

Problem Sheet 1

Probability, random processes, and noise

1. If $F_X(x)$ is the distribution function of a random variable X and $x_1 \leq x_2$, show that $F_X(x_1) \leq F_X(x_2)$.
2. Use the definition of the cumulative distribution function to write an expression for the probability of a random variable to take values between x_1 and x_2 , and take limiting cases to arrive at the definition of the probability density function as the derivative of the distribution function.
3. Show that if two random variables are independent, they are also uncorrelated.
4. Show that the covariance of two random variables $c_{ij} \triangleq E[(X_i - \mu_{X_i})(X_j - \mu_{X_j})]$ is equal to:

$$c_{ij} = E\{X_i X_j\} - \mu_{X_i} \mu_{X_j} \quad (1)$$

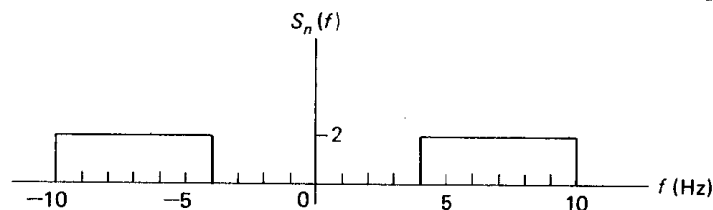
Then show that if the covariance of two random variables is zero, the two random variables are uncorrelated.

5. Consider the randomly-phased sinusoid

$$n(t) = A \cos(2\pi f_c t + \theta)$$

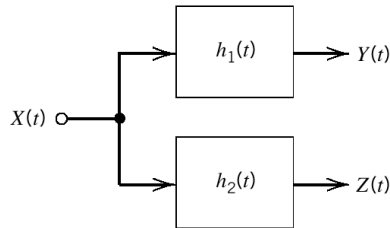
where A and f_c are constant amplitude and frequency, respectively, and θ is a random phase angle uniformly distributed over the range $[0, 2\pi]$. Calculate the mean and mean square of $n(t)$ using ensemble averages.

6. Consider a bandpass noise signal having the power spectral density shown below. Draw the PSD of $n_c(t)$ if the center frequency is chosen as:
 - (a) $f_c = 7$ Hz
 - (b) $f_c = 5$ Hz



Problem Sheet 2
Effects of noise on AM

1. A stationary zero-mean Gaussian random process $X(t)$ is passed through two linear filters with impulse response $h_1(t)$ and $h_2(t)$, yielding processes $Y(t)$ and $Z(t)$, as shown in the following figure.



Show that $Y(t_1)$ and $Z(t_2)$ are statistically independent if the transfer functions $H_1(f)$ and $H_2(f)$ do not overlap in the frequency domain (for example, when they are narrowband filters at different frequency bands).

2. Consider a message signal with a bandwidth of 10 kHz and an average power of $P = 10$ watts. Assume the transmission channel attenuates the transmitted signal by 40 dB, and adds noise with a power spectral density of:

$$S(f) = \begin{cases} N_o \left(1 - \frac{|f|}{200 \times 10^3}\right), & |f| < 200 \times 10^3 \\ 0, & \text{otherwise} \end{cases}$$

where $N_o = 10^{-9}$ watts/Hz.

What is the predetection SNR at the receiver if each of the following modulation schemes is used? Assume that a suitable filter is used at the input of the receiver to limit the out-of-band noise.

- (a) Baseband
- (b) DSB-SC with a carrier frequency of 100 kHz and a carrier amplitude of $A_c = 1$ v
- (c) DSB-SC with a carrier frequency of 150 kHz and a carrier amplitude of $A_c = 1$ v

[17.1 dB, 14 dB, 17 dB]

3. Given the baseband signal-to-noise ratio SNR_{Baseband} , consider an AM envelope detector for single-tone modulation, that is, the modulating wave is a sinusoidal wave

$$m(t) = A_m \cos(\omega_m t). \quad (2)$$

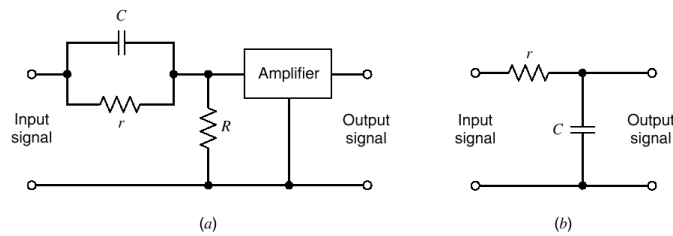
Assume that the noise power is small. Compute the output SNR in terms of the modulation index μ . What value of μ gives the maximum output SNR ?

Problem Sheet 3 Effects of noise on FM

1. Given the baseband signal-to-noise ratio SNR_{Baseband} , consider an FM detector for single-tone modulation, that is, the modulating wave is a sinusoidal wave

$$m(t) = A_m \cos(\omega_m t). \quad (3)$$

- (a) Compute the output SNR in terms of the modulation index β .
 (b) Comparing with the figure of merit for a full AM system, at what value of β will FM start to offer improved noise performance?
2. Consider the pre/de-emphasis filters for FM in the following figure.



(a) Pre-emphasis filter; (b) De-emphasis filter.

- (a) Assuming the gain of the amplifier is r/R . Derive the transfer function of the pre-emphasis filter, and show that it can be approximated as

$$H_{pe}(f) = 1 + j \frac{f}{f_0} \quad (4)$$

when $R \ll r$ and $2\pi frC \ll 1$.

- (b) Given the improvement factor

$$I = \frac{(W/f_0)^3}{3[(W/f_0) - \tan^{-1}(W/f_0)]}, \quad (5)$$

verify the gain of 13 dB for the parameters $W = 15$ kHz and $f_0 = 2.1$ kHz.

3. Suppose the modulating signal for FM is modelled as a zero-mean Gaussian random process $m(t)$ with standard deviation σ_m . One can make the approximation $m_p = 4\sigma_m$ as the overload probability $|m(t)| > 4\sigma_m$ is very small. Determine the output SNR for the FM receiver in the presence of additive white Gaussian noise, in terms of the deviation ratio β and baseband SNR.

Problem Sheet 4
Quantization and baseband transmission

1. A PCM output is produced by a uniform quantizer that has 2^n levels. Assume that the input signal is a zero-mean Gaussian process with standard deviation σ .
 - (a) If the quantizer range is required to be $\pm 4\sigma$, show that the quantization signal-to-noise ratio is $6n - 7.3$ dB.
 - (b) Write down an expression for the probability that the input signal will overload the quantizer (i.e., when the input signal falls outside of the quantizer range).
2. The input to a uniform n -bit quantizer is the periodic triangular waveform shown below, which has a period of $T = 4$ seconds, and an amplitude that varies between $+1$ and -1 volt.



Derive an expression for the signal-to-noise ratio (in decibels) at the output of the quantizer. Assume that the dynamic range of the quantizer matches that of the input signal.

$$[\text{SNR}_{\text{dB}} = 6.02n]$$

3. A source transmits 70% of the time 0s and 30% of the time 1s. What is the optimal threshold of detection so that the error rate is minimum? Assume that the noise is 0-mean white Gaussian with variance 9 and that the amplitude of the transmitted signal when 1 is indicated is $A = 5$ and it is 0 if 0 is indicated.

Problem Sheet 5
Digital modulation and demodulation

1. (a) Consider a binary ASK modulated-carrier system, which employs coherent demodulation. Let the carrier amplitude at the detector input be 0.7 volts. Assume an additive white Gaussian noise channel with a standard deviation of 0.125 volts. If the binary source stream has equal probabilities of occurrence of a symbol 0 and a symbol 1, estimate the probability of detection error.
(b) If PSK was used instead, what is the probability of error?
2. Consider the FSK system where symbol 0 and 1 are transmitted by frequency f_0 and f_1 , respectively. The unmodulated carrier frequency is $f_c = (f_0 + f_1)/2$. In practice, $f_c T \gg 1$ where T is the symbol period. Define the frequency separation $\Delta f \triangleq |f_1 - f_0|$. Larger Δf means larger bandwidth. FSK using $\Delta f = 1/2T$ is known as minimum-shift keying (MSK). Show that $\Delta f = 1/2T$ is the minimum separation so that the two sinusoids are orthogonal over one symbol period.
3. Consider a binary source alphabet where a symbol 0 is represented by 0 volts, and a symbol 1 is represented by 1 volt. Assume these symbols are transmitted over a baseband channel having uniformly distributed noise with a probability density function:

$$p(n) = \begin{cases} \frac{1}{2}, & |n| < 1 \\ 0, & \text{otherwise.} \end{cases}$$

Assume that the decision threshold T is within the range of 0 to 1 volt. If the symbols are equally likely, derive an expression for the probability of error.

Problem Sheet 6
Information theory

1. By what fraction is it theoretically possible to compress a binary bit stream if the probability of occurrence of one of the symbols is:
 - (a) 0.5
 - (b) 0.2
 - (c) 0.01

2. Using the Huffman coding procedure, construct a coding scheme for each of the 5-symbol alphabets having the following probabilities. Also calculate the average codeword length and compare it with the source entropy.
 - (a) 0.5, 0.25, 0.125, 0.0625, 0.0625
 - (b) 0.4, 0.25, 0.2, 0.1, 0.05

3. Alphanumeric data are entered into a computer from a remote terminal through a voice-grade telephone channel. The remote terminal has 110 characters on its keyboard, and that each character is represented using binary words. The telephone channel has a bandwidth of 3.2 kHz and signal-to-noise ratio at the receiver of 20 dB.
 - (a) Assuming the characters are equally likely and successive transmissions are statistically independent, how many bits are required to represent each character?
 - (b) Assuming the capacity is achieved, how fast can the characters be sent (in characters per second)?

[7 bits, 3043 characters / second]

4. An i.i.d. discrete source produces the symbols A and B with probabilities $p_A = \frac{3}{4}$ and $p_B = \frac{1}{4}$. The symbols are grouped into blocks of two and encoded as follows:

Grouped symbols	Binary code
AA	1
AB	01
BA	001
BB	000

Is this code optimum? If not, how efficient is it?

[No; 96% efficient]

Problem Sheet 7 Coding

1. Repetition codes represent the simplest type of linear block codes. In particular, a single message bit is encoded into a block of n identical bits, producing an $(n, 1)$ code.
 - (a) Write down the generator matrix and parity-check matrix of a $(5, 1)$ repetition code.
 - (b) List all the codewords of this code.
 - (c) Verify that the codewords satisfy the parity-check condition $xH^T = 0$.
2. Consider a binary symmetric channel with raw error probability $p = 10^{-2}$. Let us use an $(n, 1)$ repetition code where $n = 2m + 1$ is an odd integer and apply a majority-rule for decoding: if in a block of n received bits, the number of 0s exceeds the number of 1s, the decoder decides in favor of a 0; otherwise, it decides in favor of a 1.
 - (a) Find the general expression of the error probability P_e in terms of p .
 - (b) Compute the error probability for $n = 1, 3, 5, 7, 9, 11$.
3. Given the generator matrix of a linear block code,

$$G = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix} \quad (6)$$

- (a) Calculate the associated parity-check matrix, H .
- (b) Determine the minimum Hamming distance of this code.
- (c) Calculate a syndrome table for single bit errors.
- (d) Suppose the codeword 1000011 is sent, while the second and third bits are corrupted after transmission over the channel so that the received code vector is 1110011. Decode it using the syndrome table.