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Tutorial: Sparse Signal Processing
Part 1: Sparse Signal Representation

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Outline

- **Part 1:** Sparse Signal Representation ~90min
- **Part 2:** Sparse Sampling ~90min

Outline

- **Signal Representation Problem and Notion of Sparsity**
- **Mathematical Background:**
  - Bases and Frames
  - Analysis and Synthesis Models
  - Wavelet Theory Revisited
- **Sparsity in Union of Bases:**
  - l_0 and l_1 optimizations
  - Sparse Representation Key Bounds
- **Sparsity according to Prony**
- **Approximate sparsity and iterative shrinkage algorithms**
- **Applications**
- **Beyond Traditional Sparsity**

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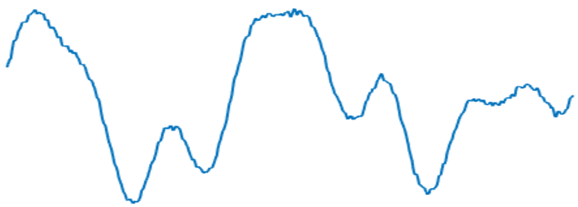
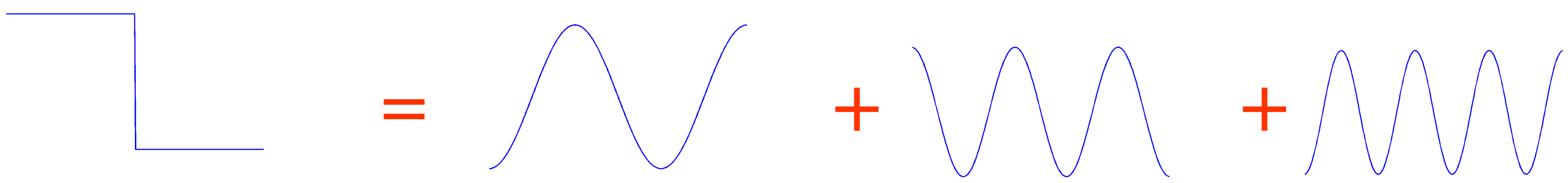
The Signal Representation Problem

Signal Processing aims to decompose complex signals using elementary functions which are then easier to manipulate



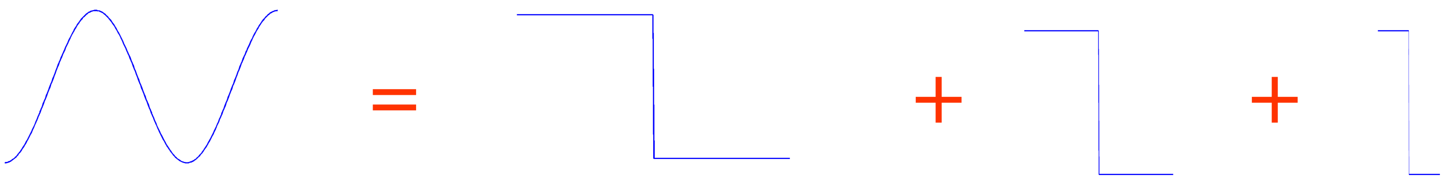
$$x(t) = \sum_{i=-\infty}^{\infty} \alpha_i \varphi_i(t)$$

Signal Representation: Fourier Series



Any signal defined on a finite interval is represented by a sum of sinusoids at different frequencies.

Signal Representation: Haar Wavelet



Any function of finite energy is given by the sum of the Haar function and its translated and scaled versions.

Haar function is the first example of a wavelet!

Sparse Signal Representations

- Given two competing signal representations, which one is better?
- Signal Processing uses **Occam's razor** to answer this question:
  "among competing *representations* that predict equally well,
  the one with the *fewest number of components* should be selected."
- Wavelets are *better* because they provide sparse representations of most natural signals
- This is why wavelets are used successfully in many signal processing applications (e.g., image compression)

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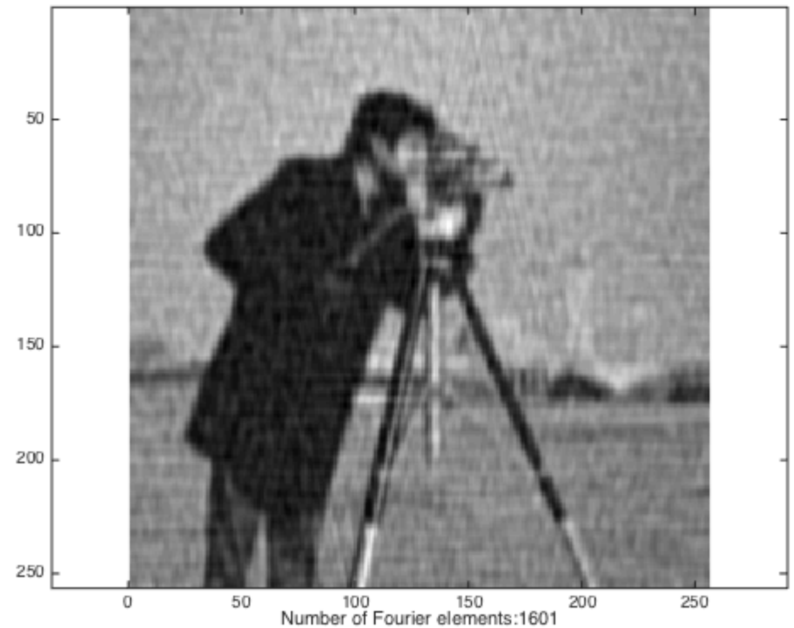
Wavelets vs Fourier

Coefficients used: 2%

Wavelets



Fourier

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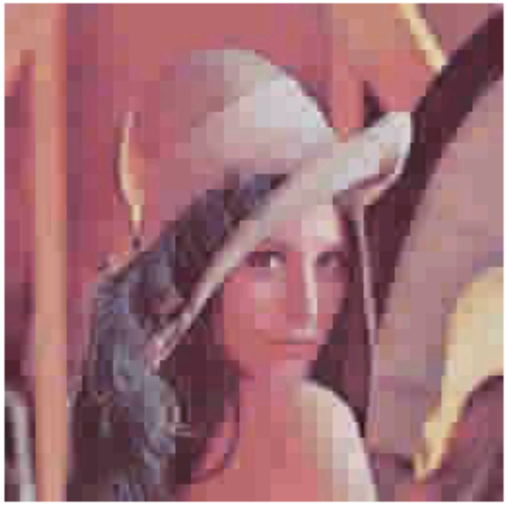
Why Sparsity?

Image Compression

Original Lena Image (256 × 256 pixels)



JPEG (Compression Ratio 43:1)

JPEG2000 (Compression Ratio 43:1)

Note: images courtesy of dspworx.com

Why Sparsity?

- In signal processing we often have to solve ill-conditioned inverse problems
- Approach: given partial and noisy knowledge of your signal, amongst all possible valid solutions, pick the sparsest one
- This sparsity-driven principle has lead to state-of-the-art algorithms in denoising, inpainting, deconvolution, sampling etc.

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Why Sparsity? Inpainting

The usual suspect

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Why Sparsity? Inpainting

Inpainting based on Scholefield-Dragotti. IEEE Trans. Image Processing 2014

Signal Representations

$$x(t)=\sum_{i=-\infty}^{\infty}\alpha_i\varphi_i(t)$$

Key Ingredients:

- a set of 'atoms': $\{\varphi_i\}$
- a inner product: $\langle x,\varphi_i\rangle=\int x(t)\varphi_i(t)dt$
- a synthesis formula: $x(t)=\sum_{i=-\infty}^{\infty}\alpha_i\varphi_i(t)$

Many choices of $\{\varphi_i\}$

- orthonormal bases (e.g., Fourier series, Haar wavelet, Daubechies wavelets)
- biorthogonal bases (e.g., splines)
- overcomplete expansions or frames

Signal Representation: Bases and Frames

Definition 1: The set of 'atoms' $\{\varphi_i\}_{i\in\mathcal{K}} \in V$ is called a basis of *V* when it is *complete* meaning that for any signal *x* there exists a sequence α_i such that

$$x(t) = \sum_{i=-\infty}^{\infty} \alpha_i \varphi_i(t)$$

and the sequence is unique.

Definition 2: The set of 'atoms' $\{\varphi_i\}_{i\in\mathcal{K}} \in V$ is called an *orthogonal* basis of the subspace *V* when

a) it is a basis of *V*

b) It is orthogonal: $\langle \varphi_i, \varphi_k \rangle = \delta_{i-k}$

Note that in the case of orthogonal bases we have that $\alpha_i = \langle x(t), \varphi_i(t) \rangle$

Consequently $x(t) = \sum_i \langle x(t), \varphi_i(t) \rangle \varphi_i(t)$

Signal Representation: Bases and Frames

Definition 3: The set of ‘atoms’ $\{\varphi_i\}_{i \in \mathcal{K}} \in V$ is called a *biorthogonal* basis of the subspace *V* when

a) it is a basis of *V*
b) it is not orthogonal: $\langle \varphi_i, \varphi_k \rangle \neq \delta_{i-k}$

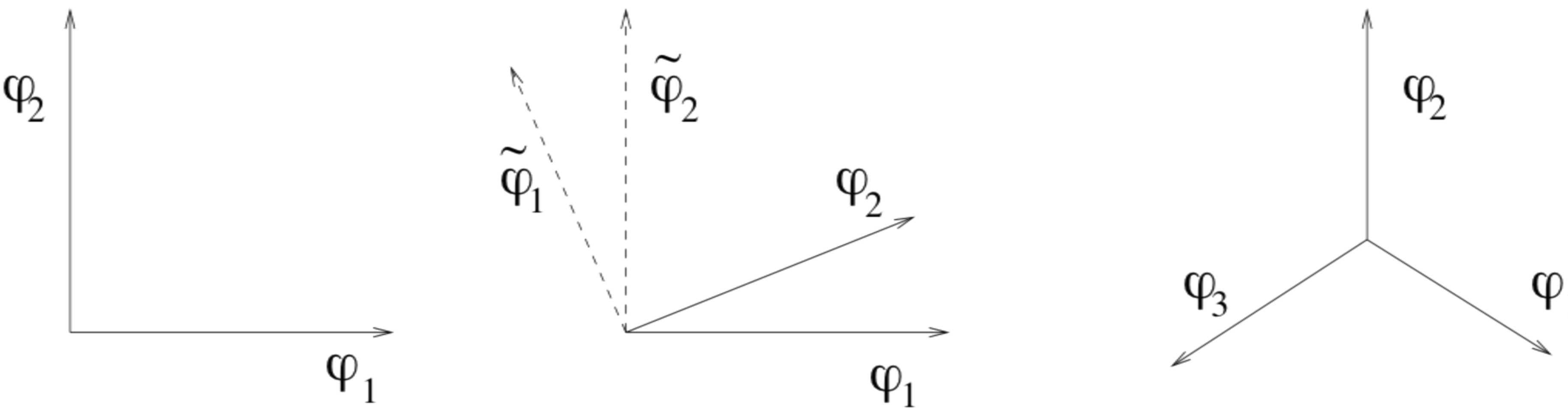
In this case we have that $\alpha_i = \langle x(t), \tilde{\varphi}_i(t) \rangle$
where the dual basis $\{\tilde{\varphi}_i\}_{i \in \mathcal{K}} \in V$ is such that: $\langle \tilde{\varphi}_i, \varphi_k \rangle = \delta_{i-k}$

Definition 4 (informal): The set of ‘atoms’ $\{\varphi_i\}_{i \in \mathcal{K}} \in V$ is called a *frame* of V when it is *overcomplete* meaning that for any signal *x* there exists a sequence α_i such that

$$x(t) = \sum_{i=-\infty}^{\infty} \alpha_i \varphi_i(t)$$

but the sequence is **not** unique.

Bases and Frames: Geometric Interpretation



a) Orthogonal Basis b) Biorthogonal Basis c) Frame

Bases and Frames: Matrix Interpretation

- Assume the 'atoms' $\{\varphi_i\}$ are finite dimensional column vectors of size *N*
- Stack them one next to the other to form the *synthesis* matrix *M*:

$$M = \begin{bmatrix} \uparrow & \cdots & \uparrow & \cdots \\ \varphi_1 & \cdots & \varphi_i & \cdots \\ \downarrow & \cdots & \downarrow & \cdots \end{bmatrix}$$

- If *M* is square and invertible then $\{\varphi_i\}$ is a basis (of R^N or C^N)
- If the inverse of *M* satisfies $M^{-1} = M^H$ the basis is orthogonal
- In the case of frames, *M* is invertible but is *'fat'*

Analysis and Synthesis Formulas

- Remember the synthesis formula $x(t) = \sum_{i=-\infty}^{\infty} \alpha_i \varphi_i(t)$

- In the finite dimensional case the inverse of *M* is the *analysis* matrix and we have: $\alpha = M^{-1}x$

- Expanded: $$\begin{pmatrix} \langle x, \tilde{\varphi}_1 \rangle \\ \vdots \\ \langle x, \tilde{\varphi}_i \rangle \\ \vdots \end{pmatrix} = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_i \\ \vdots \end{pmatrix} = \underbrace{\begin{bmatrix} \leftarrow & \tilde{\varphi}_1 & \rightarrow \\ \cdots & \cdots & \cdots \\ \leftarrow & \tilde{\varphi}_i & \rightarrow \\ \cdots & \cdots & \cdots \end{bmatrix}}_{M^{-1}} \begin{pmatrix} x_1 \\ \vdots \\ x_i \\ \vdots \end{pmatrix}$$

- The synthesis formula is $x = M\alpha \Leftrightarrow x(t) = \sum_{i=-\infty}^{\infty} \alpha_i \varphi_i(t)$

Analysis and Synthesis Formulas: Frames Case

- In the frame case the synthesis matrix *M,* in the the synthesis formula $x = M\alpha$, is "fat"

- The 'pseudo-inverse' matrix, M^{-1}, is the analysis matrix and is "tall"

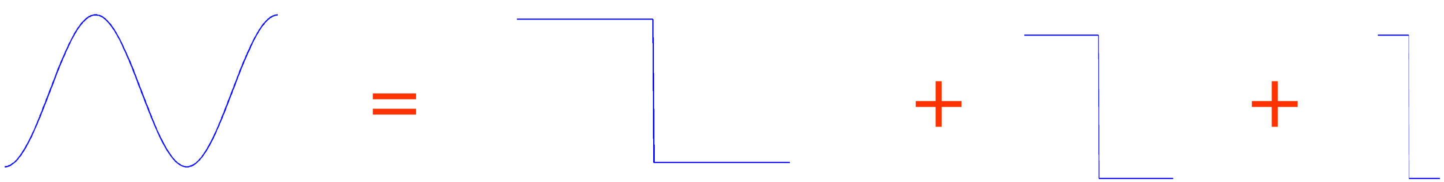
Analysis and Synthesis Formulas

- **Sparse Signal Representation** is mostly about the **synthesis** formula

- **Sparse Sampling** is mostly about the **analysis** formula

- We have been moving freely between continuous-time (infinite dimensional) case and discrete-time (finite-dimensional) case.This is usually legitimate, however, some frameworks (e.g. compressed sensing) are limited to the finite dimensional case.

Wavelet Representation Revisited



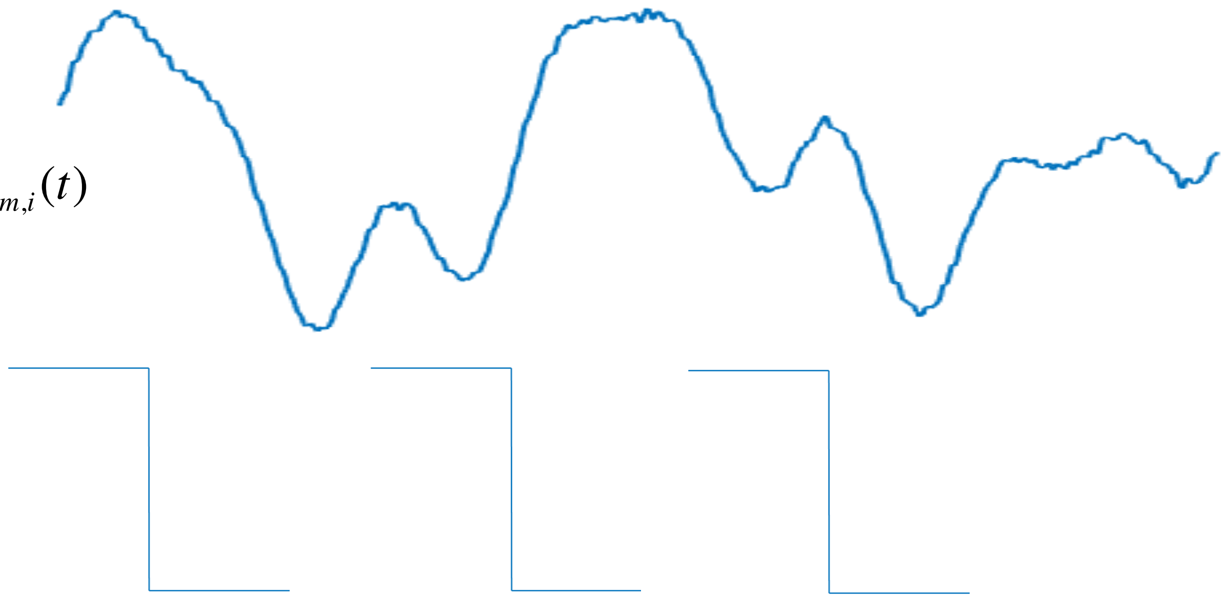
Any function of finite energy is given by the sum of the Haar function and its translated and scaled versions:

$$x(t) = \sum_{m=-\infty}^{\infty} \sum_{i=-\infty}^{\infty} \alpha_{m,i} \psi_{m,i}(t) \quad \textit{with} \quad \psi_{m,i}(t) = \frac{1}{\sqrt{2^m}} \psi(2^{-m} t - i)$$

Wavelet Representation Revisited

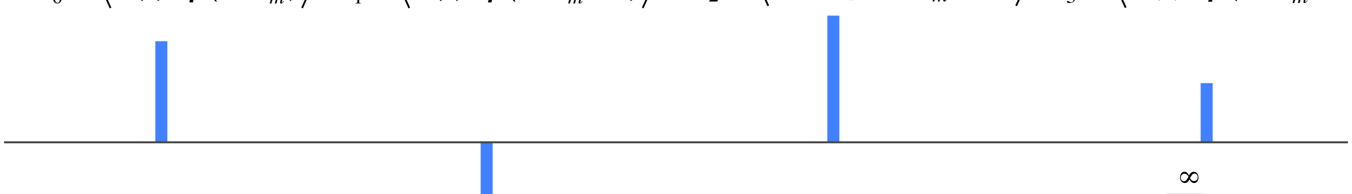
At scale m:

$$x_m(t) = \sum_{i=-\infty}^{\infty} \alpha_{m,i}\psi_{m,i}(t)$$



$$T_m = 2^m$$

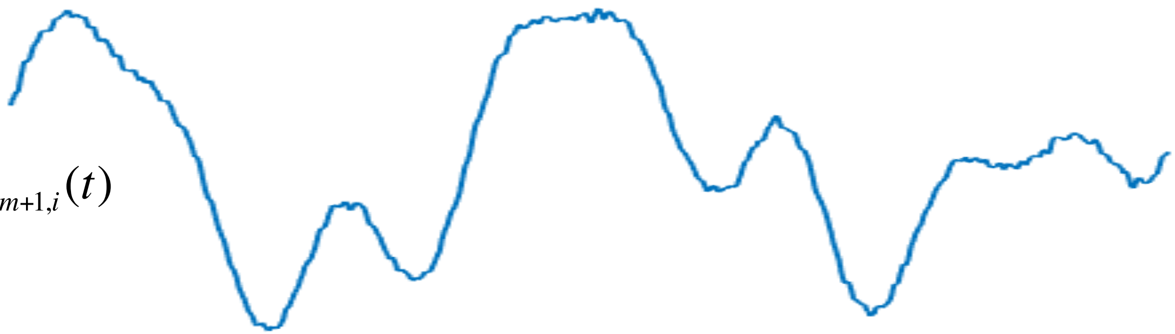
$$\alpha_0 = \langle x(t), \psi(t/T_m)\rangle \quad \alpha_1 = \langle x(t), \psi(t/T_m - 1)\rangle \quad \alpha_2 = \langle x(t), \psi(t/T_m - 2)\rangle \quad \alpha_3 = \langle x(t), \psi(t/T_m - 3)\rangle$$



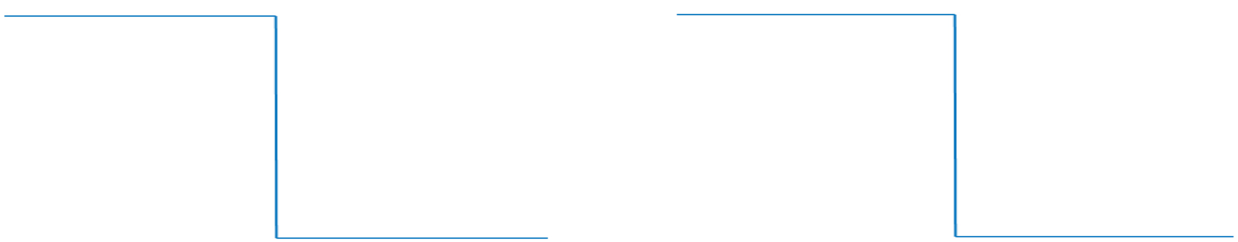
$$x_m(t) = \sum_{i=-\infty}^{\infty} \alpha_i \psi(t/T_m - i)$$

Wavelet Representation Revisited

At scale m+1:



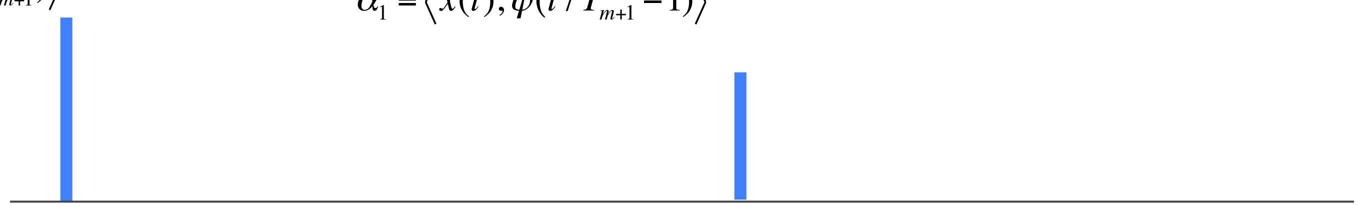
$$x_{m+1}(t) = \sum_{i=-\infty}^{\infty} \alpha_{m+1,i}\psi_{m+1,i}(t)$$



$$\alpha_0 = \langle x(t), \psi(t/T_{m+1}) \rangle$$

$$\alpha_1 = \langle x(t), \psi(t/T_{m+1} - 1) \rangle$$

$$T_{m+1} = 2^{m+1}$$



$$x_{m+1}(t) = \sum_{i=-\infty}^{\infty} \alpha_i \psi(t/T_{m+1} - i)$$

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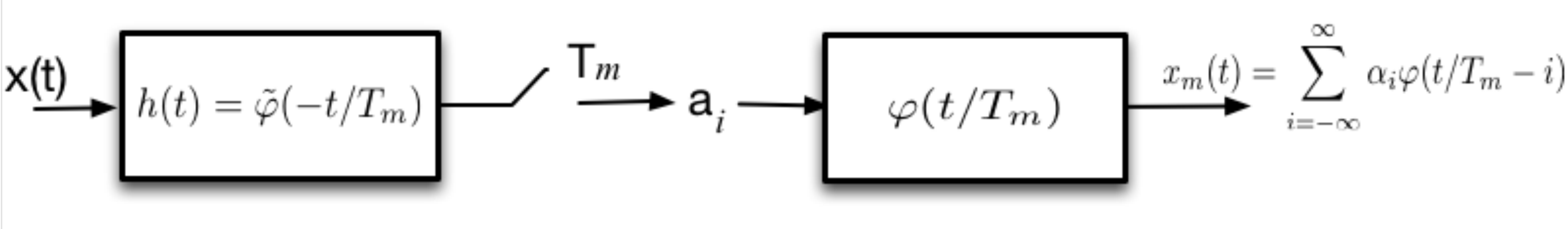
Wavelet Representation Revisited

- At each scale the approximation of x(t) is obtained by using a prototype function and its uniform shifts
- This fact allows us to use filters to implement the wavelet transform



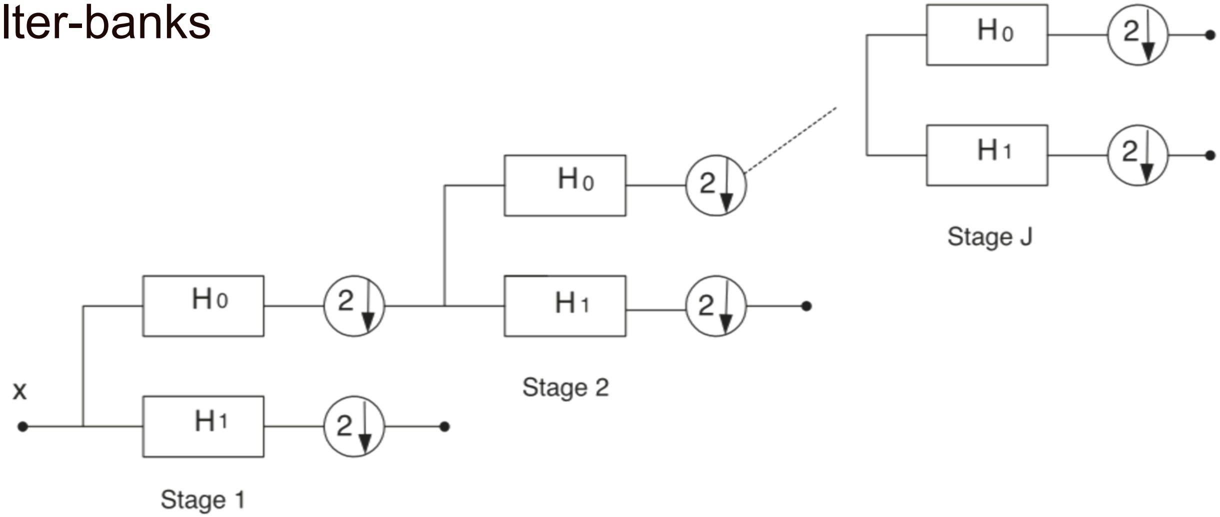
$$\alpha_i=\int x(\tau)h(iT_m-\tau)d\tau=\int x(\tau)\tilde{\varphi}(\tau/T_m-i)d\tau=\langle x(t),\tilde{\varphi}(t/T_m-i)\rangle$$

Wavelet Representation Revisited

- The sub-space generated by the prototype function $\varphi(t)$ and its uniform shifts is called a **shift-invariant sub-space**.

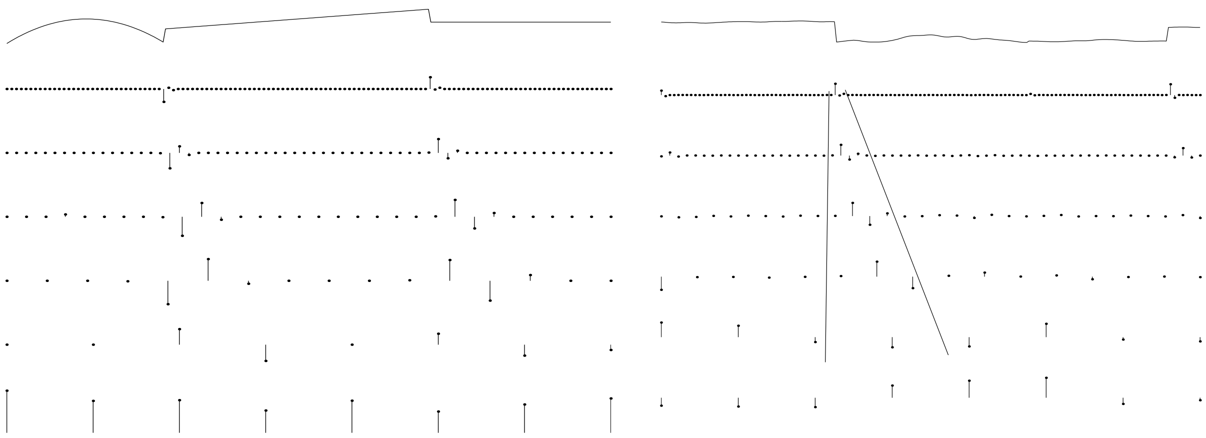
- The fact that shifts are uniform allows us to mix continuous-time and discrete-time processing

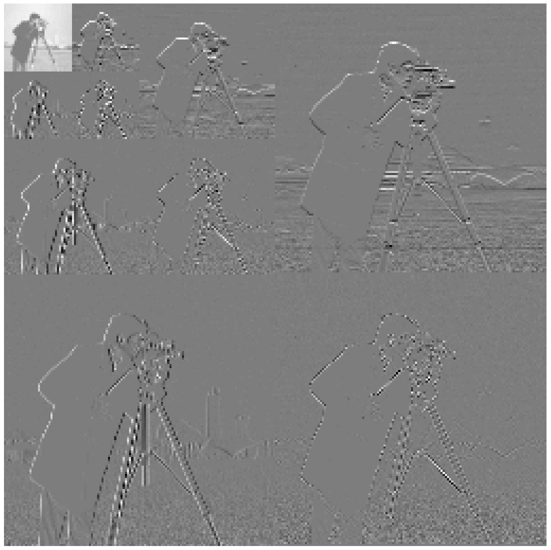
- The discrete-time version of the wavelet transform is implemented with iterated filter-banks



Wavelets and Sparsity

- Wavelets annihilates polynomials
- Wavelets provide sparse representation of piecewise smooth functions





Sparse Representation in a union of two bases

- Wavelets provide sparse representations of piecewise smooth images.
- In matrix/vector form ***y=Wα***
- Here the matrix ***W*** has size N × N and models the discrete-time wavelet transform of finite dimensional signals.

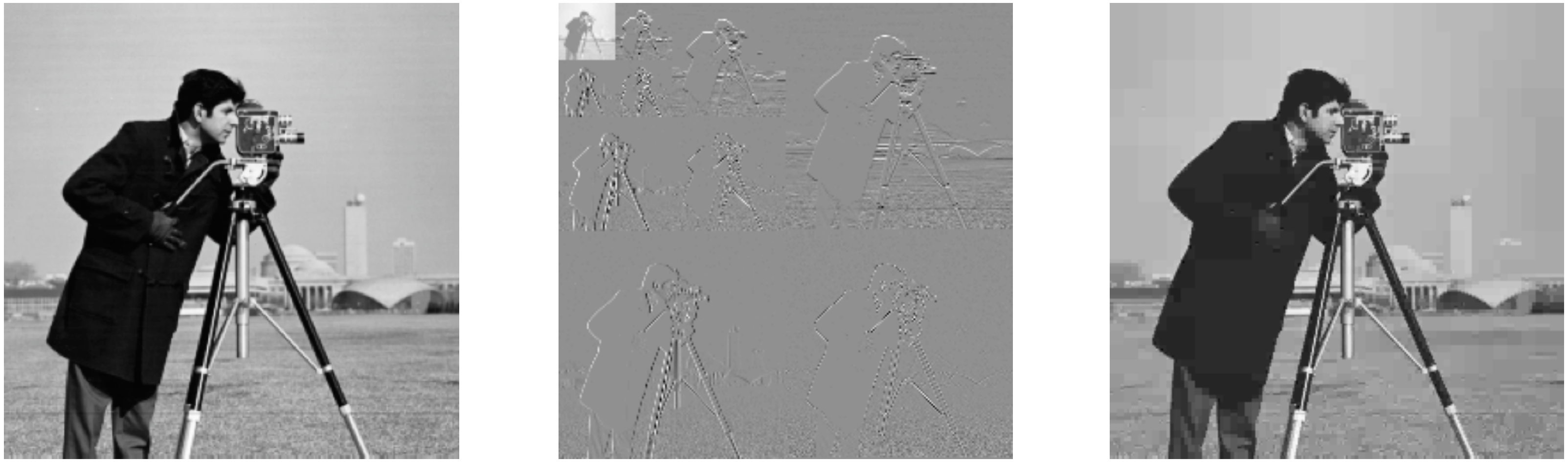
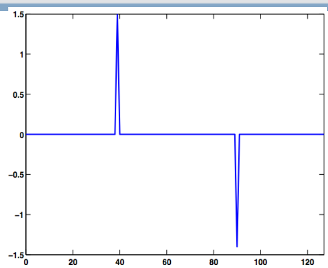


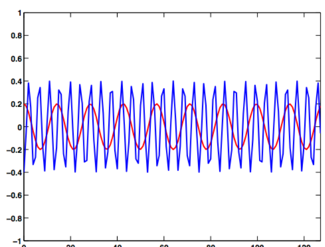
Figure: Cameraman is reconstructed using only 8% of the wavelet coefficients

- How about textures? The DCT is maybe better for textured regions
- **Key insight**: use an overcomplete dictionary (frame) ***D*** made of the union of two bases to obtain even sparser representations of images

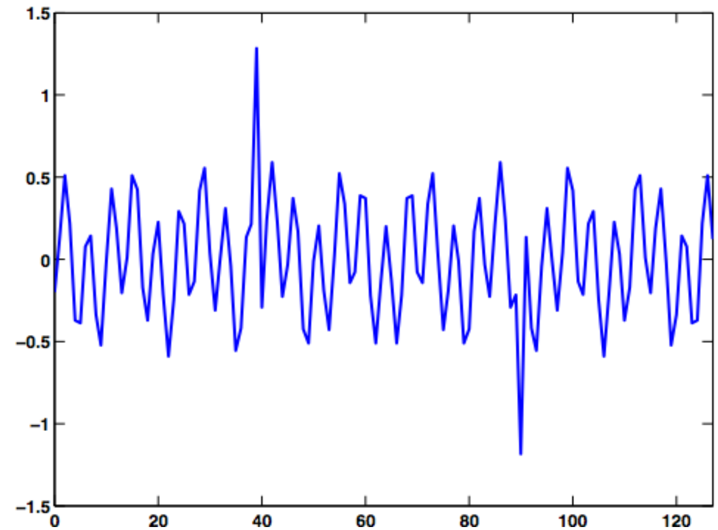
Sparse Representation in Fourier and Canonical Bases



two spikes



two complex exponentials



$\boldsymbol{y}$ *(real part plotted)*

The above signal, ***y***, is a combination of two spikes and two complex exponentials of different frequency (real part of ***y*** plotted). In matrix vector form:

$$\mathbf{y} = [\ \mathbf{I} \quad \mathbf{F}]\ \alpha = \mathbf{D}\alpha$$

where ***I*** is the N × N identity matrix and ***F*** is the N × N Fourier transform. The matrix ***D*** models the over-complete dictionary and has size N × 2N

- *Source separation: decompose signals into a smooth part and local innovations*

- *Prototype for the following problem:*

> Given two bases (or frames) $\boldsymbol{D} = [\boldsymbol{\Psi}, \boldsymbol{\Phi}]$. Represent an observed signal as a superposition of a few atoms from $\boldsymbol{\Psi}$ and a few atoms from $\boldsymbol{\Phi}$.

Example: *(Curvelets + DCT)*

images from [Elad, Starck, Querre, Donoho, 2005]

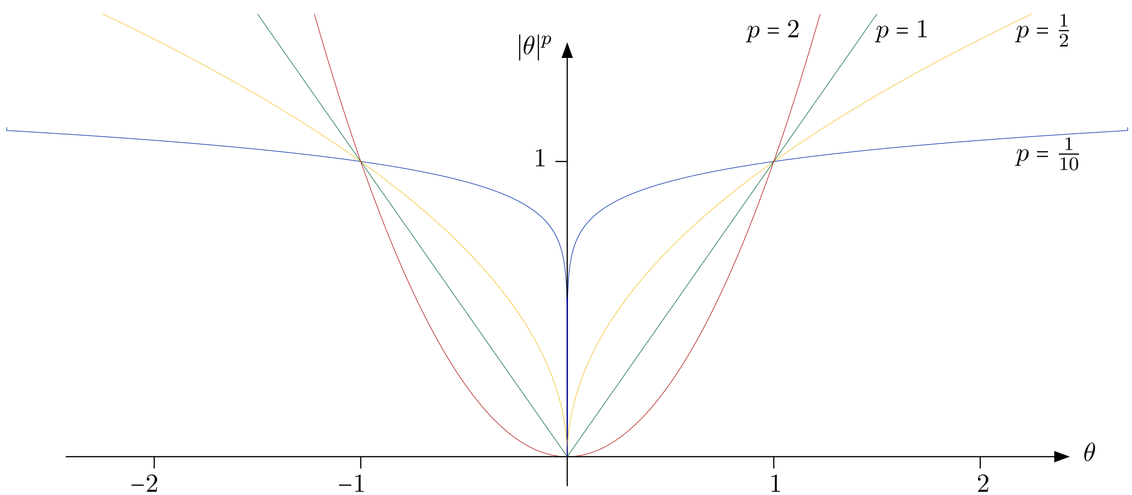
Sparse Representation in Fourier and Canonical Bases

- Given ***y***, you want to find its sparse representation
- Ideally you want to solve

$$(P_0): \quad \min \|\alpha\|_0 \quad \text{s.t.} \quad \mathbf{y} = \mathbf{D}\alpha.$$

- Alternatively you may consider the following convex relaxation

$$(P_1): \quad \min \|\alpha\|_1 \quad \text{s.t.} \quad \mathbf{y} = \mathbf{D}\alpha.$$



Sparse Representation in Fourier and Canonical Bases

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$$(P_1): \quad \min \|\alpha\|_1 \quad \text{s.t.} \quad \mathbf{y} = \mathbf{D}\alpha.$$

- The problem (P_1) can be solved using convex optimization methods (i.e., linear programming) or greedy algorithms

Sparse Representation via OMP

Algorithm 5 *OMP*—Orthogonal Matching Pursuit

Input: Dictionary $\boldsymbol{D} = [\boldsymbol{d}_1, \ldots, \boldsymbol{d}_L] \in \mathbb{C}^{N \times L}$, observation $\boldsymbol{y} \in \mathbb{C}^N$ and error threshold η [optional argument, maximum number of iterations $K_{\max}$].

Output: Sparse vector $\boldsymbol{x} \in \mathbb{C}^L$.

Initialise index $k = 0$.
Initialise solution $\boldsymbol{x}^{(0)} = \boldsymbol{0}$.
Initialise residual $\boldsymbol{r}^{(0)} = \boldsymbol{y} - \boldsymbol{D}\,\boldsymbol{x}^{(0)} = \boldsymbol{y}$.
Initialise support $\mathcal{S}^{(0)} = \emptyset$.
while $\left\|\boldsymbol{r}^{(k)}\right\|_2^2 > \eta$ [optional: $k < K_{\max}$] **do**
    $k \leftarrow k + 1$
    Compute $e[i] = \left\|z[i]\,\boldsymbol{d}_i - \boldsymbol{r}^{(k-1)}\right\|_2^2$ for $i \in \{1, \ldots, L\} \setminus \mathcal{S}^{(k-1)}$, where $z[i] = \frac{\boldsymbol{d}_i^H\,\boldsymbol{r}^{(k-1)}}{\|\boldsymbol{d}_i\|_2^2}$.
    Find index $i_0 = \arg\min_{i \in \{1,\ldots,L\} \setminus \mathcal{S}^{(k-1)}} \{e[i]\}$.
    Update support $\mathcal{S}^{(k)} = \mathcal{S}^{(k-1)} \cup \{i_0\}$.
    Compute solution $\boldsymbol{x}^{(k)} = \arg\min_{\tilde{\boldsymbol{x}} \in \mathbb{C}^L} \|\boldsymbol{D}\,\tilde{\boldsymbol{x}} - \boldsymbol{y}\|_2^2$ subject to $\operatorname{supp}\{\tilde{\boldsymbol{x}}\} = \mathcal{S}^{(k)}$.
    Update residual $\boldsymbol{r}^{(k)} = \boldsymbol{y} - \boldsymbol{D}\,\boldsymbol{x}^{(k)}$.
end while

Sparse Representation in Fourier and Canonical Bases

- Given ***y***, you want to find its K-sparse representation
- Ideally you want to solve

$$(P_0): \quad \min \|\alpha\|_0 \quad \text{s.t.} \quad \mathbf{y} = \mathbf{D}\alpha.$$

- Alternatively you may consider the following convex relaxation

$$(P_1): \quad \min \|\alpha\|_1 \quad \text{s.t.} \quad \mathbf{y} = \mathbf{D}\alpha.$$

- Key result due to Donoho-Huo [IEEE Trans. on Information Theory 2001]
  - (P_0) is unique when the sparsity K satisfies $K < \sqrt{N}$
  - (P_0) and (P_1) are equivalent when $K < \frac{1}{2}\sqrt{N}$

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Sparsity in Pairs of Orthogonal Bases

- Key extensions due to Elad-Bruckstein [IEEE Trans. on Information Theory 2002]
- Given an arbitrary pair of orthonormal bases Ψ_N and Φ_N and the mutual coherence

$$\mu(\mathbf{D}) = \max_{1 \leq k,j \leq N, k \neq j} \frac{|\mathbf{d}_k^T \mathbf{d}_j|}{\|\mathbf{d}_k\| \|\mathbf{d}_j\|},$$

- (P0) is unique when $K < 1/\mu(\mathbf{D})$
- (P0) and (P1) are equivalent when

$$2\mu(\mathbf{D})^2 K_p K_q + \mu(\mathbf{D}) \max\{K_p, K_q\} - 1 < 0, \qquad (\textbf{Tight Bound})$$

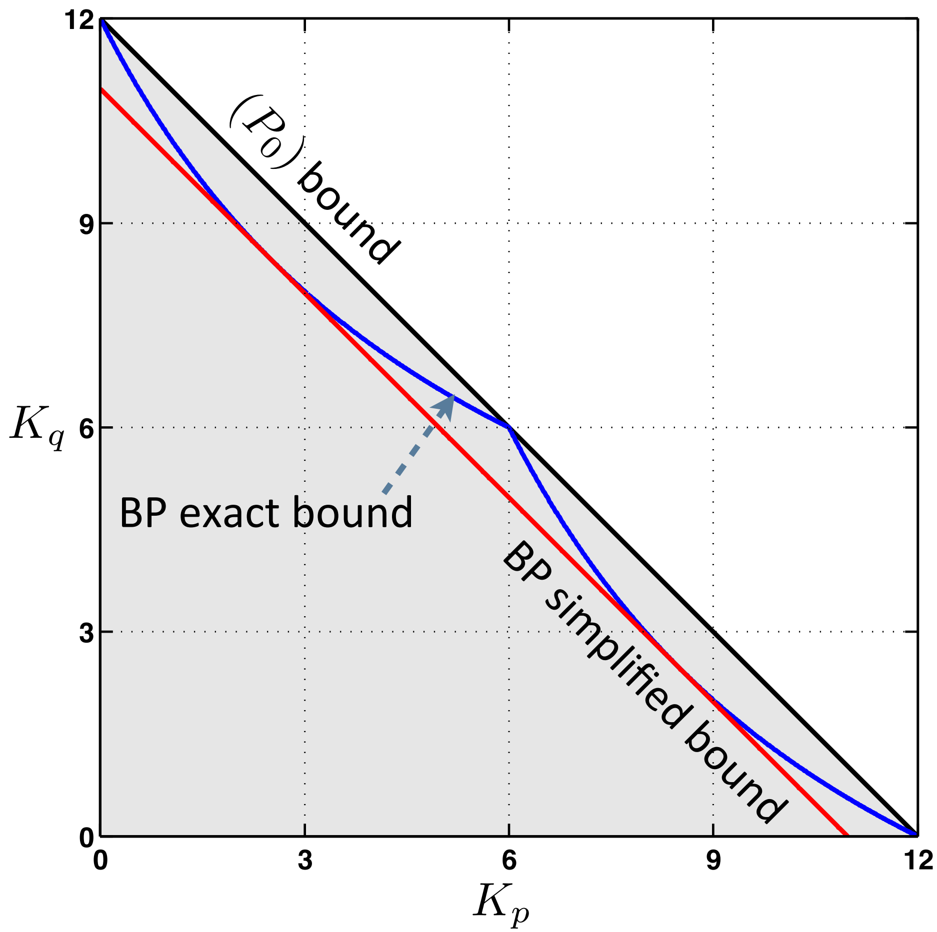
- Alternatively (P0) and (P1) are equivalent when

$$K < (\sqrt{2} - 0.5)/\mu(\mathbf{D}) \sim 0.9/\mu(\mathbf{D}) \qquad (\textbf{Weak Bound})$$

Sparsity in Pairs of Orthogonal Bases

- (P0) is unique when $$K < 1/\mu(\mathbf{D})$$
- (P0) and (P1) are equivalent when
$$2\mu(\mathbf{D})^2 K_p K_q + \mu(\mathbf{D}) \max\{K_p, K_q\} - 1 < 0, \qquad (\textbf{Tight Bound})$$
- Alternatively (P0) and (P1) are equivalent when
$$K < (\sqrt{2} - 0.5)/\mu(\mathbf{D}) \sim 0.9/\mu(\mathbf{D}) \qquad (\textbf{Weak Bound})$$
- Please note:
  - $K = K_p + K_q$
  - In Fourier and Canonical case $\mu(\mathbf{D}) = \sqrt{N}$

Sparsity Bounds in Pairs of Orthogonal Bases



Sparsity Bounds in Ovecomplete Dictionaries

Extensions [Tropp-04, GribonvalN:03, Elad-10]

- For a generic over-complete dictionary D, (P_1) is equivalent to (P_0) when[2]

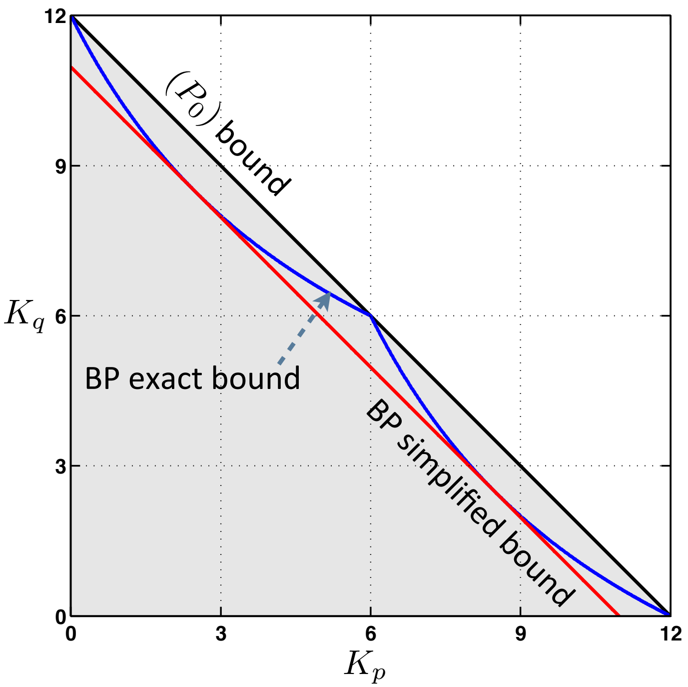
$$K < \frac{1}{2}\left(1 + \frac{1}{\mu}\right).$$

- When D is a concatenation of J orthonormal dictionaries (P_1) is equivalent to (P_0) when

$$K < \left[\sqrt{2} - 1 + \frac{1}{2(J-1)}\right]\mu^{-1}$$

The Tyranny of l_1: Is there life beyond BP?

- There is still a gap between l_0 and l_1 minimizations. Can we do better than Basis Pursuit?
- (P0) is NP-Hard for unrestricted dictionary. Can we say the same for structured dictionaries like Fourier and Identity?
- What happens when the unicity constraint is not met?



Sparsity according to Prony: Overview of Prony's Method

- Consider the case when the signal ***y*** is made only of *K* Fourier atoms, i.e., ***y=Fc*** for some K-sparse vector ***c***
- The sparse vector ***c*** can be reconstructed from only *2K* *consecutive* entries of ***y***
- *Sketch of the Proof:*
  - The *n*th entry of ***y*** is of the form

$$y_n = \frac{1}{\sqrt{N}} \sum_{k=0}^{K-1} c_{m_k} \, e^{j2\pi m_k n/N} = \sum_{k=0}^{K-1} \alpha_k u_k^n,$$

where m_k is the index of the *k*th nonzero element of ***c***

Overview of Prony's Method

- Consider the polynomial:

$$P(x) = \prod_{k=1}^{K}(x - u_k) = x^K + h_1 x^{K-1} + h_2 x^{K-2} + \ldots + h_{K-1}x + h_K.$$

- It is easy to verify that $h_n * y_n = 0$

- In matrix-vector form, we get

$$\begin{bmatrix} y_{l+K} & y_{l+K-1} & \cdots & y_l \\ y_{l+K+1} & y_{l+K} & \cdots & y_{l+1} \\ \vdots & \ddots & \ddots & \vdots \\ y_{l+2K-2} & \ddots & \ddots & \vdots \\ y_{l+2K-1} & y_{l+2K-2} & \cdots & y_{l+K-1} \end{bmatrix} \begin{bmatrix} 1 \\ h_1 \\ h_2 \\ \vdots \\ h_K \end{bmatrix} = \mathbf{T}_{K,l}\mathbf{h} = \mathbf{0}$$

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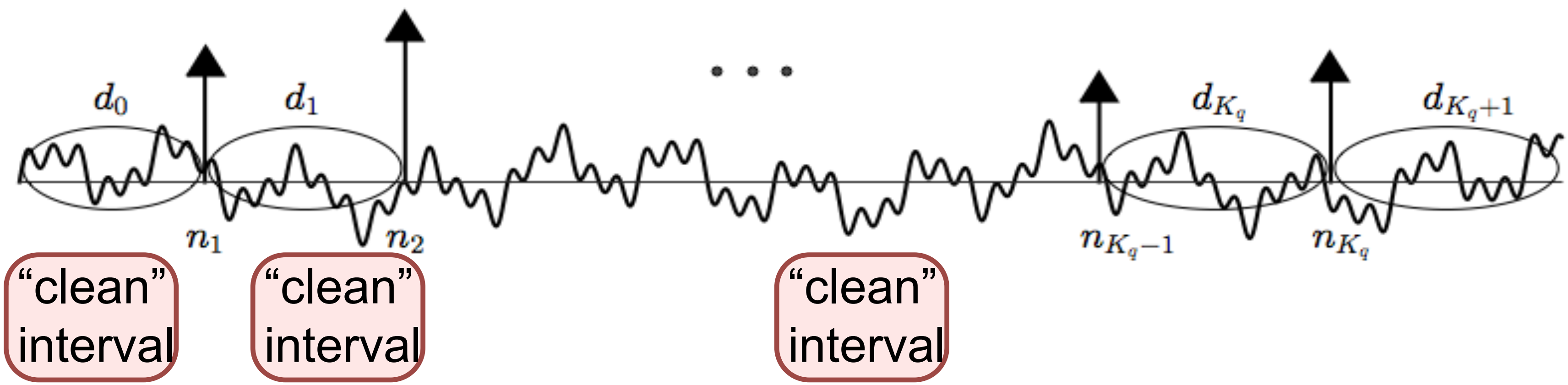
Overview of Prony's Method

- The vector of polynomial coefficients $\boldsymbol{h}=[1, h_1, \ldots, h_k]^T$ is in the null space of $\boldsymbol{T}_{k,l}$. Moreover, $\boldsymbol{T}_{k,l}$ has size $K \times (K+1)$ and has full rank, therefore, $\boldsymbol{h}$ is unique.

- Prony's method summary:
  - Given the input y_n, build the Toeplitz matrix $\boldsymbol{T}_{k,l}$ and solve for $\boldsymbol{h}$.
  - Find the roots $P(x) = 1 + \sum_{n=1}^{K} h_k x^{K-k}$. These roots are exactly the exponentials $\{u_k\}_{k=0}^{K-1}$. They give you the locations of the non-zero entries of your vector.
  - Given the locations of the non-zero entries find their amplitudes (this is a linear problem)

ProSparse – Prony's based Sparsity

- Given $\boldsymbol{y} = [\boldsymbol{F}, \boldsymbol{I}] = \boldsymbol{D}\boldsymbol{x}$ where $\boldsymbol{x}$ is (K_p, K_q)-sparse



- K_p Fourier atoms —> need a "clean" interval of length $2K_p$
- For sufficiently sparse signals, such intervals always exist
- Sequential search and test: polynomial complexity

ProSparse Properties

Theorem ***[Dragotti & Lu, IEEE IT. 2014]*** *Let* $\boldsymbol{D} = [\boldsymbol{F}, \boldsymbol{I}]$ *and* $\boldsymbol{y} \in \mathbb{C}^N$ *an arbitrary signal. There exists an algorithm, with a worst-case complexity of* $\mathcal{O}(N^3)$ *, that finds* ***all*** *(K_p,K_q)-sparse signals* $\boldsymbol{x}$ *such that*

$$\boldsymbol{y} = \boldsymbol{D}\boldsymbol{x} \quad \text{and} \quad K_p K_q < N/2$$

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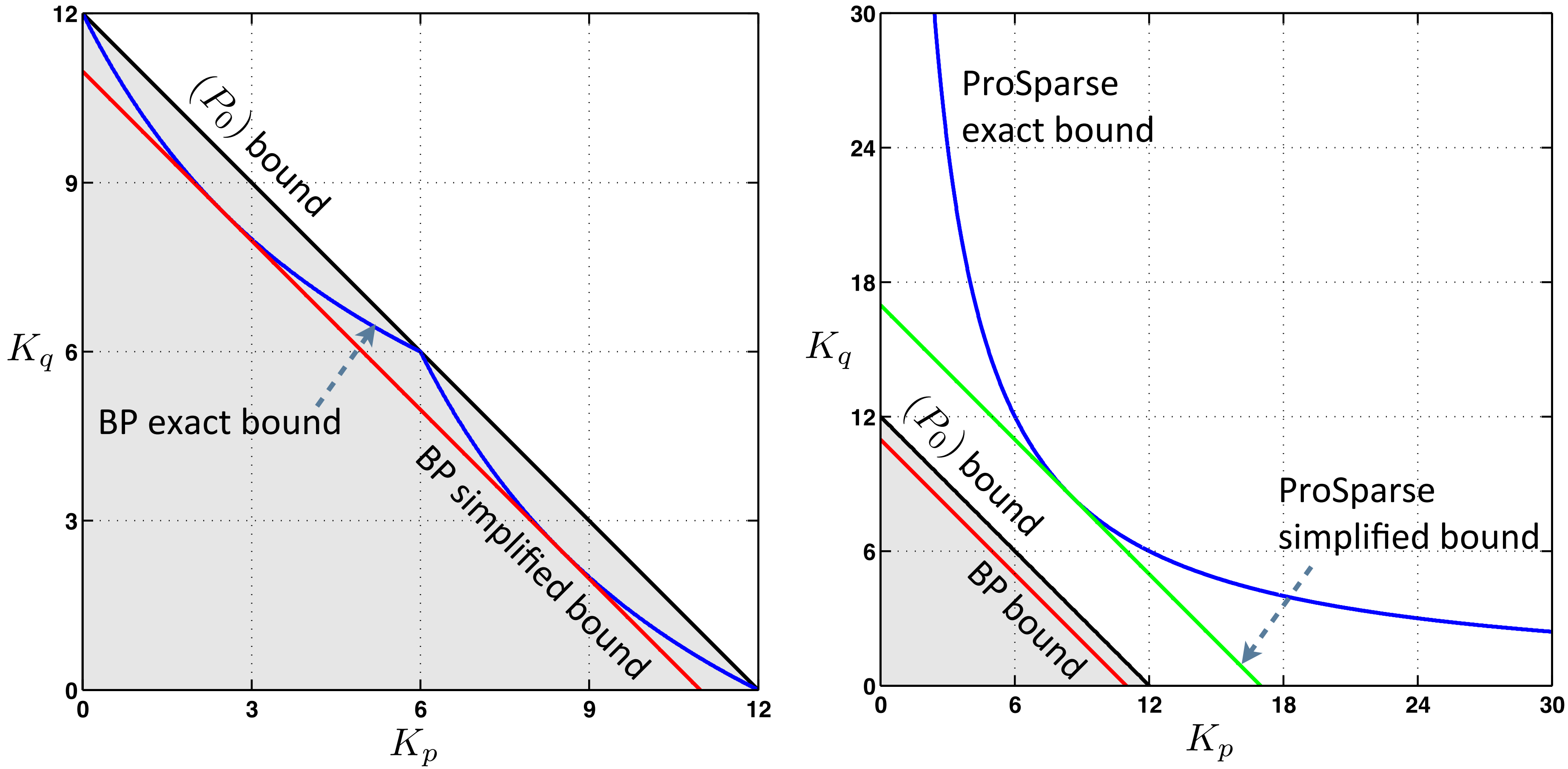
ProSparse Bounds vs BP



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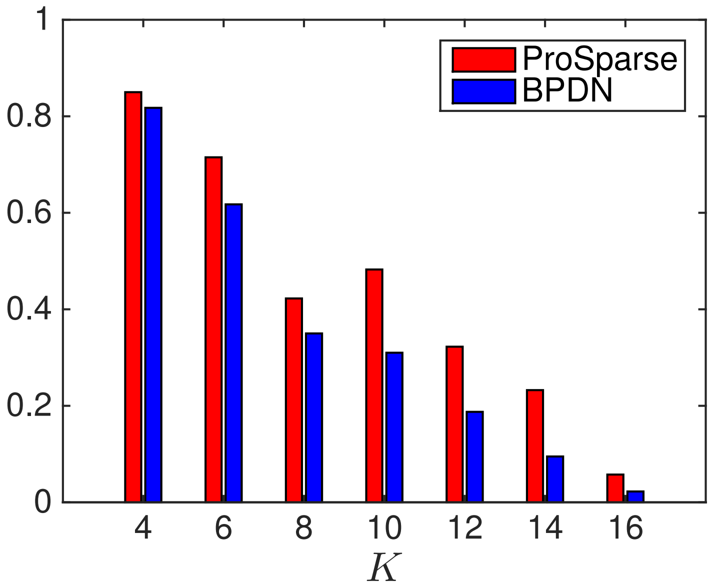
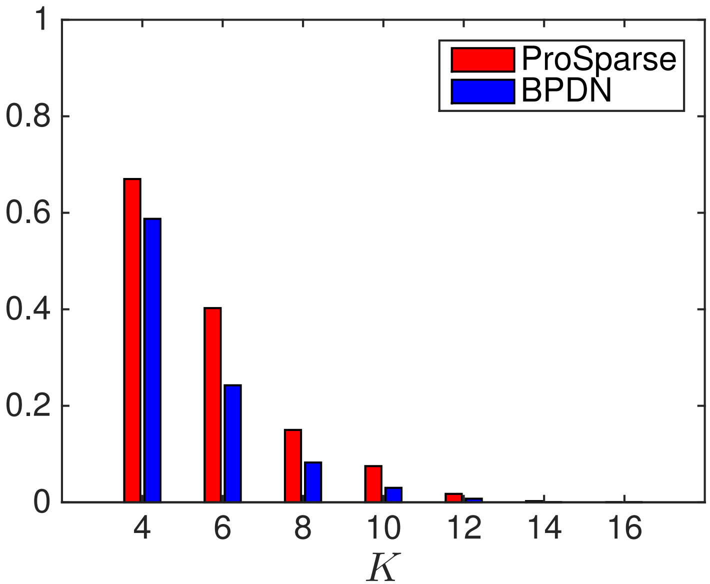
ProSparse vs BP (noisy case)

- Support Recovery



(a) SNR = 10 dB



(b) SNR = 5 dB

Recovery of Approximately Sparse Signals

- Assume ***y*** is not exactly sparse. You may also observe a corrupted (e.g., noisy version) of ***y***

- Rather than solving (P_1)

$$(P_1): \quad \min \|\alpha\|_1 \quad \text{s.t.} \quad \mathbf{y} = \mathbf{D}\alpha.$$

- Consider the following relaxed version:

$$\min_{\alpha} \|\mathbf{y} - \mathbf{D}\alpha\|_2^2 + \lambda\|\alpha\|_1$$

- This basic formulation applies to denoising, deconvolution, inpainting (see for example Elad et al. SPIE 2007)

Recovery of Approximately Sparse Signals

- Consider the following relaxed version:

$$\min_{\alpha} \|\mathbf{y} - \mathbf{D}\alpha\|_2^2 + \lambda\|\alpha\|_1$$

- This basic formulation applies to denoising, deconvolution, inpainting (see for example Elad et al. SPIE 2007)

- Can be solved iteratively as follows

$$\alpha_{i+1} = \mathcal{S}_{\lambda/c}\left(\frac{1}{c}\mathbf{D}^T(\mathbf{y} - \mathbf{D}\alpha_i) + \alpha_i\right)$$

- Here $\mathcal{S}$ is a shrinkage operator (e.g., soft-threshold)

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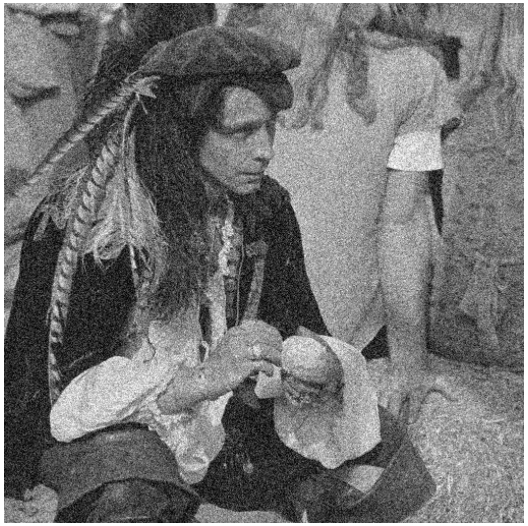
Application: Image Denoising

Original



Degraded (PSNR=20dB)

State-of-the-Art BM3D-SAPCA (PSNR = 29.81 dB)

Application: Image Forensic

Spot the difference ;-)

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Application: Image Forensic – Recapture Detection

Application: Image Forensic – Alias Free Recapture

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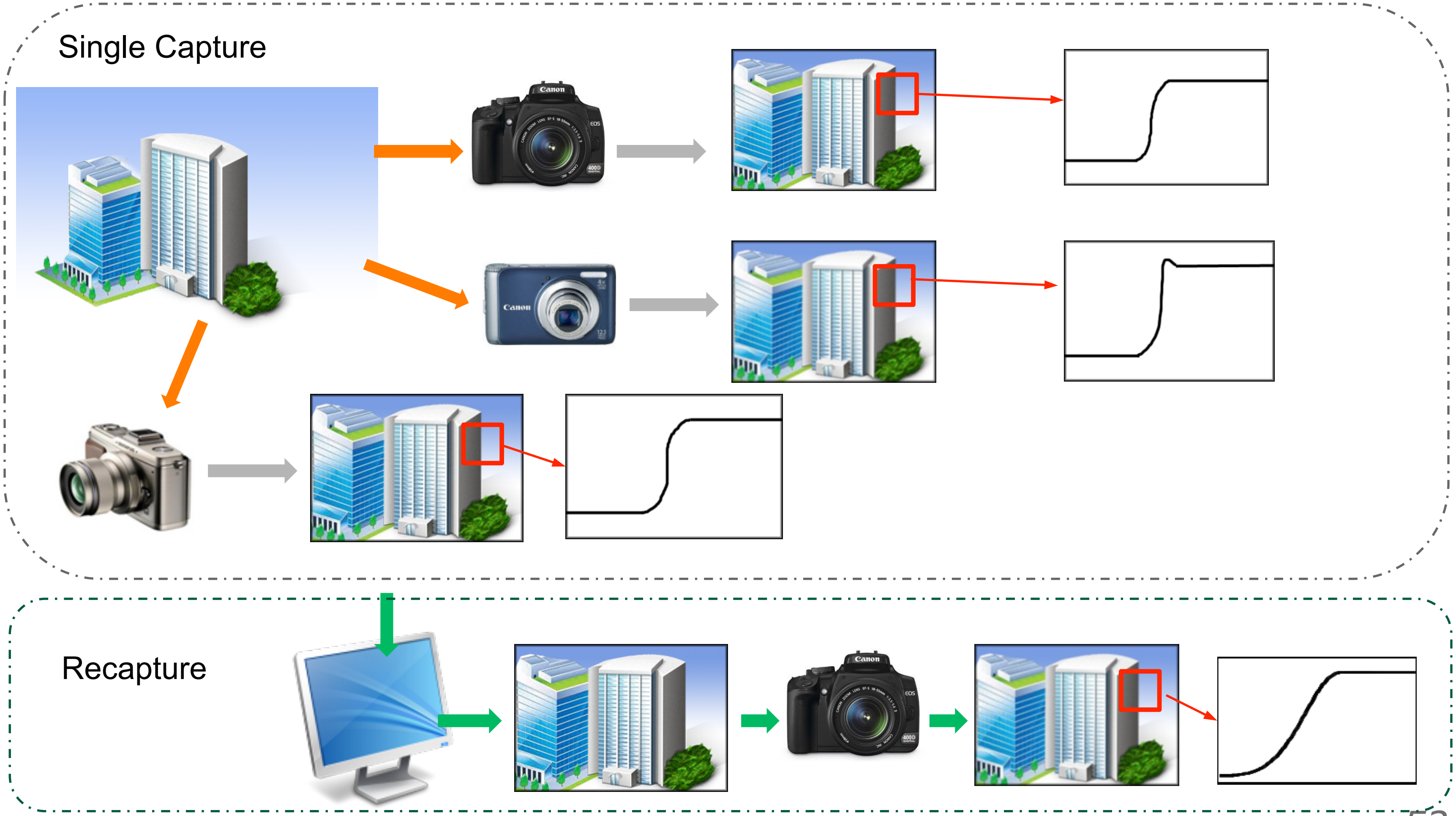
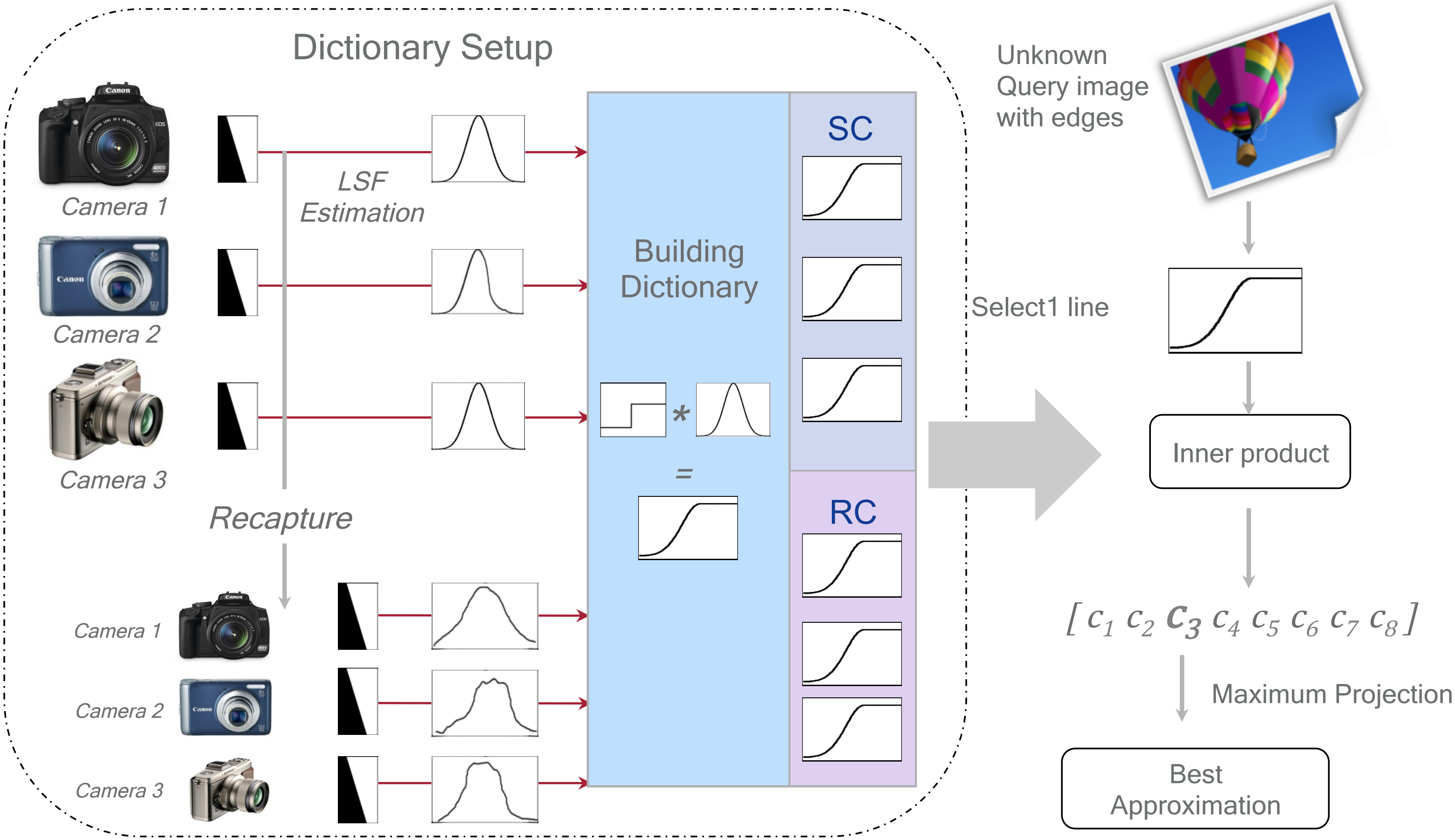
Recapture Footprints: Blurring

Our key footprint: unique blurred patterns introduced by acquisition devices



Proposed Scheme

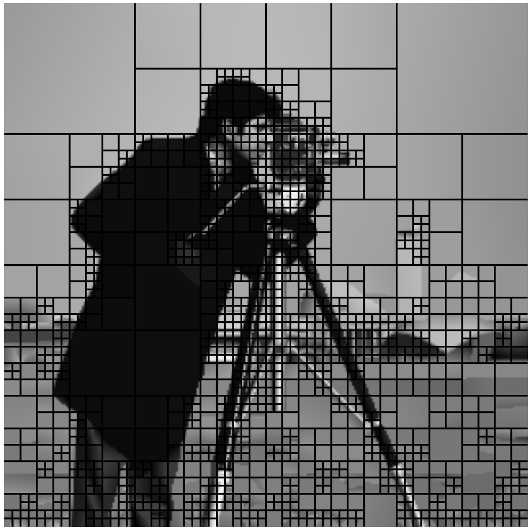


Thongkamwitoon, Muammar and Dragotti, IEEE Trans. on INFS 2015

Beyond Traditional Sparsity Models

- Traditional sparsity models are essentially linear and apply essentially only to 1-D signals

- **Possible 2-D extension:** decompose the image with tiles of different size each being made of two smooth regions separated by a straight edge (semi-parametric model)



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Beyond Traditional Sparsity Models (cont'd)

- The better the sparsity, the better the results ;-)
  Scholefield-Dragotti. IEEE Trans. Image Processing 2014

Degraded
(PSNR=10.6dB)

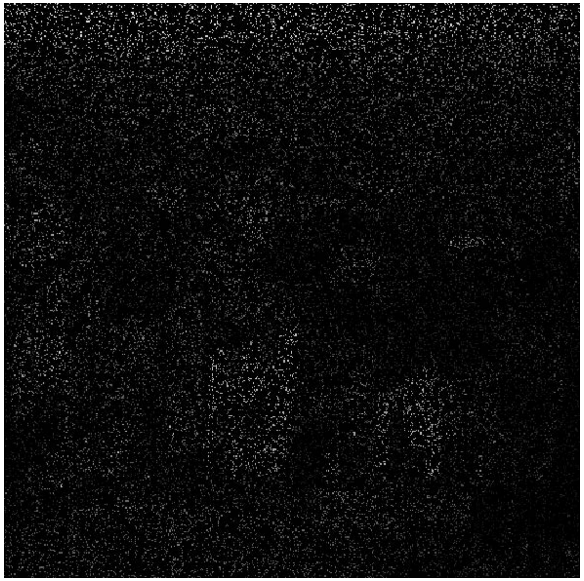
State-of-the-Art
(PSNR = 26.8 dB)

New Sparsity Model
(PSNR = 27.1 dB)

Beyond Traditional Sparsity Models (cont'd)

- Inpainting: Scholefield-Dragotti. IEEE Trans. Image Processing 2014

Original



90% missing pixels

Inpainted using new Sparsity Model

Summary

- “Hey Hey My My the notion of Sparsity will never die” ;-)
- Traditional sparsity is based around two pillars:
  - An expansion-based sparsity model
  - Reconstruction based on convex programming (e.g., BP)
- Room for more ***creative solutions*** both in terms of sparsity and reconstruction method
  - E.g. Reconstruction using *ProSparse* outperforms Convex Programming in specific settings
  - E.g. Semi-Parametric sparsity models for images outperforms state-of-the-art image processing algorithms

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References

Key papers and books on Sparse Signal Representation

- M. Elad,'Sparse and Redundant Representations', Springer, 2010
- D. L. Donoho and P.B. Starck, 'Uncertainty principles and signal recovery', SIAM J. Appl. Math., 1989, pp. 906-931.
- D.L. Donoho and X. Huo, 'Uncertainty principles and ideal atomic decomposition', IEEE Trans. on Info.. Theory, vol.47(7, pp.2845-62, November 2001.
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ProSparse:

- P. L. Dragotti and Y. M. Lu, "On Sparse Representation in Fourier and Local Bases," IEEE Transactions on Information Theory, vol. 60, no. 12, pp. 7888-7899, 2014.

BM3D:

- K. Dabov, A. Foi, V. Katkovnik, and K. Egiazarian, "Image denoising by sparse 3D transform-domain collaborative filtering," IEEE Trans. Image Process., vol. 16, no. 8, pp. 2080-2095, August 2007.

New Image Sparsity Models:

- A. Scholefield and P.L. Dragotti *Quadtree Structured Image Approximation for Denoising and Interpolation*, IEEE Transactions on Image Processing, vol. 23, no. 3, March 2014. (*Software*)

Image Recapture Detection:

- T. Thongkamwitoon, H. Muammar and P. L. Dragotti, An image recapture detection algorithm based on learning dictionaries of edge profiles, IEEE Trans on Info. Forensics and Security, vol. 10 (5), May 2015.