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Wavelets and Sparse Sampling on Circulant and Complex Graphs

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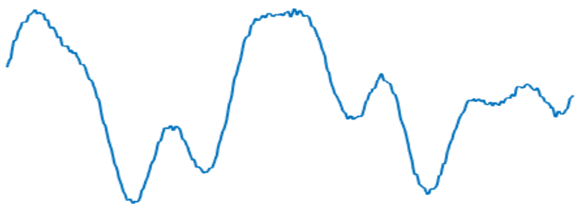
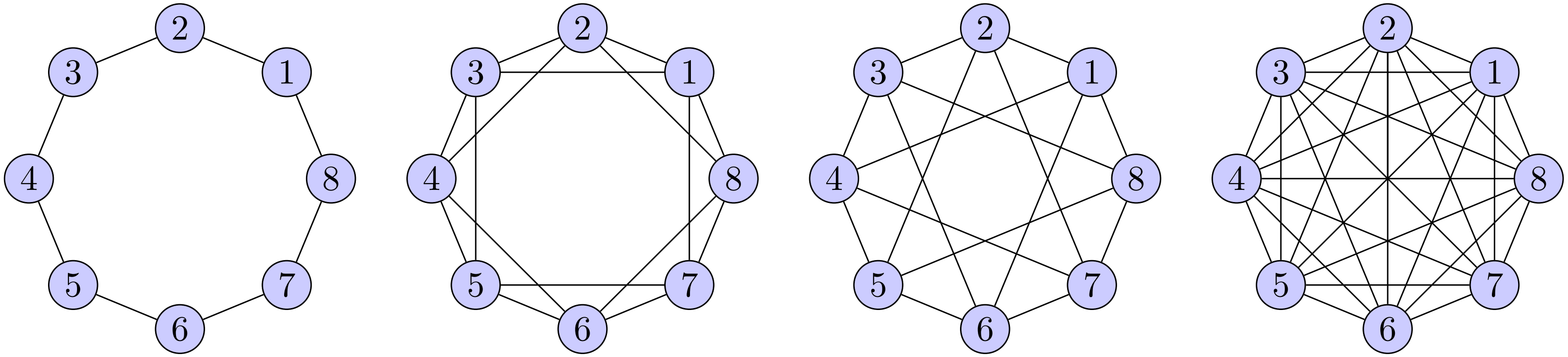
Motivation: Elementary Building Blocks

Signal Processing aims to decompose complex signals using elementary functions which are then easier to manipulate



$$x(t) = \sum_{i=-\infty}^{\infty} \alpha_i \varphi_i(t)$$

Elementary Building Blocks: Circulant Graphs



Insight: Use circulant graphs as elementary building blocks for graph signal processing

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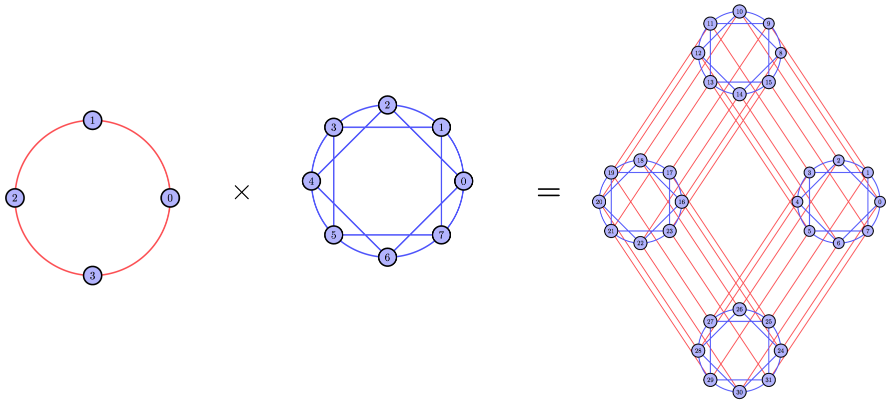
Elementary Building Blocks: Circulant Graphs



Complex Graph can be decomposed in (approximately) circulant components

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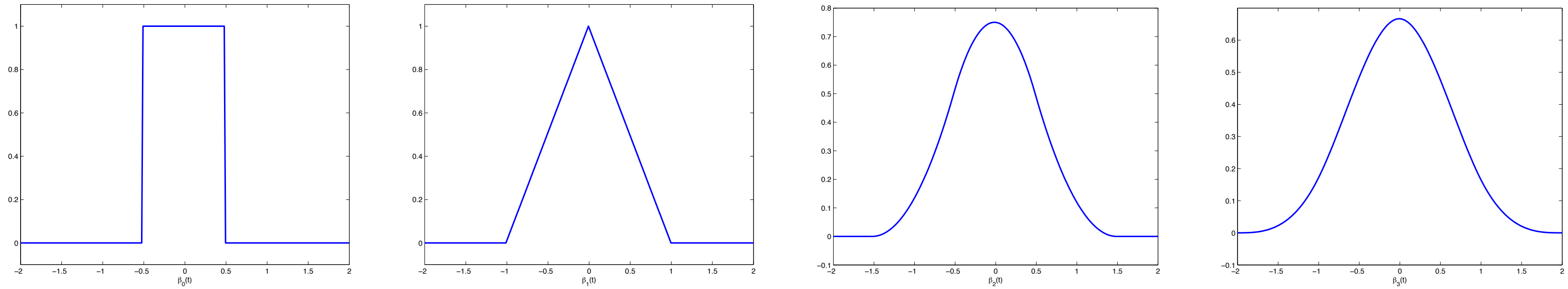
Roadmap

- **Goal: Develop a complete graph signal processing theory for circulant graphs:**
  - B-Splines and E-Splines Wavelets on Graphs
  - Sparse Sampling on Circulant Graphs
- **Apply the above theory to signals on Complex Graphs by decomposing them into elementary Circulant Graphs**

Polynomial Splines

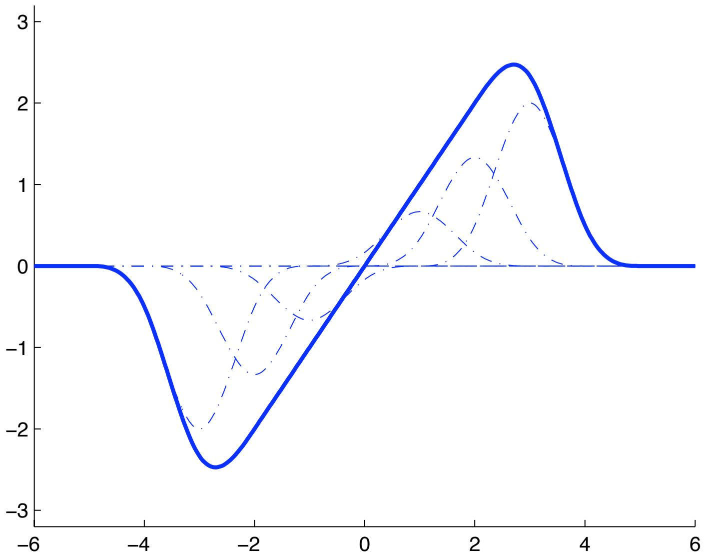
- Polynomial Splines are at the heart of any wavelet construction[1]

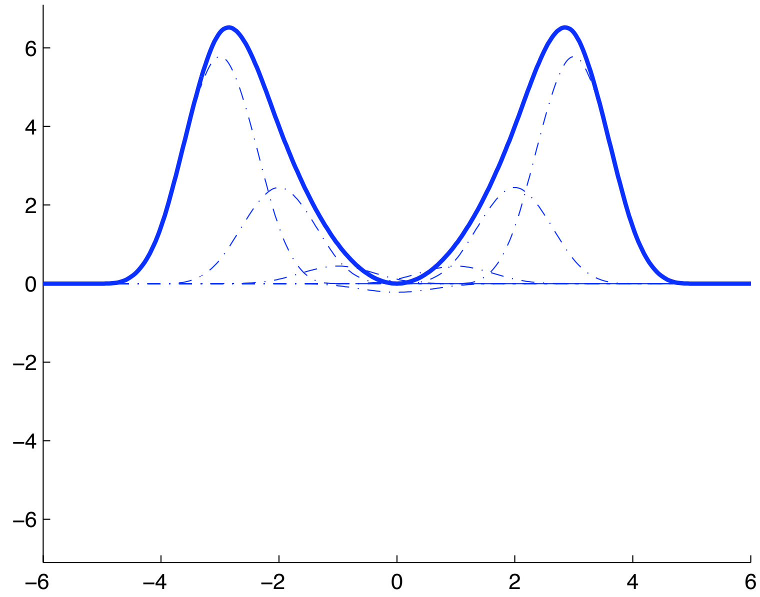


[1] M. Unser and T. Blu, "Wavelet Theory Demystified" IEEE Trans. Signal processing, 2003.

Polynomial Splines

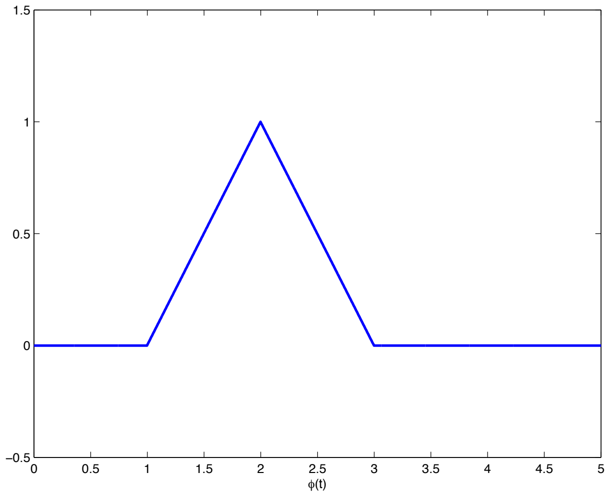
- Splines Reproduce Polynomials

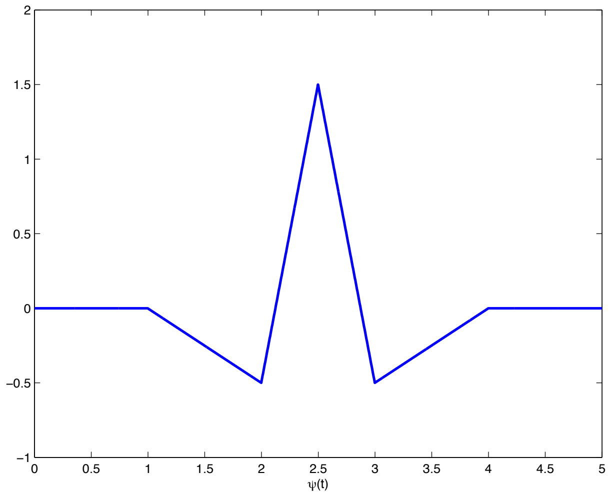




Polynomial Splines

- Splines Reproduce Polynomials
- Spline Wavelets Annihilate Polynomials (vanishing moments property)





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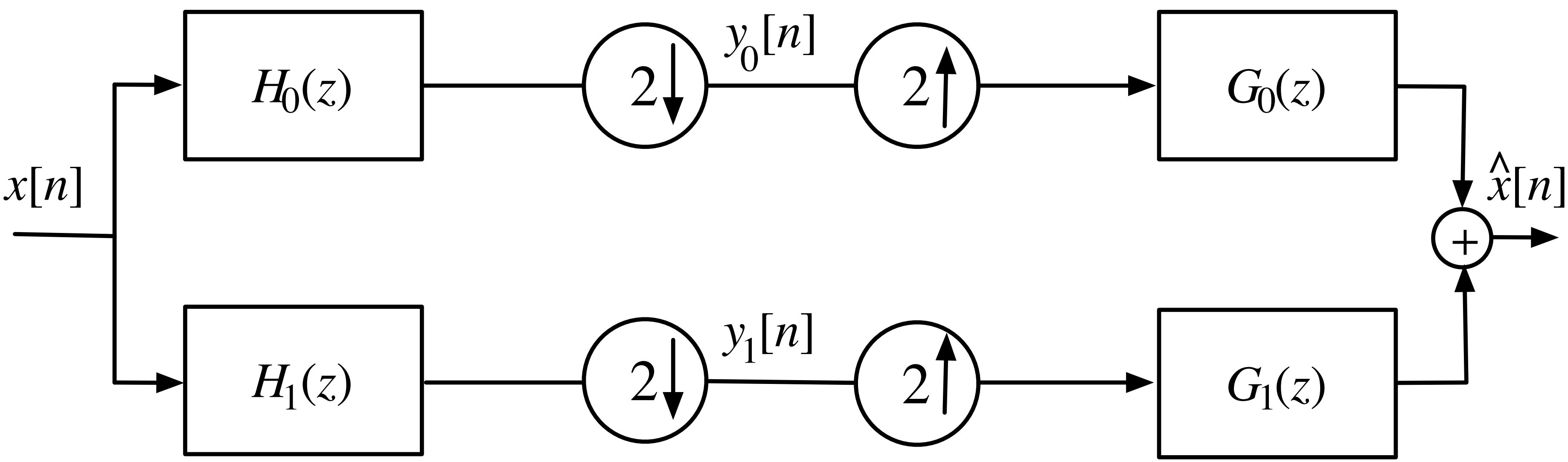
Discrete-Time Polynomial Splines



- **The low-pass filter is the polynomial spline**
- **The high-pass filter annihilates polynomial:**

$$\sum_k h_1[k]x[n-k] = 0$$

when x[n] is a polynomial of the right degree.

Splines on Circulant Graphs

- For a undirected circulant graph with adjacency matrix A, the graph Laplacian L=D-A has two vanishing moments. Thus, L annihilates up to linear polynomial graph signals. Moreover, $(D\text{-}A)^k$ has 2k vanishing moments.

- **Theorem [KotzagiannidisD:16]:** Given the undirected, and connected circulant graph *G* with adjacency matrix *A* and degree *d* per node, the higher-order graph-spline wavelet transform is given by:

$$\mathbf{H}_{LP} = \frac{1}{2^k}\left(\mathbf{I}_N + \frac{\mathbf{A}}{d}\right)^k$$

$$\mathbf{H}_{HP} = \frac{1}{2^k}\left(\mathbf{I}_N - \frac{\mathbf{A}}{d}\right)^k$$

and is invertible for any down-sampling pattern[2]

[2]Similar result for k=1 in [Ekambaram et al, '13].

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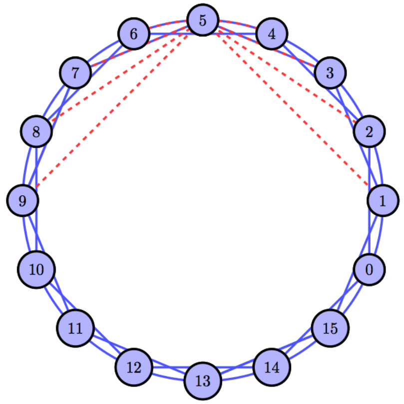
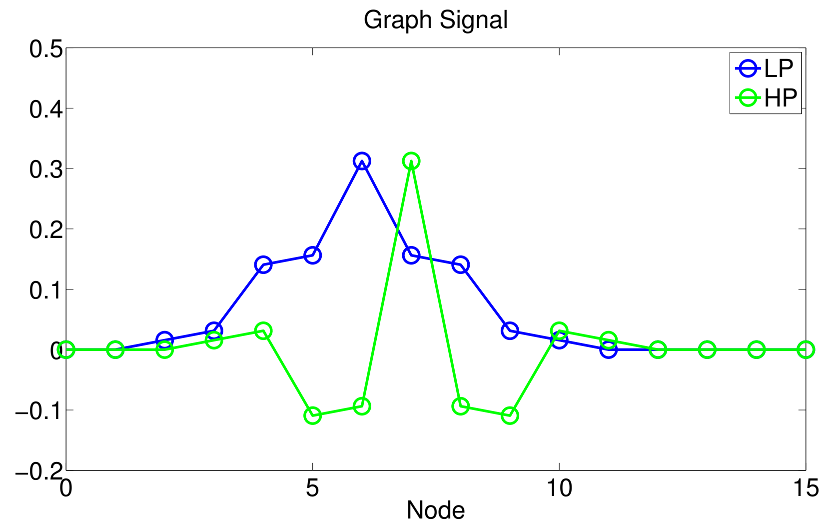
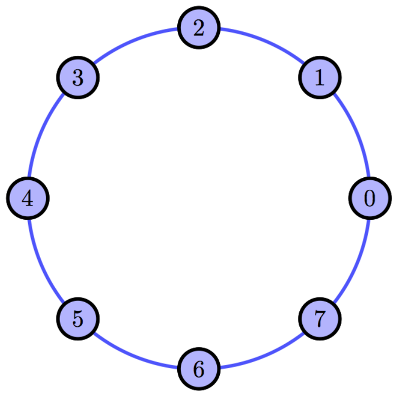
Splines on Circulant Graphs

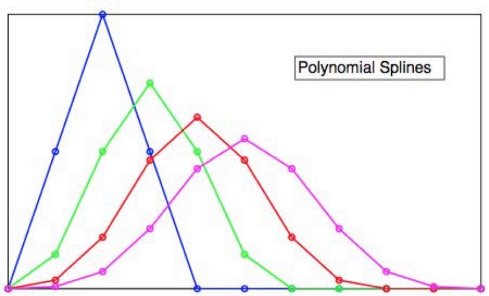




- The proposed splines annihilates polynomials, however, a polynomial signal is **NOT** in the null space of H_{HP}
- The low-pass filters reproduces polynomials when the graph is bipartite
- The low-pass filter converges to the traditional discrete spline for simple cycles

Splines on Circulant Graphs





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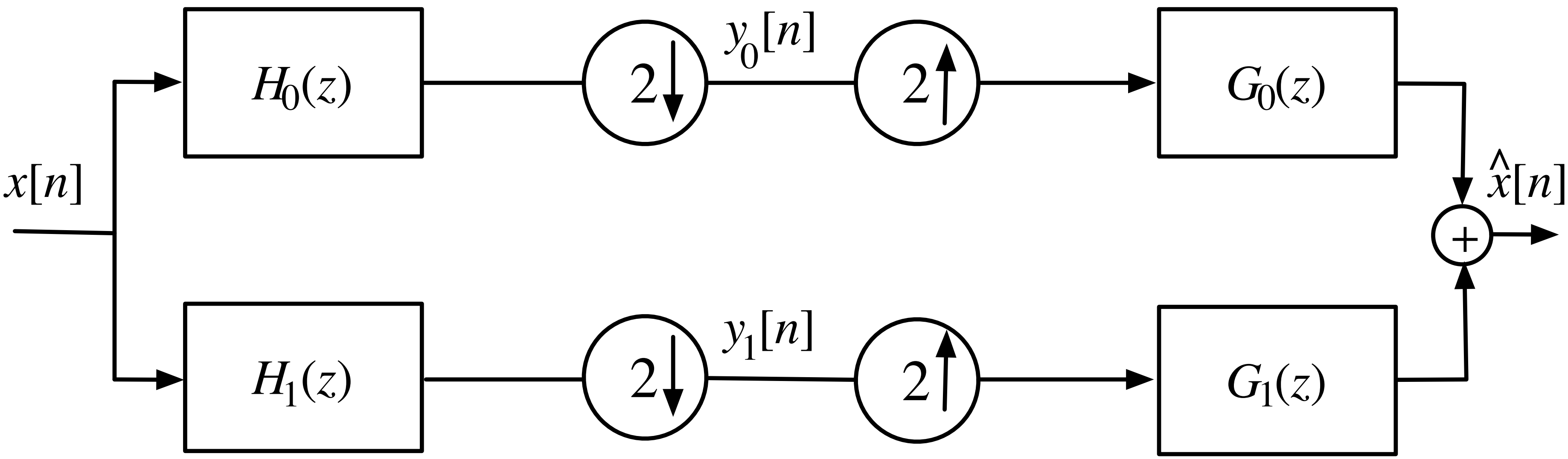
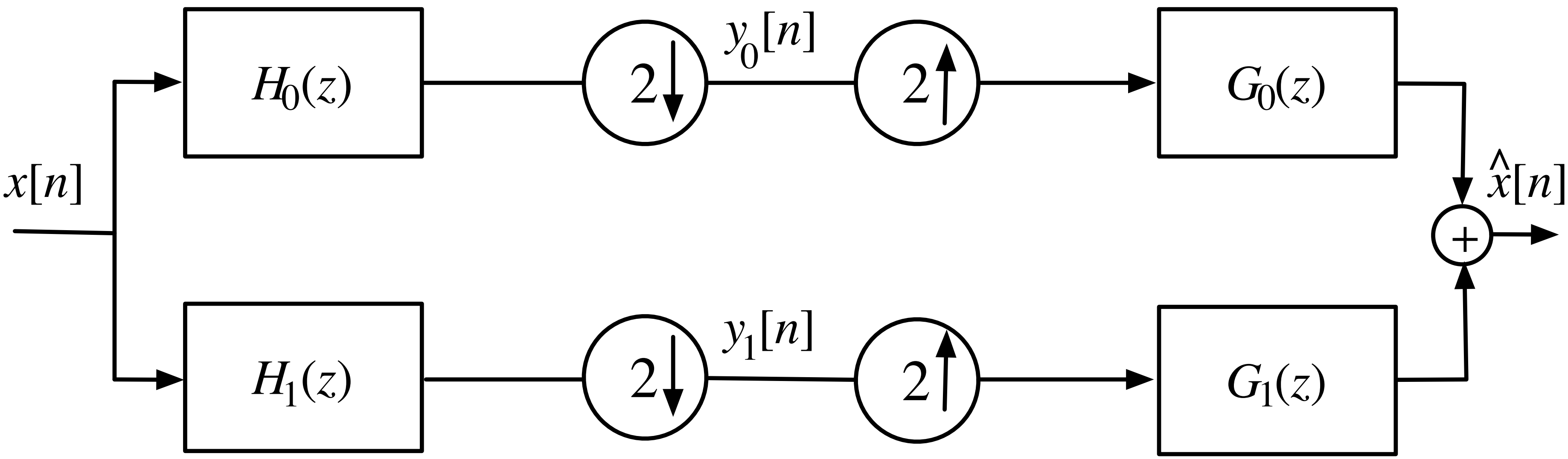
Complementary Splines on Circulant Graphs



- The previous construction does not impose constraint on the synthesis graph filters
- There is a need to reproduce polynomials also on non-bipartite graphs
- Construct complementary filters all of compact support and with polynomial reproduction capabilities by using spectral factorization

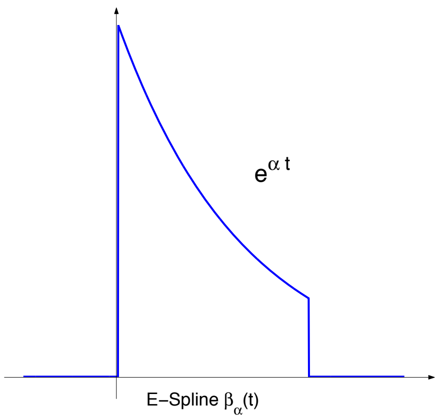
Complementary Splines on Circulant Graphs



$$\mathbf{H}_{LP,an} \stackrel{(*)}{=} \mathbf{C}\bar{\mathbf{H}}_{LP} = \frac{1}{2^k}\mathbf{C}\left(\mathbf{I}_N + \frac{\mathbf{A}}{d}\right)^k$$

$$\mathbf{H}_{HP,an} = \frac{1}{2^k}\left(\mathbf{I}_N - \frac{\mathbf{A}}{d}\right)^k$$

E-Splines[3]



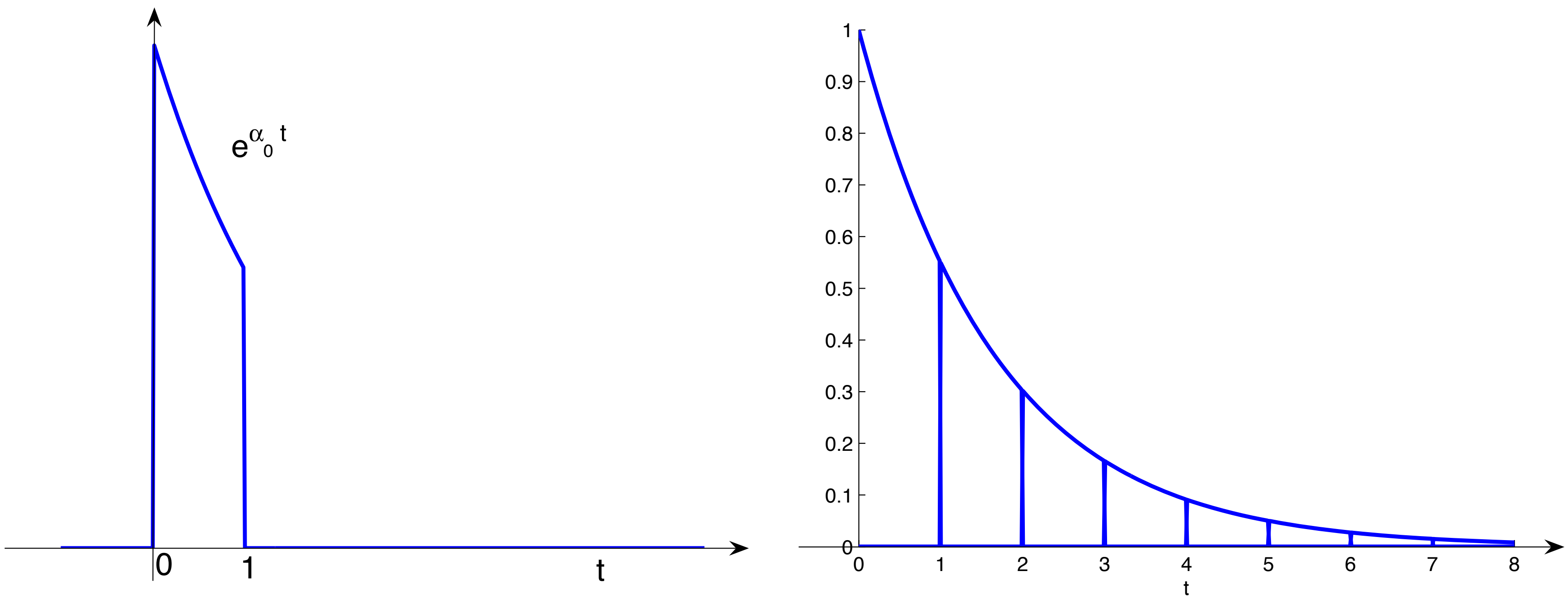
$$\beta_\alpha(t) \Leftrightarrow \hat{\beta}(\omega) = \frac{1 - e^{\alpha - j\omega}}{j\omega - \alpha}$$

Notice:

- α can be complex.
- E-Spline is of compact support.
- E-Spline reduces to the classical polynomial spline when $\alpha = 0$.

[3] M. Unser and T. Blu, ‘Cardinal Exponential Splines: Part I - Theory and Filtering Algorithms’, IEEE Trans. on Signal Proc., 2005

E-Splines



E-Splines Reproduce Exponentials
E-Spline Wavelets Annihilate Exponentials

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E-Splines on Circulant Graphs

- *The E-spline construction is based on a differential operator different from the one of the polynomial spline*[4]
- *We therefore need to change the graph Laplacian in order to design graph E-Splines*

Definition: Let G be an undirected, circulant graph with adjacency matrix A and degree d per node with symmetric weights $d_k=A_{i,k}$. Then the parameterized e-graph Laplacian of G is $L=D_a-A$ with exponential degree

$$d_\alpha = \sum_k d_k \cos(\alpha k)$$

[4]See also: I. Pesenson, "Variational Splines and Paley–Wiener Spaces on Combinatorial Graphs", 2011

E-Splines on Circulant Graphs

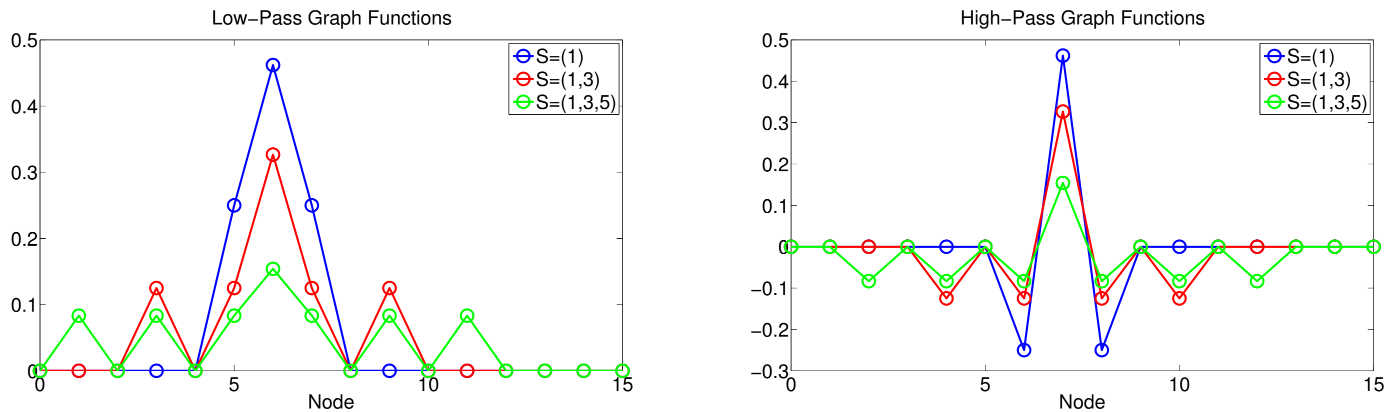
Theorem: *Given the undirected, and connected circulant graph G with adjacency matrix A and degree d per node, the higher-order e-graph-spline wavelet transform is given by:*

$$\mathbf{H}_{LP_{\vec{\alpha}}} = \prod_{n=1}^{T} \frac{1}{2^k} \left(\beta_n \mathbf{I}_N + \frac{\mathbf{A}}{d} \right)^k$$

$$\mathbf{H}_{HP_{\vec{\alpha}}} = \prod_{n=1}^{T} \frac{1}{2^k} \left(\beta_n \mathbf{I}_N - \frac{\mathbf{A}}{d} \right)^k$$

and is "normally" invertible for any down-sampling pattern.

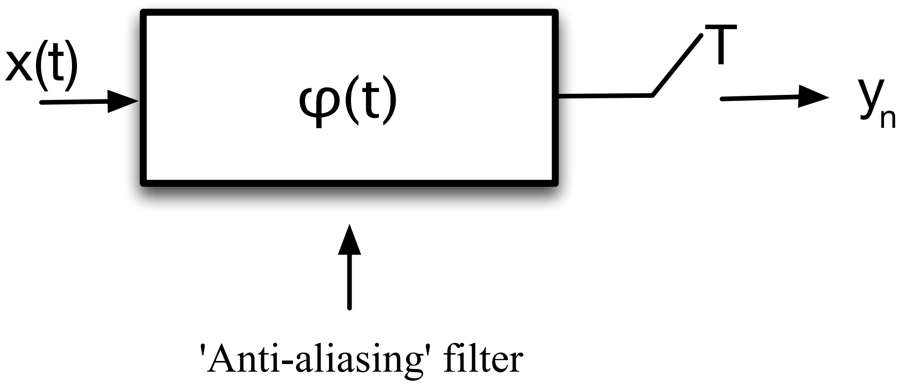
E-Splines on Circulant Graphs



- The proposed splines annihilate exponential signals
- The low-pass filter converges to the traditional discrete E-spline for simple cycles
- The low-pass filters reproduces exponentials when the graph is bipartite
- This construction converges to polynomial splines when β_n=0

Sparse Sampling

Goal: *Given a sparse graph signal x on a circulant graph, sample it and coarsen its graph without losing information*
Note: *Sparse signals do not belong to fixed sub-spaces like bandlimited signals*



Challenges:

- *Find a deterministic sampling and reconstruction strategy for x*
- *Find a good strategy for graph coarsening*

Insight: *leverage from the theory of sampling Finite Rate of Innovation (FRI) signals*[5]

[5] P.L. Dragotti, M. Vetterli and T. Blu, 'Sampling Moments and Reconstructing Signals of Finite Rate of Innovation: Shannon meets Strang-Fix', IEEE Trans. on Signal Processing, 2007.

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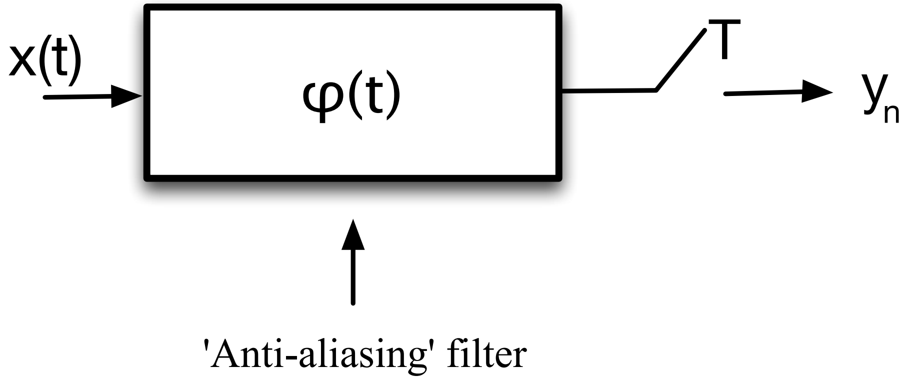
Sparse Sampling

In FRI theory

- *x* can be discrete or continuous
- the anti-aliasing filter is an E-spline because it can reproduce exponentials
- Reconstruction is achieved using Prony's method
- Exact deterministic results



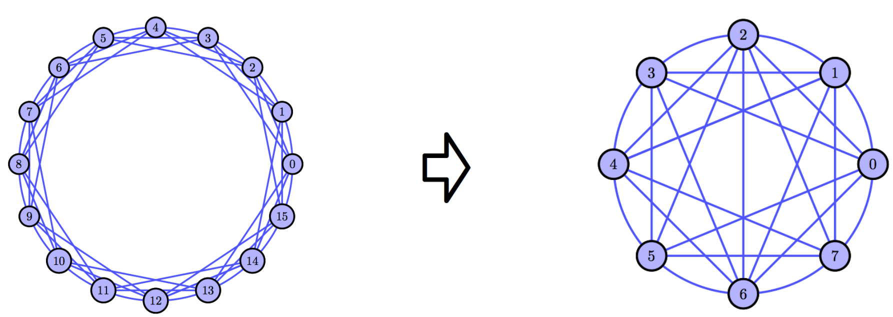
For Graph Signals

- Replace the traditional E-spline with the graph E-spline that preserves reproduction of exponential property
- Graph coarsening obtained by downsampling the eigenbasis of the original graph (this preserve the generating set of the original graph)

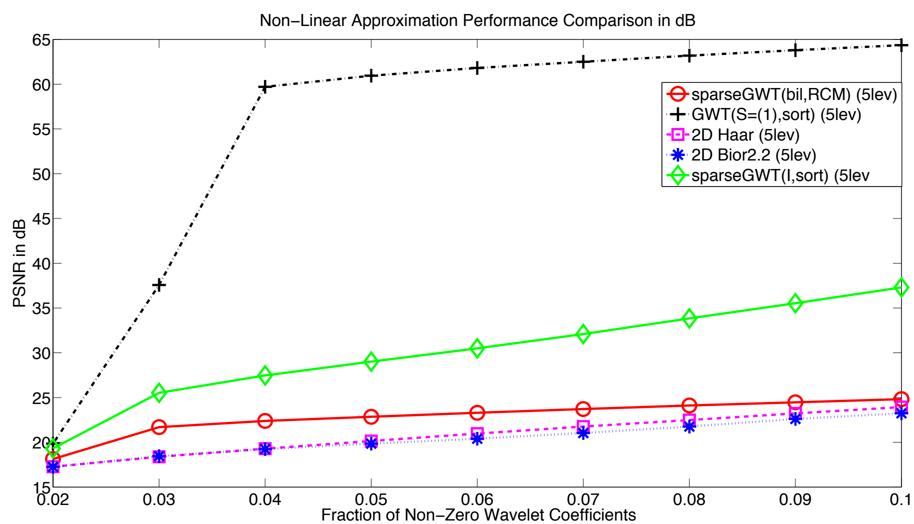
Sparse Sampling

$$\mathbf{x}_W \rightarrow \underset{(N\times N)}{\overset{\textbf{GWT}}{\boxed{\mathbf{W}}}} \rightarrow \mathbf{x}$$

$$\mathbf{x} \rightarrow \underset{(N/2\times N)}{\overset{\textbf{LP}}{\boxed{\mathbf{E}\downarrow \mathbf{2}}}} \rightarrow \tilde{\mathbf{y}} \rightarrow \underset{(M\times N/2)}{\overset{\textbf{GFT}}{\boxed{\mathbf{C}}}} \rightarrow \mathbf{y}$$



Preliminary Applications: Image Approximation



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Conclusions

Summary:

- Introduced new families of high order graph wavelets
- Extension of sparse sampling theory to the case of graph signals
- Preliminary encouraging results on non-linear image approximation

Papers:

- M.S. Kotzagiannidis and P.L. Dragotti, 'Splines and Wavelets on Circulant Graphs', http://arxiv.org/abs/1603.04917
- M.S. Kotzagiannidis and P.L. Dragotti,"The graph FRI framework: spline wavelet theory and sampling on circulant graphs, ICASSP 2016