IMPERIAL COLLEGE LONDON

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2009

EEE/ISE PART II: MEng, BEng and ACGI

COMMUNICATIONS 2

Corrected Copy

Wednesday, 10 June 2:00 pm

Time allowed: 2:00 hours

There are FOUR questions on this paper.

Q1 is compulsory.
Answer Q1 and any two of questions 2-4.
Q1 carries 40% of the marks. Questions 2 to 4 carry equal marks (30% each).

Any special instructions for invigilators and information for candidates are on page 1.

Examiners responsible	First Marker(s) :	K.K. Leung, K.K. Leung
	Second Marker(s) :	J.A. Barria, J.A. Barria

Special Instructions for Invigilator: **None**

Information for Students: **None**

1. This is a compulsory question.

a. Consider two random variables X and Y.
   i. If X and Y are uncorrelated, express $E[XY]$ in terms of $E[X]$ and $E[Y]$. [2]
   ii. Comment on the relationship between the following two statements: "X and Y are independent" and "X and Y are uncorrelated". [3]
   iii. A property is defined as follows: If X and Y are uncorrelated, then X and Y are independent. Name one form of the joint probability distribution functions (PDFs) for X and Y that possesses such a property. [2]

b. Channel noise.
   i. What are the three most important parameters that define the capacity of a communication channel? [2]
   ii. Name the theoretical principle (theorem) that supports the validity of modeling channel noise as a Gaussian distribution in most cases and why? [4]
   iii. What is it meant by "white noise"? [3]
   iv. What are the *in-phase* and *quadrature-phase components* (which are denoted by $n_c(t)$ and $n_s(t)$, respectively) of the band-limited white noise $n(t)$ where the centre of the frequency band is $\pm f_c$ Hz? [4]
   v. Compare the average power of the noise waveforms: $n(t)$, $n_c(t)$ and $n_s(t)$. [2]

c. Digital communications.
   i. To transmit a signal with a bandwidth of W Hz in a digital communication system, what is the minimum sampling rate so that the original signal can be recovered at the receiver and why? Use a frequency-domain diagram to explain. [4]
   ii. Let Δ be the uniform gap between two adjacent quantizing levels and assume that the quantization error (noise) is uniformly distributed from $-\Delta/2$ to $\Delta/2$. Write down the probability density function (pdf) for the quantization noise. [2]
   iii. Using the result in the above part ii, derive the quantization noise power. [5]

d. Source entropy.
   i. For a communication system with three-symbol alphabet {A, B, C}, the probabilities of transmitting these symbols, P_A, P_B and P_C, are 0.7, 0.2 and 0.1, respectively. Calculate the source entropy of the system. [4]
   ii. What is the main objective of the Huffman coding algorithm? [3]

2. Noise effects on amplitude modulation.

Consider an amplitude modulation (AM) signal $s(t)$ with the following notation:

A is the amplitude of the carrier,
f_c is the carrier frequency,
$m(t)$ is the message (information) signal,
$n(t)$ is the noise signal,
$n_c(t)$ is the in-phase component of the noise signal,
$n_s(t)$ is the quadrature phase component of the noise signal, and
N_o is the power spectral density for the noise in the baseband transmission.

a. Provide an expression for the AM signal $s(t)$. [3]

b. Assuming that noise is negligible, state and explain a condition under which an envelop detection method can be employed to recover the message signal at the receiver. [3]

c. What is "synchronous detection"? What is the key technical difficulty in using the synchronous detection to receive the AM signal? [3]

d. Draw a block diagram to depict the AM transmission, the additive-white-Gaussian-noise (AWGN) channel and the receiver structure using the synchronous detection. [3]

e. Give an expression for the pre-detection signal as a function of the AM signal, $n_c(t)$, and $n_s(t)$ of the noise signal. [3]

f. Let the signal power at the receiver output be denoted by $P = E\{m^2(t)\}$. Derive the signal-to-noise ratio (SNR) of the receiver output for the synchronous detection. [3]

g. Derive a relationship between the SNR of the receiver output for the synchronous detection and that for the baseband transmission with the same transmitted power. [5]

h. Now consider the envelop detection for the AM signal with an assumption that noise is much smaller when compared with the carrier and message signal. By approximating the envelop of the pre-detection signal under the assumption, derive a relationship between the SNR of the receiver output for the envelop detection and that for the baseband transmission with the same transmitted power. [5]

i. Compare and comment on your results in parts g and h. [2]

3. a. Digital communications.

Consider a digital communication channel that transmits a polar signal of 0 or 1. When a 0 and 1 are sent, the pulse amplitudes are $-A_P$ and A_P, respectively. The channel has additive zero-mean white Gaussian noise with the rms value of the noise denoted by σ_n (i.e., the noise power is σ_n^2). Assume that the detection threshold *T* is *0* at the receiver. That is, if the received signal is equal to or great than *0*, the signal is considered to be a 1. Otherwise, a 0 is considered to be sent.

i. Given that a 0 is being sent, derive the error probability $P_E(0)$ in terms of A_P, σ_n and

$$Q(y) = \frac{1}{\sqrt{2\pi}} \int_y^\infty e^{-x^2/2} dx. \qquad [4]$$

ii. Similarly, given that a 1 is being sent, obtain the error probability $P_E(1)$ in terms of A_P, σ_n and $Q(y)$. [4]

iii. Using results in parts i and ii and assuming equal probability for sending 0 and 1, obtain the error probability P_E in terms of A_P, σ_n and $Q(y)$. [3]

iv. Now, let the probabilities of sending a 0 and 1 be denoted by p_0 and p_1 where $p_1 > p_0$. Discuss whether the new detection threshold *T* for $p_1 > p_0$ should be greater or less than 0 in order to minimize the overall error probability. Provide a physical interpretation for your answer. [4]

b. Power spectral density.

Consider a linear system with the input, output and impulse-response functions denoted by $x(t)$, $y(t)$ and $h(t)$, respectively. It is known that $y(t) = h(t) * x(t)$ where * represents convolution. The corresponding relationship in Fourier transforms is $Y(f) = H(f)X(f)$. Let $R_x(\tau)$ and $R_y(\tau)$ be the respective autocorrelation function for $x(t)$ and $y(t)$. Furthermore, let $P_x(f)$ and $P_y(f)$ be the power spectral density (PSD) for $x(t)$ and $y(t)$, respectively. Recall that the PSD is the Fourier transform of the autocorrelation function of a given signal.

i. Using the definition of autocorrelation function, namely $R_y(\tau) = E[y(t)y(t+\tau)]$, and the fact that $y(t) = h(t) * x(t)$, prove that $R_y(\tau) = h(-\tau) * u(-\tau)$ where $u(-\tau) \equiv h(\tau) * R_x(\tau)$. [10]

ii. Based on the result in part i above, show that $P_y(f) = |H(f)|^2 P_x(f)$. [5]

4. Error control code.

   a. For any error control code, express the maximum number of errors the coding scheme can detect in terms of the minimum Hamming distance (denoted by $d_{\min}$) and explain why. [3]

   b. To continue with part a, express the maximum number of errors t the coding scheme can correct in terms of $d_{\min}$ and explain why. [4]

   c. For the (7,4) Hamming code, the generator matrix is given by

$$G = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{pmatrix}$$

      i. Using the generator matrix G, calculate the code words for two source blocks: 1010 and 1101. [4]

      ii. Obtain the Hamming distance between the code words obtained in part i. [2]

      iii. Write the parity check matrix H associated with the above generator matrix G. [5]

      iv. For a received code word y, which may contain errors, describe how the associated syndrome s can be determined. [2]

      v. For the error vector 0000010, find the corresponding syndrome. [3]

   d. For five symbols S_0, S_1, S_2, S_3, S_4 of the alphabet of a discrete memory-less source, their probabilities are 0.4, 0.2, 0.2, 0.1 and 0.1, respectively. Use the Huffman encoding algorithm to generate the code words for the source. [7]

E2.4 Communications II

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1. a. i. Given X, Y are uncorrelated,

$$E\left[(X-m_X)(Y-m_Y)\right]=0$$

$$\Rightarrow \quad E\left[XY - m_X Y - m_Y X + m_X m_Y\right]=0$$

$$\Rightarrow \quad E\left[XY\right] - \cancel{m_X E[Y]} - m_Y E[X] + \cancel{m_X m_Y} = 0$$

$$\Rightarrow \quad E[XY] = m_Y E(X)$$

i.e. $\quad E(XY) = E(X)E(Y)$.

ii. If "X & Y are independent," then "X and Y are uncorrelated."
However, the converse is not necessarily true.
That is, if "X & Y are uncorrelated," "X & Y are not necessarily independent.

iii. One such PDF is jointly normal distribution for X & Y.

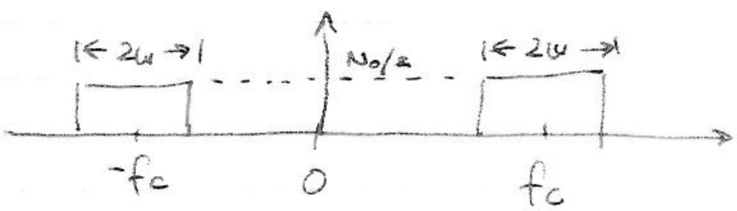
1. b. i. The three parameters are:

bandwidth, signal power and noise power.

ii. The Central Limit Theorem states that the normalized sum of independent R.V.'s has a Gaussian distribution. Since channel noise is originated from many independent sources, thus the total noise behaves like a Gaussian distribution as the Theorem suggests.

iii. White noise has the constant power sprectrum density (PSD) over all frequencies.

iv. The power spectral density of the band-limited white noise $n(t)$ is depicted as follows



The bandpass represtation of $n(t)$ is

$$n(t) = n_c(t)\cos(2\pi f_c t) - n_s(t)\sin(2\pi f_c t)$$

where $$n_c(t) = \sum_k a_k \cos(2\pi(f_k - f_c)t + \theta_k)$$

$$n_s(t) = \sum_k a_k \sin(2\pi(f_k - f_c)t + \theta_k)$$

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. b. iv. (Cont'd) In words, $n_c(t)$ is the baseband noise component that is in-phase with the carrier at frequency f_c, while $n_s(t)$ is that component that is $90°$ out of phase with the carrier.

v. The average power of $n(t)$, $n_c(t)$ and $n_s(t)$ is identical (because $n_c(t)$ & $n_s(t)$ are $90°$ out of phase with each other, although $n(t) = n_c(t)\cos(2\pi f_c t) - n_s(t)\sin(2\pi f_c t)$.

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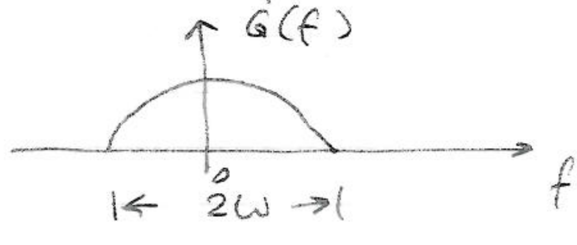
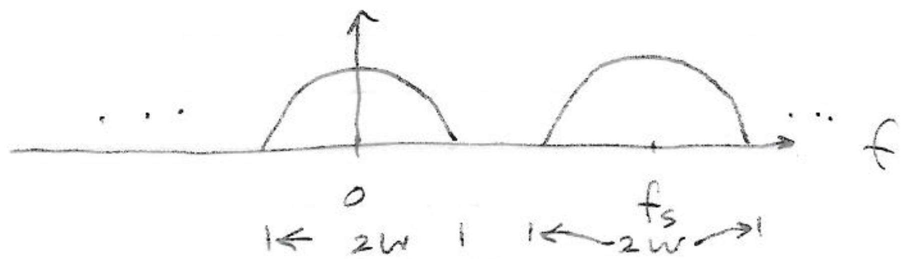
1. c. i. The minimum sampling rate is $2W$ Hz, referred to as Nyquist rate.

The physical interpretation of the Nyquist rate is given as follows.

The frequency domain representation of the signal:



The frequency domain representation of the sampled signal:



From the above diagram, it is clear that if the sampling frequency f_s is greater than $2W$ Hz, the replica of the signal do not overlap with each other i.e., with no changes to the signal due to the sampling operations. Therefore, the original signal can be recovered by a low-pass filter.

1.c. ii. Let N denote the noise level and $f_N(t)$ be the pdf for the noise.

Then

$$f_N(t) = \begin{cases} 1/\Delta & \text{for } -\Delta/2 \leq t \leq \Delta/2 \\ 0 & \text{otherwise} \end{cases}$$

iii. Noise power $P_N = E\left((N - E(N))^2\right)$

Since $E(N) = 0$, $P_N = E(N^2)$

Then, $P_N = E(N^2) = \int_{-\Delta/2}^{\Delta/2} t^2 \cdot \frac{1}{\Delta} \cdot dt^2$

$$\Rightarrow P_N = \frac{1}{\Delta} \cdot \frac{t^3}{3} \Big|_{-\Delta/2}^{\Delta/2}$$

$$\Rightarrow P_N = \frac{1}{\Delta} \cdot \left[\frac{\Delta^3}{8} + \frac{\Delta^3}{8} \right] \cdot \frac{1}{3}$$

$$\Rightarrow P_N = \frac{\Delta^2}{12}.$$

1. d. i. Source entropy

$$H(S) = -\sum_{k} p_k \log_2 p_k$$

$$\Rightarrow \quad H(S) = -\sum_{k=1}^{3} p_k \log_2 p_k$$

$$= -\left[0.7 \log_2 0.7 + 0.2 \log_2 0.2 + 0.1 \log_2 0.1 \right]$$

$$\Rightarrow \quad H(S) = 0.7 \times 0.515 + 0.2 \times 2.322 + 0.1 \times 3.322$$

$$H(S) = 1.157 \text{ bits/symbol}$$

ii. The main objective of the Huffman coding algorithm is to choose codeword lengths for a given source alphabet so that more probable sequences have shorter codewords. On the contrary, less likely sequences will have longer codewords.

2. a. $S(t) = [A + m(t)] \cos(2\pi f_c t)$

b. Let $m_p = \max_t |m(t)|$

With negligible noise, a conditi- under which the envelop detection can be employed to recover the signal is.

$$\mu = \frac{m_p}{A} \leq 1 \iff A \geq m_p$$

That is, the amplitude of the carrier is larger than the maximum (peak) amplitude of the message signal. Given $A \geq m_p$, the envelop of the received signal is the message information.

c. Synchronous detection is a detection method where the receiver multiplies the received signal with the carrier with exactly identical phase and frequency of that at the transmitter, in order to recover the transmitted signal.

The key technical challenge for synchronous detection is to generate the carrier with precisely identical frequency and phase of the carrier at the transmitting side.

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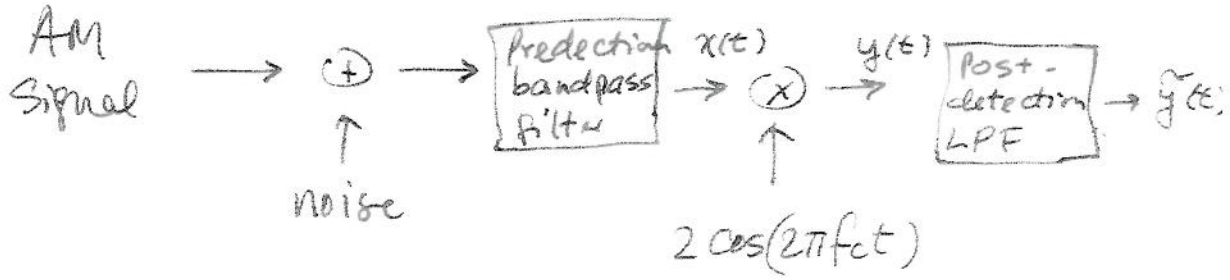
2. d.



e. Pre-detection signal

$$x(t) = s(t) + n(t)$$

$$= [A + m(t)] \cos(2\pi f_o t) + n(t)$$

$$\Rightarrow x(t) = [A + m(t) + n_c(t)] \cos(2\pi f_c t) - n_s(t) \sin(2\pi f_c t)$$

f. The signal power at the receiver output

$$P = E[m^2(t)]$$

Noise power : $P_N = 2 N_o W$ ($\because$ The bandwidth used by AM is 2W.)

$$\Rightarrow SNR_{AM/sync.\ det.} = \frac{P}{2 N_o W}$$

g. Transmitting power $P_T = E[s(t)^2] = E[(A + m(t))^2 \cos^2(2\pi f_o t)]$

$$\Rightarrow P_T = \frac{A^2}{2} + \frac{E[m^2(t)]}{2}$$

$$\Rightarrow P_T = \frac{A^2}{2} + \frac{P}{2}$$

Therefore, the SNR for baseband transmission

$$SNR_{baseband} = \frac{P_T}{N_0 W}.$$

← Bandwidth is W for baseband transmission.

$$\Rightarrow SNR_{baseband} = \frac{A^2+P}{2} \cdot \frac{1}{N_0 W}$$

From part f, $SNR_{AM/sync\ det} = \frac{P}{2 N_0 W}$

Therefore, $SNR_{AM/sync\ det} = \frac{P}{A^2+P} \cdot SNR_{baseband}$

h. The envelope of the signal $x(t)$

$$y(t) = \sqrt{(A + m(t) + n_c(t))^2 + n_s(t)^2}$$

When noise is much smaller than the carrier and message signal,

$$|A + m(t) + n_c(t)| >> |n_s(t)|$$

Thus, $y(t) \simeq A + m(t) + n_c(t)$

Based on this approximation,

$$SNR_{env} \doteq \frac{E[m(t)^2]}{2 N_0 W} \doteq \frac{P}{2 N_0 W}.$$

Since $SNR_{baseband} = \frac{A^2 + P}{2N_oW}$

Thus, $SNR_{env} \doteq \frac{P}{A^2+P} \cdot SNR_{baseband}$

∴ i. Results in parts g & h reveal that

$$SNR_{env} \approx SNR_{AM/\text{sync det}}.$$

That is, for low noise, the envelop detection performs very closely to that of synchronous detection.

3.a.i. $P_E(0) = P_r(-A_p + n > 0)$ where n is the random noise.

$$\Rightarrow \quad P_E(0) = P_r(n > A_p)$$

Since the noise has a zero-mean Gaussian distribution,

$$P_E(0) = P_r(n > A_p)$$

$$= \frac{1}{\sqrt{2\pi}\,\sigma_n} \int_{A_p}^{\infty} e^{-n^2/2\sigma_n^2}\, dn$$

$$= \frac{1}{\sqrt{2\pi}} \int_{A_p/\sigma_n}^{\infty} e^{-x^2/2}\, dx$$

$$\Rightarrow \quad P_E(0) = Q\left(\frac{A_p}{\sigma_n}\right)$$

ii. Similarly,

$$P_E(1) = P_r(A_p + n < 0)$$

$$= P_r(n < -A_p)$$

$$\Rightarrow \quad P_E(1) = \frac{1}{\sqrt{2\pi}\,\sigma_n} \int_{-\infty}^{-A_p} e^{-n^2/2\sigma_n^2}\, dn$$

Since $e^{-n^2/2\sigma_n^2}$ is symmetrical about the axis of $n=0$,

$$P_E(1) = \frac{1}{\sqrt{2\pi}\,\sigma_n} \int_{A_p}^{\infty} e^{-n^2/2\sigma_n^2}\, dn$$

$$\Rightarrow \quad P_E(1) = Q\left(\frac{A_p}{\sigma_n}\right).$$

3. a. iii. $$P_E = P_E(0) \cdot P_0 + P_E(1)\, P_1$$

$$= Q\left(\frac{A_P}{\sigma_n}\right) \cdot \frac{1}{2} + Q\left(\frac{A_P}{\sigma_n}\right) \cdot \frac{1}{2}$$

$$\Rightarrow \quad P_E = Q\left(\frac{A_P}{\sigma_n}\right).$$

iv. When $P_1 > P_0$, the detection threshold T should be set less than 0.

This is so because with $P_1 > P_0$, the received signal is likely to be greater than zero. When a 1 is transmitted, setting $T < 0$ will help reduce the probability of error. On the other hand, setting $T < 0$ will increase the probability of error for sending a 0. At balance, given $P_1 > P_0$, with $T < 0$, the reduction of the error prob for sending a 1 will be well compensated for the increase of the error prob for sending a 0. That is, setting $T < 0$ will reduce the overall error prob, given $P_1 > P_0$.

3.b.i. $R_y(\tau) = E[\,y(t)\, y(t+\tau)\,]$

$$= E\left[\int_{-\infty}^{\infty} h(t_1)\, x(t-t_1)\,dt_1 \cdot \int_{-\infty}^{\infty} h(t_2)\, x(t+\tau-t_2)\,dt_2\right]$$

$$\Rightarrow R_y(\tau) = E\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(t_1)\, h(t_2)\, x(t-t_1)\, x(t+\tau-t_2)\,dt_1\,dt_2\right]$$

$$\Rightarrow R_y(\tau) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(t_1)\, h(t_2)\, E\left[x(t-t_1)\, x(t+\tau-t_2)\right] dt_1\, dt_2$$

$$\Rightarrow R_y(\tau) = \int_{-\infty}^{\infty}\underline{\int_{-\infty}^{\infty}} \underbrace{h(t_1) h(t_2) R_x(\tau+t_1-t_2)\,dt_2}_{=\, h(\tau+t_1) * R_x(\tau+t_1)}\,dt_1$$

$$\Rightarrow R_y(\tau) = \int_{-\infty}^{\infty} h(t_1)\, h(\tau+t_1) * R_x(\tau+t_1)\,dt_1$$

$$\Rightarrow R_y(\tau) = \int_{-\infty}^{\infty} h(t_1)\, s(\tau+t_1)\,dt_1$$ where $s(t+\tau) \triangleq h(\tau+t_1) * R_x(\tau+t_1)$

$$\Rightarrow R_y(\tau) = \int_{-\infty}^{\infty} h(t_1)\, u(-\tau-t_1)\,dt_1$$ where $u(-t) \triangleq s(t)$

Therefore $R_y(\tau) = h(-\tau) * u(-\tau)$

or $R_y(\tau) = h(-\tau) * h(\tau) * R_x(\tau)$

3.6. ii. $R_y(\tau) = h(-\tau) * h(\tau) * R_x(\tau)$

Take the Fourier transforms on both sides of the above. We obtain

$$P_y(f) = \mathcal{F}[h(-\tau)] \cdot H(f) \cdot P_x(f)$$

By the property of Fourier transform,

$\mathcal{F}[h(-\tau)] = H^*(f)$ where $*$ is the complex conjugate

Substitute this into the expression for $P_y(f)$,

$$P_y(f) = H^*(f)\, H(f) \cdot P_x(f)$$

$$\Rightarrow \quad P_y(f) = |H(f)|^2 P_x(f).$$

QED

4. a. The number of errors that can be detected is $d_{min} - 1$.

This is so because there must be at least d_{min} errors that can turn an input code into another different but valid codeword, escaping the detection mechanism.

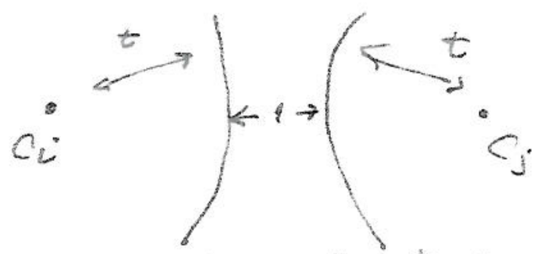
For any input codeword with less than d_{min} errors, the resultant vector cannot be a valid codeword, thus enabling detection.

b. The number of errors t that can be corrected is

$$t = \left\lfloor \frac{d_{min} - 1}{2} \right\rfloor$$ where $\lfloor x \rfloor$ is the largest

integer that is smaller or equal to x.

Consider two codewords C_i and C_j.



When there are at most t errors for both c_i and c_j, the resultant codewords still differ from each other by one Hamming distance. Therefore, the erroneous codewords can be corrected (or treated) as the "nearest" to the valid codewords.

4. c. i. The output codeword

$x = u \cdot G$ where u is the source code block

$$(1\,0\,1\,0)\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{pmatrix}$$

$$= (1\,0\,1\,0\,1\,0\,1)$$

$$(1\,1\,0\,1)\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{pmatrix}$$

$$= (1\,1\,0\,1\,0\,0\,1)$$

ii. The Hamming distance between the obtained codewords

$$\begin{array}{r} 1\,0\,1\,0\,1\,0\,1 \\ \oplus\ 1\,1\,0\,1\,0\,0\,1 \\ \hline 0\,1\,1\,1\,1\,0\,0 \end{array}$$

Hamming distance = 4.

iii. The parity check matrix H

$$H = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 \end{pmatrix}$$

iv. The syndrome $s = y \cdot H^T$

where H is the parity check matrix.

4. c. v. Syndrome $S = eH^T$

So, $$S = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\Rightarrow \quad S = \begin{pmatrix} 0 & 1 & 0 \end{pmatrix}$$

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4. d.

Symbol	Stage 1	2	3	4	Code word
S_0	0.4	0.4	0.4	0.6 0	00
S_1	0.2	0.2	0.4 0	0.4 1	10
S_2	0.2	0.2 0	0.2 1		11
S_3	0.1 0	0.2 1			010
S_4	0.1 1				011

Answer.