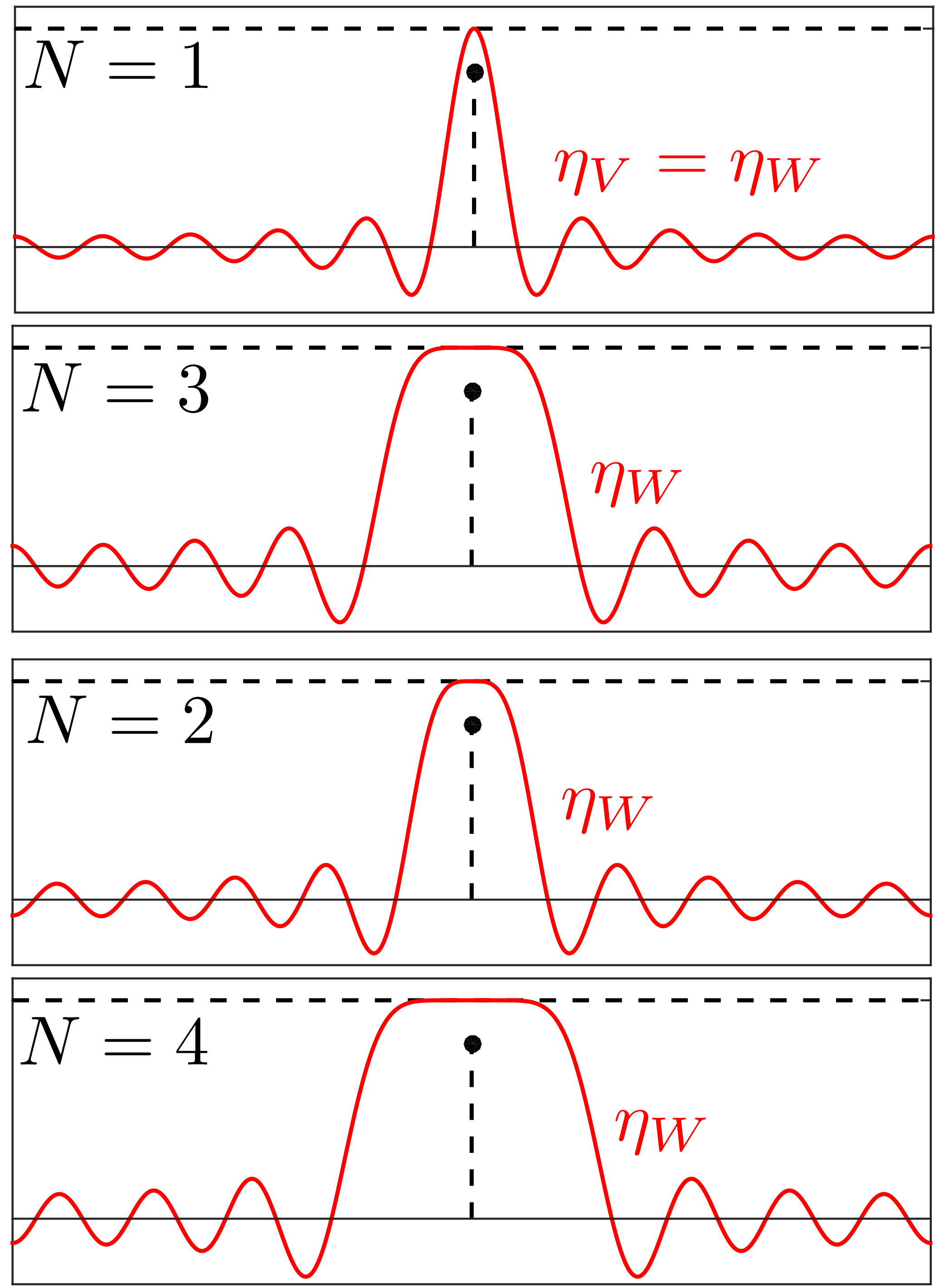
$$\Longleftrightarrow \begin{cases} \forall\, t \neq 0, |\eta_W(t)| < 1 \\ {\eta_W}^{(2N)}(0) \neq 0 \end{cases}$$

Lemma:

If $\eta_W \in \mathrm{ND}_N$, $\exists \Delta_0 > 0$,

$\forall\, \Delta < \Delta_0$, $\eta_V \in \mathrm{ND}(m_{\Delta x, a})$

$\rightarrow \eta_W$ govern stability as $\Delta \rightarrow 0$.



Asymptotic Robustness

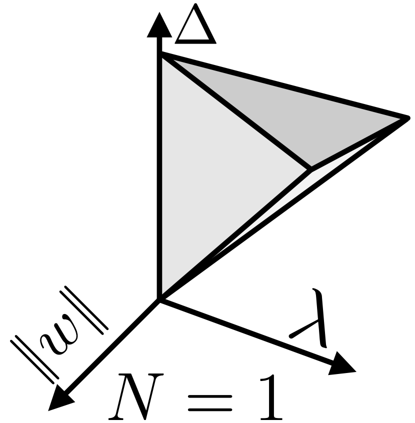
Theorem: If $\eta_W \in \mathrm{ND}_N$, letting $m_0 = m_{a,\Delta x}$, then

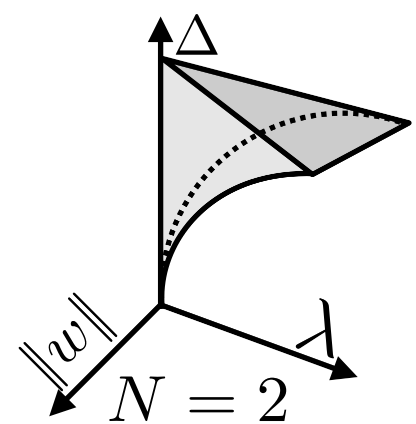
for $\left(\frac{w}{\lambda}, \frac{w}{\Delta^{2N-1}}, \frac{\lambda}{\Delta^{2N-1}}\right) = O(1)$

the solution of $\mathcal{P}_\lambda(y)$ for $y = \Phi(m_0) + w$ is

$\sum_{i=1}^N a_i^\star \delta_{\Delta x_i^\star}$ where $\|(x,a) - (x^\star, a^\star)\| = O\left(\frac{\|w\| + \lambda}{\Delta^{2N-1}}\right)$

[Denoyelle, D., P. 2015]





Asymptotic Robustness

Theorem: If $\eta_W \in \mathrm{ND}_N$, letting $m_0 = m_{a,\Delta x}$, then

for $\left(\frac{w}{\lambda}, \frac{w}{\Delta^{2N-1}}, \frac{\lambda}{\Delta^{2N-1}}\right) = O(1)$

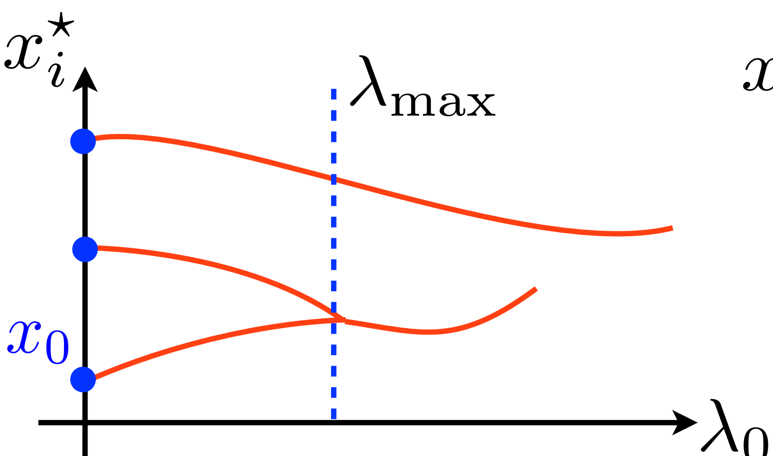
the solution of $\mathcal{P}_\lambda(y)$ for $y = \Phi(m_0) + w$ is

$\sum_{i=1}^{N} a_i^\star \delta_{\Delta x_i^\star}$ where $\|(x,a) - (x^\star, a^\star)\| = O\left(\frac{\|w\| + \lambda}{\Delta^{2N-1}}\right)$

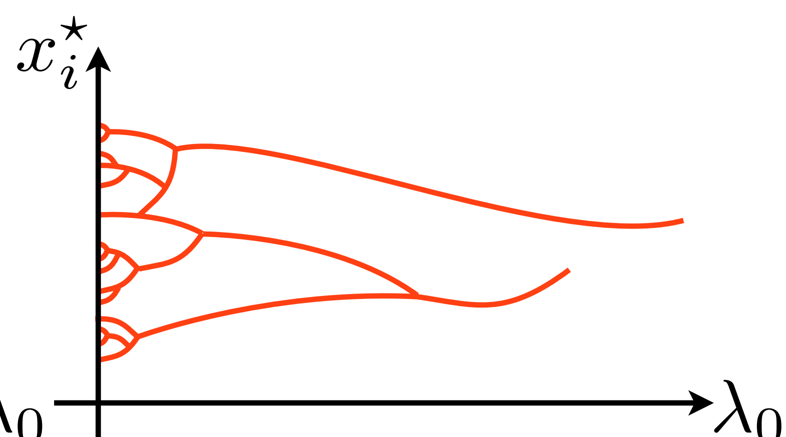
[Denoyelle, D., P. 2015]

$y = \Phi m_{a,\Delta x} + w$

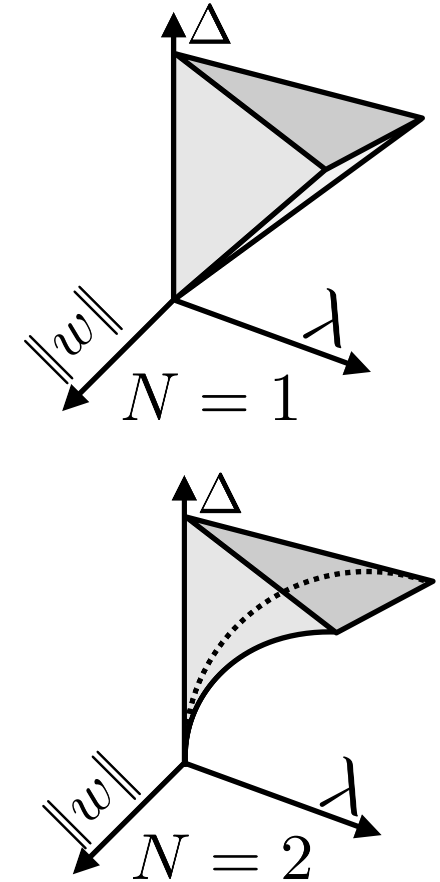
Noise: $w = \lambda w_0$.
Regularization: $\lambda = \lambda_0 \Delta^\alpha$



$\alpha = 2N - 1$



$\alpha < 2N - 1$

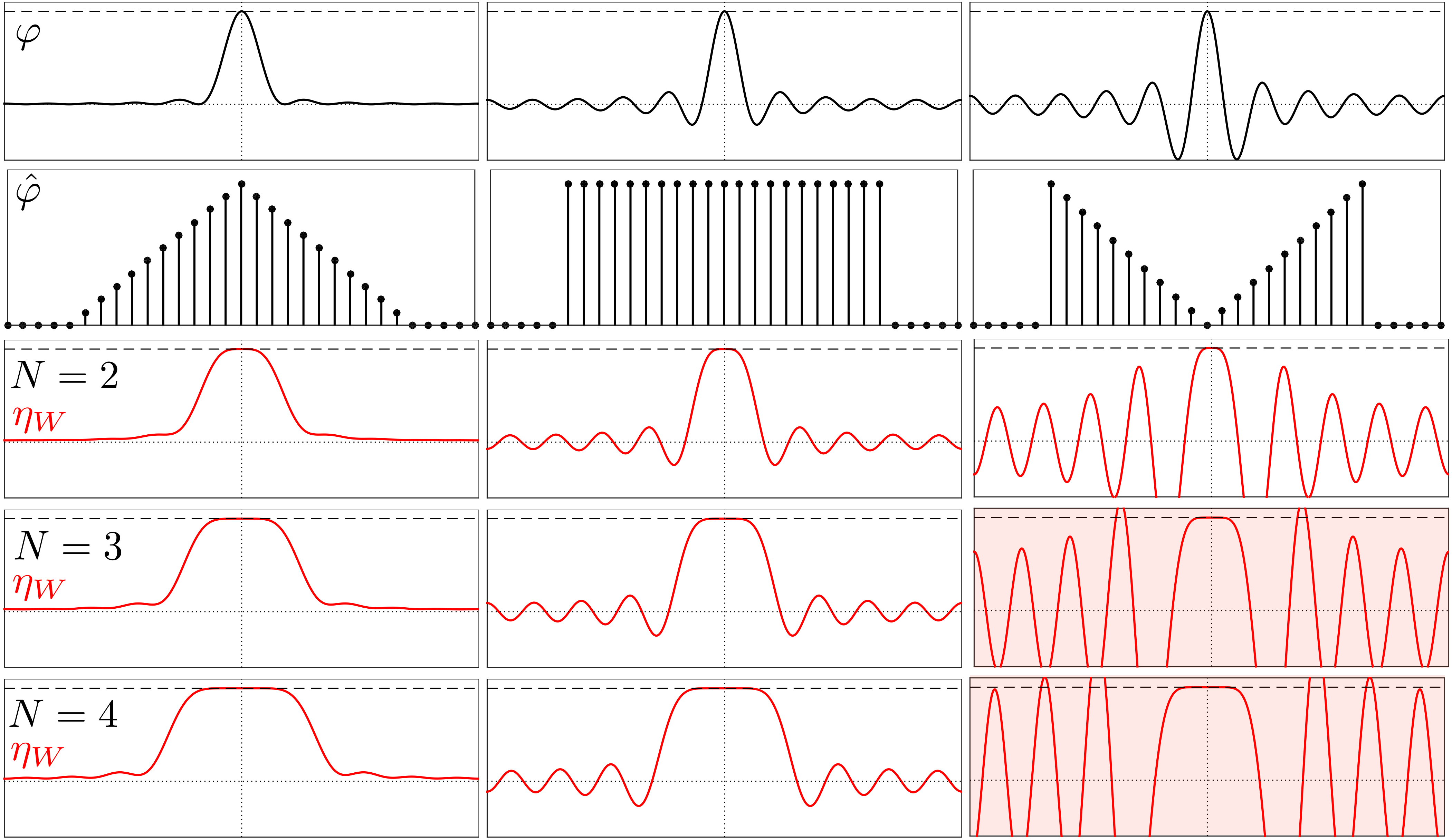


When is η_W Non-degenerate ?

Proposition: one has $\eta_W^{(2N)}(0) < 0$. $\longrightarrow$ “locally” non-degenerate.

When is η_W Non-degenerate ?

Proposition: one has $\eta_W^{(2N)}(0) < 0$. $\longrightarrow$ "locally" non-degenerate.

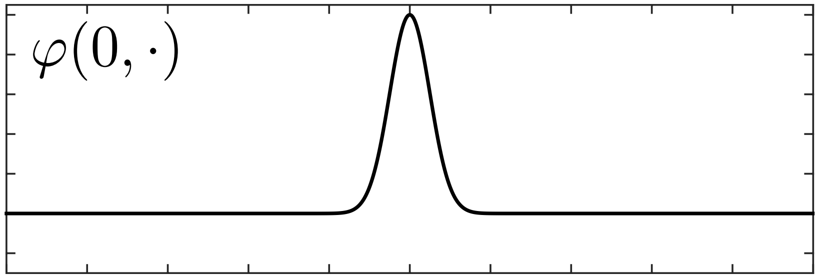


Gaussian Deconvolution

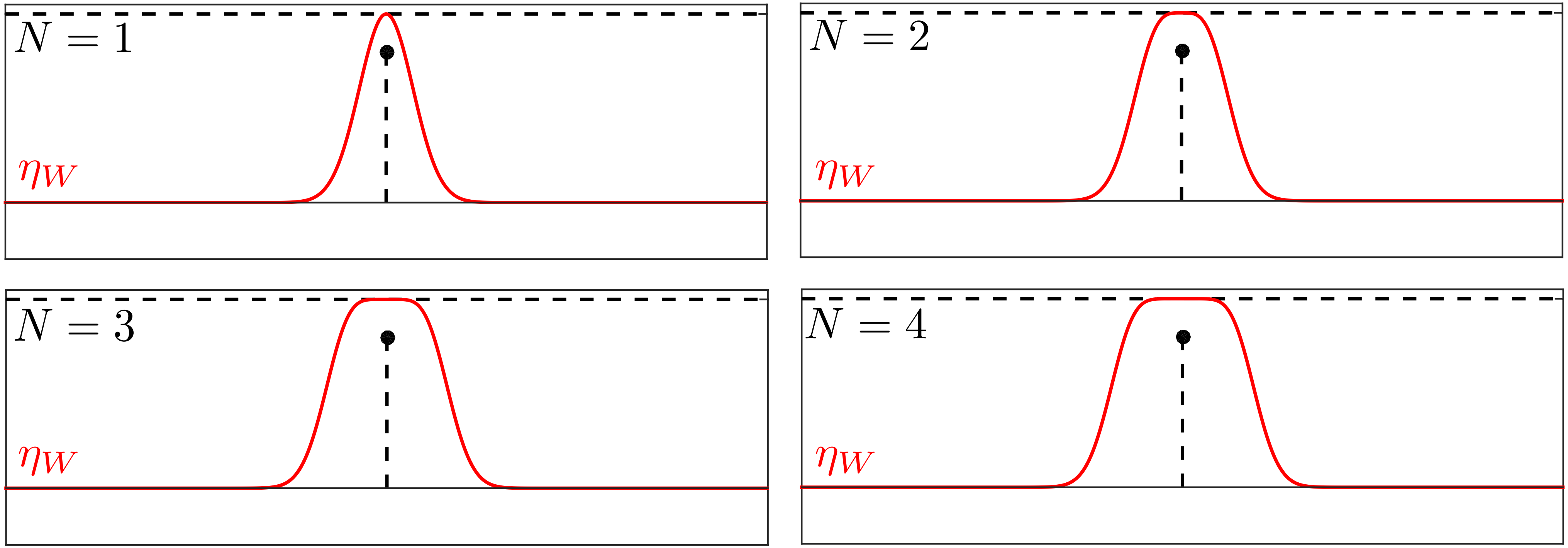
Gaussian convolution: $\varphi(x,t) = e^{-\frac{|x-t|^2}{2\sigma^2}}$ $\Phi(m) \overset{\text{def.}}{=} \int \varphi(x,\cdot)\mathrm{d}m(x)$

Proposition: $\eta_W(x) = e^{-\frac{x^2}{4\sigma^2}} \sum_{k=0}^{N-1} \frac{(x/2\sigma)^{2k}}{k!}$

In particular, η_W is non-degenerate.

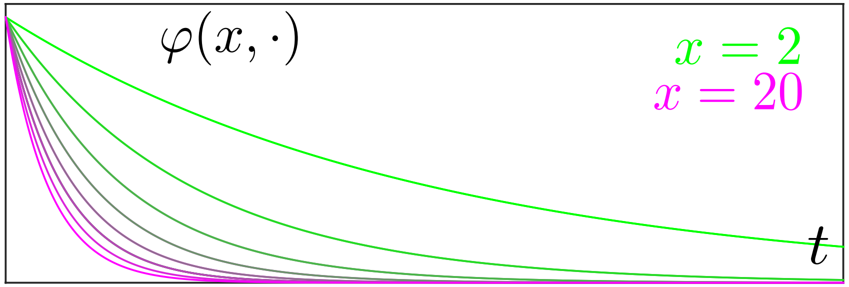


$\longrightarrow$ Gaussian deconvolution is support-stable.



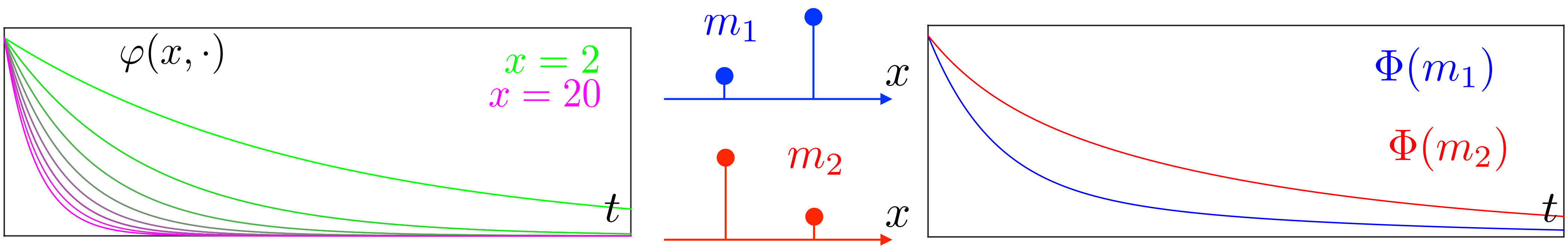
Laplace Transform Inversion

Laplace transform: $\varphi(x,t) = e^{-xt}$ $\Phi(m) \stackrel{\text{def.}}{=} \int \varphi(x,\cdot)\mathrm{d}m(x)$ [with E. Soubies]



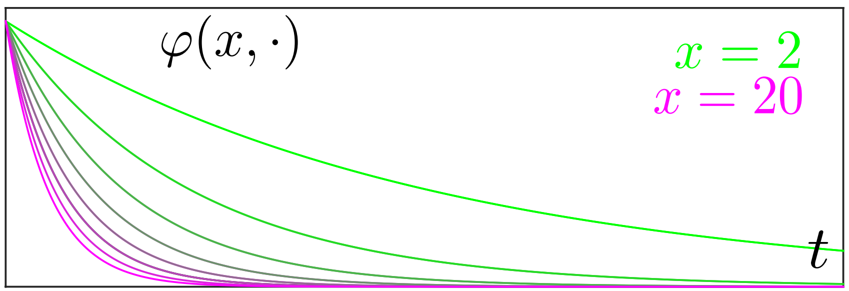
Laplace Transform Inversion

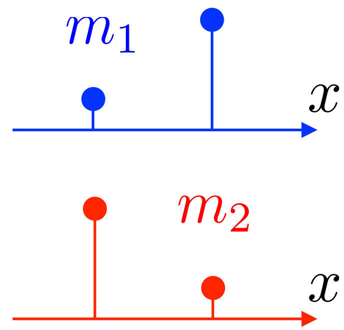
Laplace transform: $\varphi(x,t) = e^{-xt}$ $\quad \Phi(m) \overset{\text{def.}}{=} \int \varphi(x,\cdot)\mathrm{d}m(x)$ [with E. Soubies]

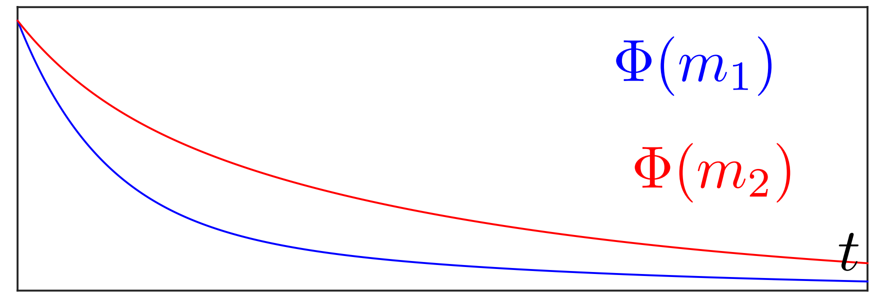


Laplace Transform Inversion

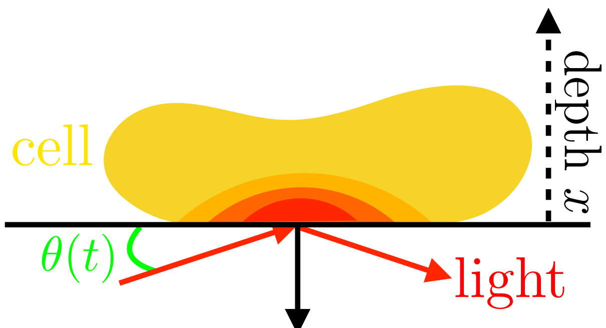
Laplace transform: $\varphi(x,t) = e^{-xt}$ $\quad \Phi(m) \overset{\text{def.}}{=} \int \varphi(x,\cdot)\mathrm{d}m(x)$ [with E. Soubies]



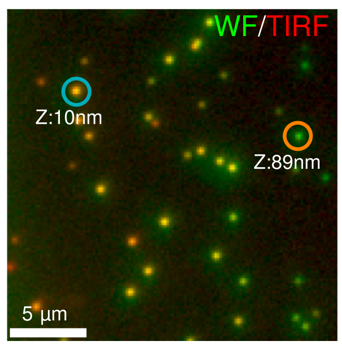


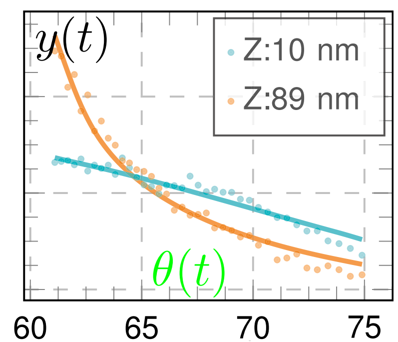


Total internal reflection fluorescence microscopy (TIRFM) → multiple angles $\theta(t)$.



$$y(t) = \Phi m(t)$$

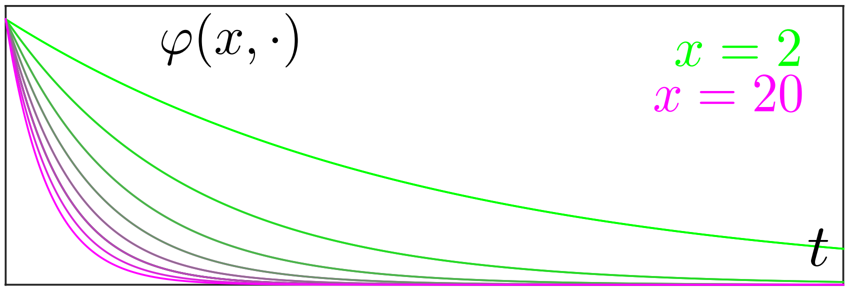


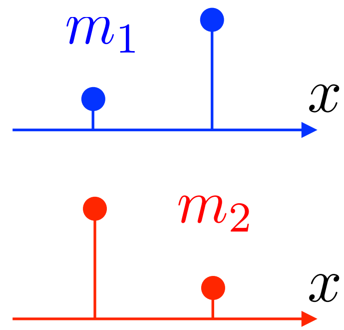


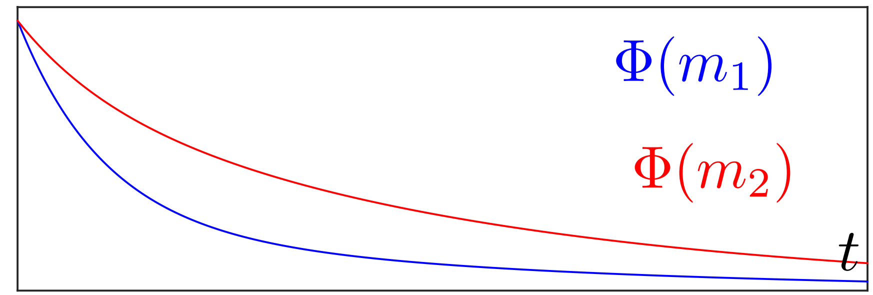
[Boulanger et al. 2014]

Laplace Transform Inversion

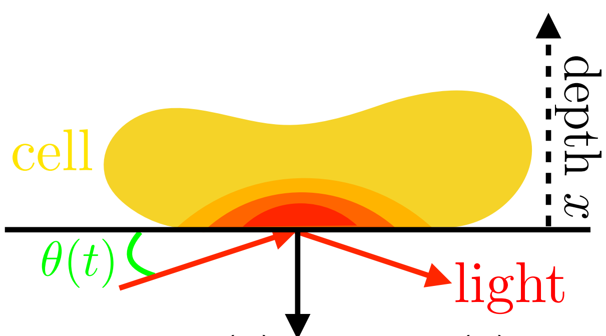
Laplace transform: $\varphi(x,t) = e^{-xt}$ $\quad \Phi(m) \overset{\text{def.}}{=} \int \varphi(x,\cdot)\mathrm{d}m(x)$ [with E. Soubies]



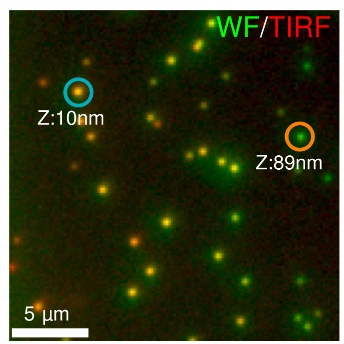


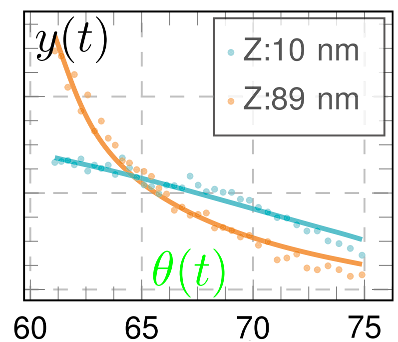


Total internal reflection fluorescence microscopy (TIRFM) $\rightarrow$ multiple angles $\theta(t)$.



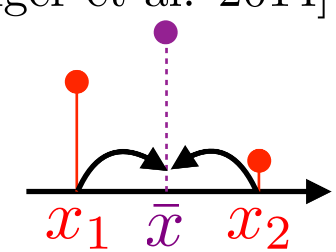
$$y(t) = \Phi m(t)$$

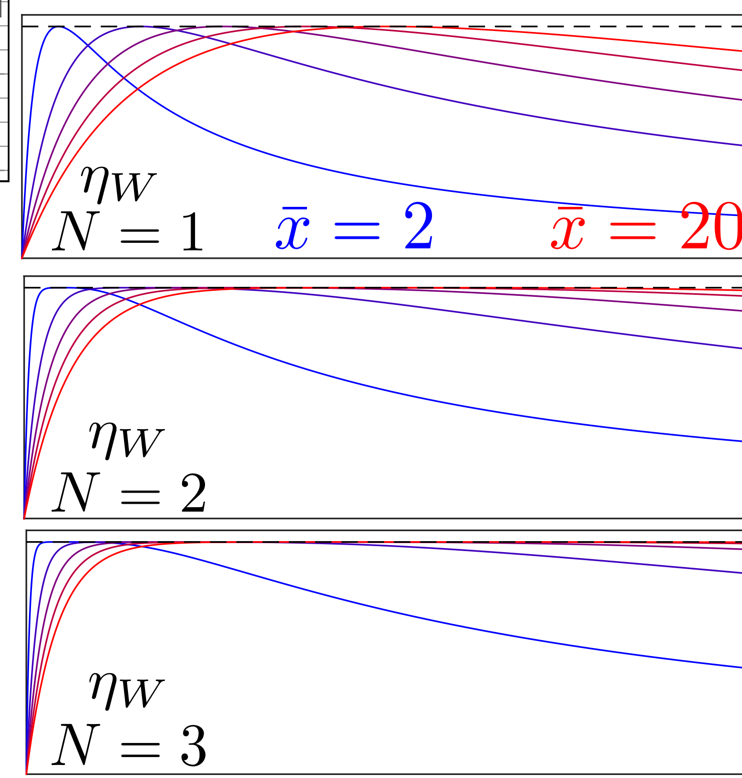




[Boulanger et al. 2014]

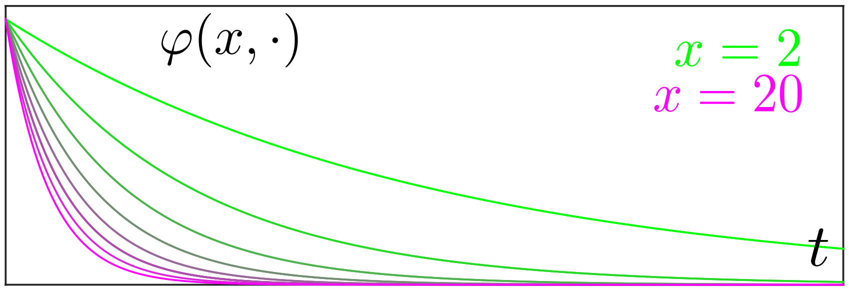
Non-translation-invariant operator
$\rightarrow \eta_W$ depends on $\bar{x}$!

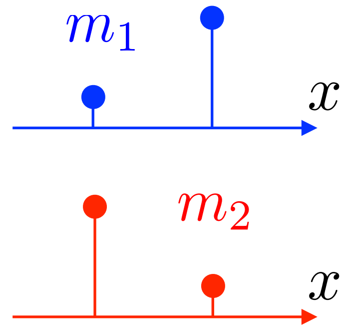


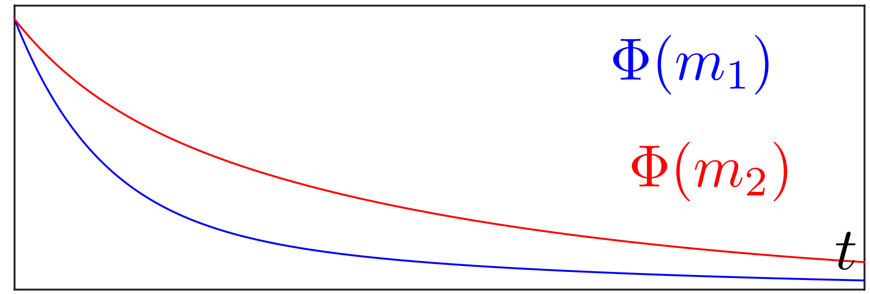


Laplace Transform Inversion

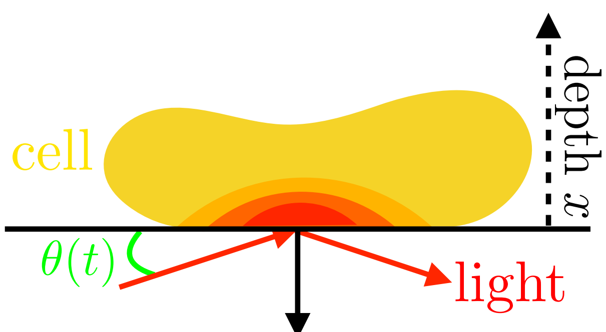
Laplace transform: $\varphi(x,t) = e^{-xt}$ $\quad \Phi(m) \overset{\text{def.}}{=} \int \varphi(x,\cdot)\mathrm{d}m(x)$ [with E. Soubies]



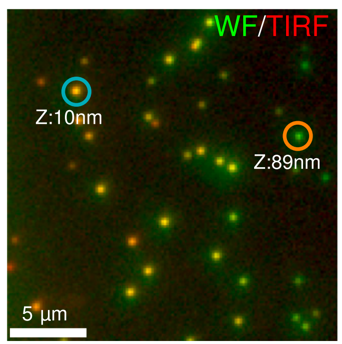


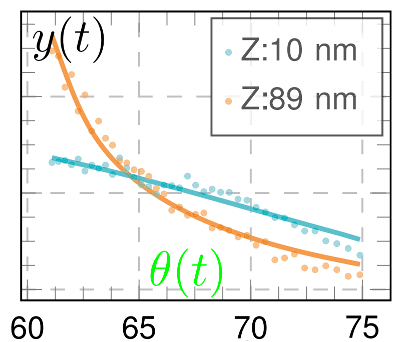


Total internal reflection fluorescence microscopy (TIRFM) → multiple angles $\theta(t)$.

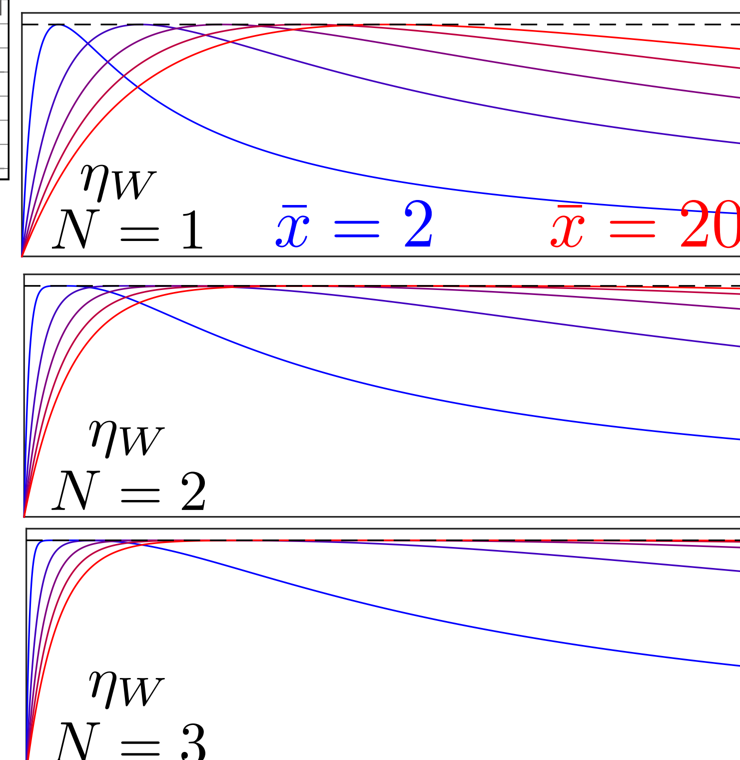


$y(t) = \Phi m(t)$

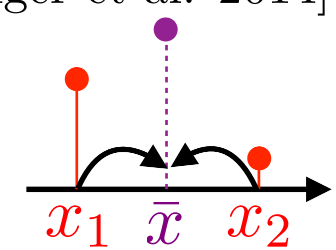




[Boulanger et al. 2014]



Non-translation-invariant operator
→ η_W depends on $\bar{x}$!



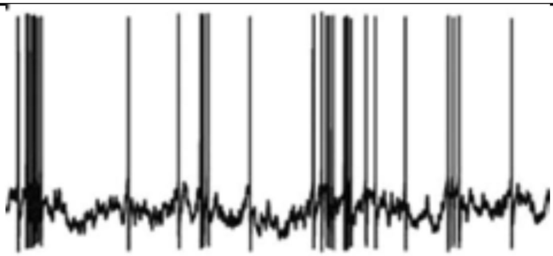
Proposition: $\eta_W(x) = 1 - \left(\frac{x - \bar{x}}{x + \bar{x}}\right)^{2N}$

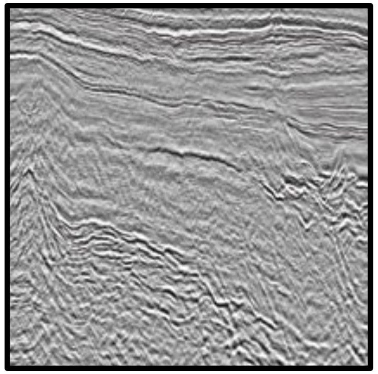
In particular, η_W is non-degenerate.

Conclusion

Deconvolution of measures:

$\longrightarrow L^2$ errors are not well-suited.





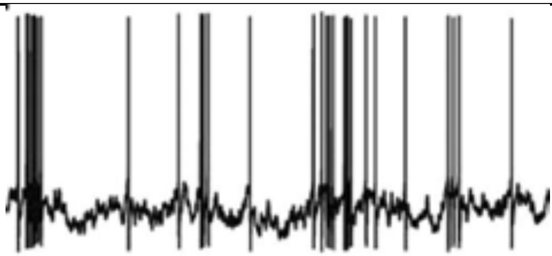
Weak-* convergence.

Optimal transport distance.

Exact support estimation.

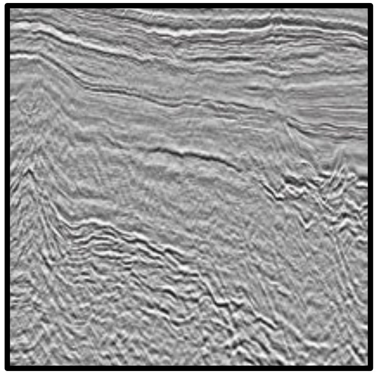
...

Conclusion



Deconvolution of measures:

$\longrightarrow$ L^2 errors are not well-suited.



- Weak-* convergence.
- Optimal transport distance.
- Exact support estimation.
- ...

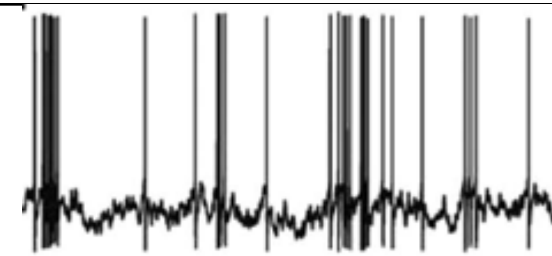
Low-noise behavior:

$\longrightarrow$ dictated by η_0.

$\longrightarrow$ checkable via η_V.

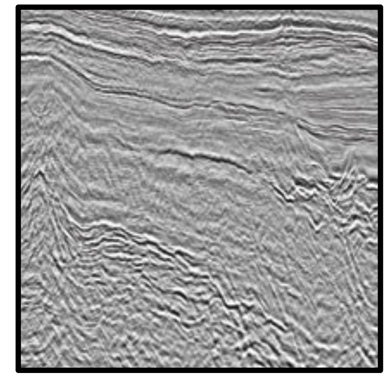
$\longrightarrow$ asymptotic via η_W.

Conclusion



Deconvolution of measures:

$\longrightarrow$ L^2 errors are not well-suited.



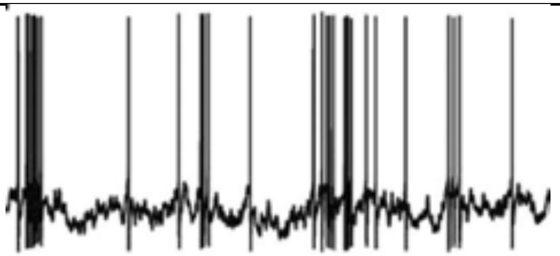
- Weak-* convergence.
- Optimal transport distance.
- Exact support estimation.
- …

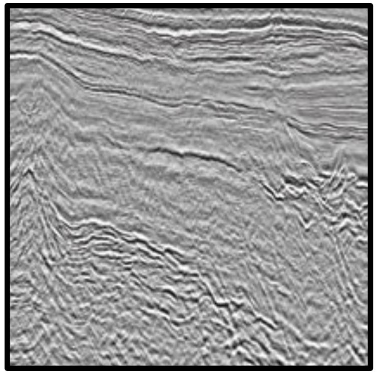
Low-noise behavior:
$\longrightarrow$ dictated by η_0.
$\longrightarrow$ checkable via η_V.
$\longrightarrow$ asymptotic via η_W.

Lasso on discrete grids: similar η_0-analysis applies.

$\longrightarrow$ Relate discrete and continuous recoveries.

Conclusion





Deconvolution of measures:

$\longrightarrow$ L^2 errors are not well-suited.

- Weak-* convergence.
- Optimal transport distance.
- Exact support estimation.
- ...

Low-noise behavior:

$\longrightarrow$ dictated by η_0.

$\longrightarrow$ checkable via η_V.

$\longrightarrow$ asymptotic via η_W.

Lasso on discrete grids: similar η_0-analysis applies.

$\longrightarrow$ Relate discrete and continuous recoveries.

Open problem: other regularizations (e.g. piecewise constant) ?
see [Chambolle, Duval, Peyré, Poon 2016] for TV *denoising*.