SAMPLING SCHEMES FOR 2-D SIGNALS WITH FINITE RATE OF INNOVATION USING KERNELS THAT REPRODUCE POLYNOMIALS

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ABSTRACT

In this paper, we propose new sampling schemes for classes of 2-D signals with finite rate of innovation (FRI). In particular, we consider sets of 2-D Diracs and bilevel polygons. As opposed to using only sinc or Gaussian kernels [7], we allow the sampling kernel to be any function that reproduces polynomials.

In the proposed sampling schemes, we exploit the polynomial approximation properties of the sampling kernels in association with other relevant techniques such as complex-moments [10], annihilating filter method [3], and directional derivatives. Specifically, for the bilevel polygons, we propose two different methods: the first uses a global reconstruction algorithm and complex moments, while the second is based on directional derivatives and local reconstruction algorithms. The trade-off between these two reconstruction modalities is also briefly discussed.

1. INTRODUCTION

We know that most natural-world phenomena are observed and analyzed through sampling, thus sampling is one of the core elements in many applications of modern science and technology. Although Shannon's sampling theory and its extensions are very powerful and have been successfully utilized for bandlimited signals, in many situations this 'bandlimited-sinc' constraint is too restrictive to abide for the available acquisition devices and processing algorithms [1]. The conventional 'bandlimited' scenario has been extended to classes of nonbandlimited signals such as uniform splines that reside in a subspace spanned by a generating function and its shifted versions [1, 2]. For a comprehensive account on the modern sampling developments, we refer to [1].

Very recently, novel sampling schemes have been presented for larger classes of 1-D signals that are neither bandlimited nor reside in a subspace [3]. Such signals belong to a class of signals with a finite number of degrees of freedom (or rate of innovation) and are classified as signals with Finite Rate of Innovation (FRI). Streams of Diracs, nonuniform splines, and piecewise polynomials are examples of such signals. These novel sampling schemes feature: sinc and Gaussian sampling kernel, Annihilating filter method, and perfect reconstruction using only a finite number of samples (lowpass estimates). Nevertheless, both sinc and Gaussian kernels pose difficulties in practice due to their slow decay and infinite support.

Extensions of the schemes in [3] to the 2-D case are examined in [7] and [8]. These 2-D extensions, however, protract with the sinc and Gaussian kernels. In [4, 5] it was shown that 1-D FRI signals can be sampled using a very rich class of kernels such as functions that reproduce polynomials, exponential splines [6], and functions with rational Fourier transforms. In this paper we furnish extensions of these results in 2-D; we focus on the case of 2-D kernels that can reproduce polynomials and show that sets of 2-D Diracs and polygonal images can be sampled and perfectly reconstructed using these kernels. For the polygonal case, we present two alternative schemes: one is based on complex-moments and annihilating filter method, the other on the link between finite differences and directional derivatives.

The paper is organized as follows: In the next section we show a generic sampling setup for 2-D, and highlight the useful properties of our sampling kernels. In Section 3, we present a local sampling scheme for sets of 2-D Diracs as an extension to the 1-D scheme given in [4]. In Section 4, we address a global scheme for sampling the bilevel polygons, inspired by the complex-moments based approach of [9, 10]. We then extend this global sampling scheme (of bilevel polygons) to sets of Diracs as an alternative to a local one presented in Section 3. In Section 5, we propose a novel directional-derivatives based sampling scheme for planar polygons. The scheme is local (involve only a corner point at a time) and holds only local complexities irrespective of the number of corner points. Finally, we conclude in Section 6.

2. SAMPLING SETUP AND KERNEL

Consider a 2-D generic sampling setup, where a continuous 2-D FRI signal $g(x, y)$ is prefiltered with a smoothing kernel $\varphi_{xy}(x, y)$. The filtered version $g(x, y) * \varphi_{xy}(-x, -y)$ is sampled uniformly to obtain a set of samples $S_{j,k}$ given by

$$S_{j,k} = \langle g(x, y), \varphi_{xy}(x/T_x - j, y/T_y - k)\rangle \tag{1}$$

where $x, y \in \mathbb{R}$, $j, k \in \mathbb{Z}$, and $T_x, T_y \in \mathbb{R}^+$ are the sampling intervals along x and y directions respectively.

Our sampling kernel $\varphi_{xy}(x, y)$ is given by the tensor product of two 1-D functions $\varphi(t), t \in \mathbb{R}$ that can reproduce polynomials. The sampling kernel $\varphi_{xy}(x, y)$ then follows the partition of unity and polynomial approximation properties as given by

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$$\sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \varphi_{xy}(x-j, y-k) = 1 \quad (2)$$

$$\sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} C_{\gamma,j}^{x}\, \varphi_{xy}(x-j, y-k) = x^{\gamma}$$

$$\sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} C_{\gamma,k}^{y}\, \varphi_{xy}(x-j, y-k) = y^{\gamma} \quad (3)$$

where $\gamma = \{0, 1, \ldots, \Gamma - 1\}$ specify numbers of degrees of polynomials that the sampling kernel $\varphi_{xy}(x, y)$ can reproduce. The coefficients $C_{\gamma,j}^{x}$ and $C_{\gamma,k}^{y}$ are kernel dependent weights along x and y directions respectively. We further assume that $\varphi_{xy}(x, y)$ is of compact support $L_x \times L_y$. Notice that orthogonal Daubechies scaling functions and biorthogonal B-Splines, among many other scaling functions, satisfy the above properties.

3. SETS OF 2-D DIRACS: AN EXTENSION OF 1-D CASE

We begin with a simple class of FRI signals, that is, a set of 2-D Diracs $g(x, y) = \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} a_{j,k}\, \delta_{xy}\,(x - x_j, y - y_k)$, where $a, x, y \in \mathbb{R}$. Each Dirac can be parameterized by an amplitude and a position (in terms of two Cartesian coordinates x and y) and thus has a finite number of degrees of freedom (or rate of innovation) which equals three.

Assume that there is at most one Dirac in an area of size $L_xT_x \times L_yT_y$, and that the sampling kernel $\varphi_{xy}(x, y)$ can reproduce polynomials of degrees zero and one. With the backdrop from [4], we are sure that only $L_x \times L_y$ inner products (kernels) overlap in any given region of size $L_xT_x \times L_yT_y$ that encloses a unique Dirac $a_{p,q}\delta_{xy}(x-x_p, y-y_q), p, q \in \mathbb{Z}$. Therefore, relating the properties of partition of unity (2) and polynomial approximations (3) as illustrated in Figure 1, the amplitude and position of a given Dirac are given by [1]

$$a_{p,q} = \sum_{j=1}^{L_x} \sum_{k=1}^{L_y} S_{j,k} \quad (4)$$

$$x_p = \left(\sum_{j=1}^{L_x} \sum_{k=1}^{L_y} C_{1,j}^{x} S_{j,k} \right) \Big/ a_{p,q}$$

$$y_q = \left(\sum_{j=1}^{L_x} \sum_{k=1}^{L_y} C_{1,k}^{y} S_{j,k} \right) \Big/ a_{p,q} \quad (5)$$

where the coefficients $C_{1,j}^{x}$ and $C_{1,k}^{y}$ are identified from equation (3).

Hence, a sampling scheme for Diracs follows

Proposition 1. *Given a sampling kernel $\varphi_{xy}(x, y)$ that can reproduce polynomials of degree zero and one along both Cartesian axis x and y and of compact support $L_x \times L_y$, a set of finite amplitude 2-D Diracs $g(x, y) = \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} a_{j,k}\, \delta_{xy}\,(x - x_j, y - y_k)$ is uniquely determined from its samples defined by $S_{j,k} = \langle g(x, y), \varphi_{xy}(x/T_x - j, y/T_y - k)\rangle$ if and only if there is at most one Dirac in any distinct rectangular area of size $L_xT_x \times L_yT_y$.*

[1] As the proof is in the line of its 1-D formulation [4], we omit it for the judicious use of the space.

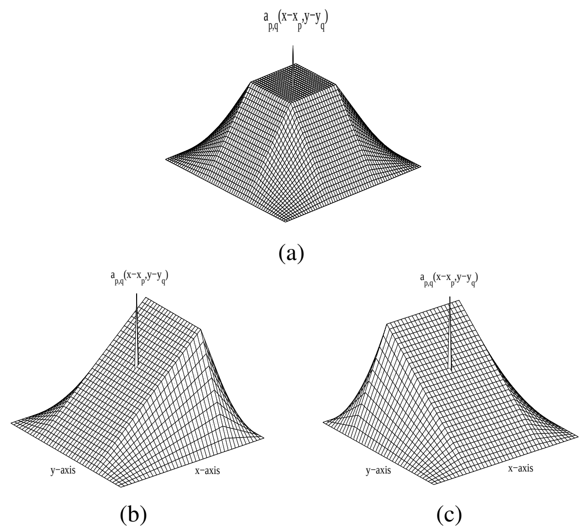


Fig. 1. Sampling of a 2-D Dirac using linear B-Spline $B_{xy}^{1}(x, y)$: (a) Partition of unity responsible for the determination of amplitude $a_{p,q}$, (b) Reproduction of polynomial of degree 1 along x direction responsible for the determination of coordinate x_p, and (c) Reproduction of polynomial of degree 1 along y direction responsible for the determination of coordinate y_q.

4. BILEVEL POLYGONS AND DIRACS: USING COMPLEX MOMENTS

Consider the other class of FRI signal $g(x, y)$ that consists of an arbitrary bilevel polygon with N corner points (vertices) such that inside the polygon is '1' and outside is '0'. Naturally, the rate of innovation in this case is $2N$. Assume that the bilevel polygon is convex and simply connected. The positions of its N corner points $\{z_n\}$, $n = 1, 2, \ldots, N$ are given by a set of N complex scalars $z = x + iy$, where $i = \sqrt{-1}$, and x, y are the Cartesian coordinates.

Following the formulations from [9, 10] and applying Davis's theorem, we have a set of N complex coefficients $\rho_n = \frac{i}{2}\left(\frac{z_{n-1}^{*} - z_n^{*}}{z_{n-1} - z_n} - \frac{z_n^{*} - z_{n+1}^{*}}{z_n - z_{n+1}}\right)$, where z^{*} is the complex conjugate of z. Since the polygon is closed and simply connected, it follows the modulo operation $z_{N+n} = z_n$. It is demonstrated in [9, 10], as a special case, that for an analytic function $f(z) = z^m, m \in \mathbb{N}$, in the closure ($\mathcal{O}$) of the polygon, the weighted complex moments $\tau_m = \int_{\mathcal{O}} f^{''}(z)\ dz = m(m-1) \int_{\mathcal{O}} z^{m-2}\ dz = \sum_{n=1}^{N} \rho_n z_n^m$. Notice that $\tau_0 = \tau_1 = 0$ from the above relation. The result of [10] states that at least $2N - 1$ complex moments can uniquely determine the N corner points of a given bilevel polygon.

Let $g(x, y)$ be a bilevel polygon, and $\varphi_{xy}(x, y)$ be the sampling kernel that satisfies the properties outlined in equations (2) and (3) with $\Gamma \geq 2N$. The polynomial approximation property of $\varphi_{xy}(x, y)$ allows us to obtain the complex moments of $g(x, y)$ directly from its samples $S_{j,k}$. For instance, τ_3 is given by

$$\tau_3 = 6 \int_{\mathcal{O}} z\ dz = 6 \sum_{j} \sum_{k} \left(C_{\gamma,j}^{x} + iC_{\gamma,k}^{y}\right) S_{j,k}$$

where $\gamma = m - 2 = 1$. Similarly, we can obtain any other moment τ_m for $m = 2, \ldots, \Gamma + 1$.

Applying the annihilating filter method [3], a filter $A[l],\ l = 0, 1, \ldots, N$ is designed such that $\tau_m * A[l] = 0$. The N complex roots of the annihilating filter $A[l]$ provide positions (in $x + iy$ form) of N corner points of the given bilevel polygon. Assumptions of convexity and simply connectedness guarantee a uniqueness of the polygonal reconstruction. Consequently, a sampling perspective to the result of [10] follows

Proposition 2. *Given a sampling kernel $\varphi_{xy}(x, y)$, a simply connected and convex bilevel polygon $g(x, y)$ with at most N corner points is uniquely determined by its samples $S_{j,k} = \langle g(x, y), \varphi_{xy}(x/T_x - j, y/T_y - k)\rangle$, provided that $\varphi_{xy}(x, y)$ can reproduce polynomial up to degree $2N - 1$ along both the Cartesian axis x and y.*

Now imagine a case where, instead of a bilevel polygon, signal $g(x, y)$ consists of a set of N 2-D Diracs. Then, it is easy to show that the above proposition can be extended to the set of N Diracs.

The complex-moments based approach provides a global solution for the reconstruction of bilevel polygons and sets of Diracs. The complexity of the solution, however, increases with the complexity of the signal $g(x, y)$ (i.e. with the numbers of corner points). Polygons with very close corner points still pose a reconstruction challenge due to numerical instabilities in the algorithmic implementations.

Simulation results, for a simple scenario, are shown in Figure 2. Figure 2(a) shows the original bilevel image $g(x, y)$ that consists of three polygonal shapes. Assume that the polygons are enough apart such that in the sampled version of $g(x, y)$ as shown in part (b), they do not overlap. From these samples, using complex-moments, we can retrieve the exact locations of the corner points for each polygonal shape individually. For instance, a set of samples around pentagon is given in part (c). The reconstructed corner points of the pentagon are indicated with $+$ in part (d).

5. PLANAR POLYGONS: DIRECTIONAL DERIVATIVES BASED APPROACH

Consider a continuous planar polygon $g(x, y)$ with N corner points. All N sides (boundaries) of the polygon are identified by the 2-D lines $y_i = \tan(\theta_i)x_i + b_i,\ i = 1, 2, \ldots, N$ where b_i are shifts (offsets), and θ_i denotes the orientations.

We understand that for a continuous polygon $g(x, y)$, in theory, two successive directional derivatives $\mathcal{D}_{\theta_1}$ and $\mathcal{D}_{\theta_2}$ along the two adjacent polygonal sides with orientations θ_1 and θ_2 result into a 2-D Dirac at the corner point formed by the two respective sides.

In practice, we do not have direct access to $g(x, y)$ but only to the samples $S_{j,k} = \langle g(x, y), \varphi_{xy}(x/T_x - j, y/T_y - k)\rangle$, where $\varphi_{xy}(x, y)$ is the sampling kernel. A discrete equivalent to the directional derivatives is the evaluation of directional differences over the set of samples $S_{j,k}$. The connection between these two dwells in the lattice theory, and in particular, involves subsampling over a rectangular lattice of $\mathbb{Z}^2$. A quick overview on the fundamentals of the lattice theory can be found in [11]. For a detailed treatment we refer to [12].

Assume that the base lattice $\Lambda = \{\lambda : \lambda = n_1\vec{v}_1 + n_2\vec{v}_2\}$ with $n_i, \vec{v}_i \in \mathbb{Z}$, where $\vec{v}_i = \{v_{i,1}, v_{i,2} \mid i = 1, 2\}$. The base lattice Λ is characterized by a (non-unique) sampling matrix $V_\Lambda = \begin{pmatrix} v_{1,1} & v_{1,2} \\ v_{2,1} & v_{2,2} \end{pmatrix}$ with its determinant defined as $\det(V_\Lambda)$. The row vectors $\vec{v}_1$ and $\vec{v}_2$ determine the orientations θ_1 and θ_2 of the two adjacent polygonal sides at a given corner point such that $\theta_1 = \tan^{-1}\left(\frac{v_{1,2}}{v_{1,1}}\right)$, and $\theta_2 = \tan^{-1}\left(\frac{v_{2,2}}{v_{2,1}}\right)$.

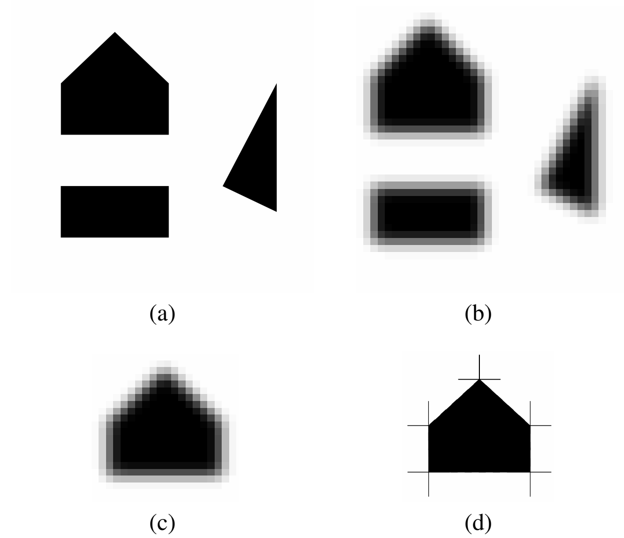


Fig. 2. (a) An original image $g(x, y)$ of size 3767×3767 pixels consists of three bilevel polygons: triangle, rectangle, and pentagon. (b) The set of 50×50 samples produced by the inner products of $g(x, y)$ with a B-Spline sampling kernel $\varphi_{xy}(x, y) = \beta^9_{xy}(x, y)$ with support 631×631 pixels that can reproduce polynomials up to degree nine. (c) Sampled version of the pentagon. (d) Original pentagon and five reconstructed corner points (with $+$).

Following the modus operandi depicted in Figure 3, apply a pair of directional differences along θ_1 and θ_2 over the two pairs of samples $S_{j,k}$ (identified by Λ) around an arbitrary corner point A of the planar polygon $g(x, y)$. It follows that

$$
\begin{aligned}
&\mathcal{D}_{\theta_2}\left[\mathcal{D}_{\theta_1}\left[S_{j,k}\right]\right] \\
&\quad = \left\{S_{(v_{2,1}+v_{1,1}),(v_{2,2}+v_{1,2})} - S_{(v_{2,1}),(v_{2,2})}\right\} - \\
&\quad\quad \left\{S_{(v_{1,1}),(v_{1,2})} - S_{0,0}\right\} \\
&\quad = \Big\langle g(x, y), \left\{\varphi_{xy}\big(x - (v_{2,1} + v_{1,1}), y - (v_{2,2} + v_{1,2})\big) - \right. \\
&\quad\quad \left.\varphi_{xy}\big(x - v_{2,1}, y - v_{2,2}\big)\right\} - \left\{\varphi_{xy}\big(x - v_{1,1}, y - v_{1,2}\big) - \right. \\
&\quad\quad \left.\varphi_{xy}\big(x - 0, y - 0\big)\right\}\Big\rangle.
\end{aligned}
$$

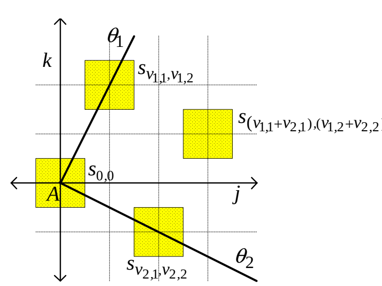


Fig. 3. Two successive directional differences $\mathcal{D}_{\theta_1}$ and $\mathcal{D}_{\theta_2}$ evaluated along θ_1 and θ_2 over the two pairs of samples $S_{j,k}$ around an arbitrary corner point A of the planar polygon $g(x, y)$.

By using Parseval's identities, and after certain manipulations, we derive that

$$\frac{\mathcal{D}_{\theta_2}\left[\mathcal{D}_{\theta_1}\left[S_{j,k}\right]\right]}{|\det(V_\Lambda)|} = \left\langle \frac{\partial}{\partial\theta_2}\left(\frac{\partial}{\partial\theta_1}\left(g(x,y)\right)\right), \zeta_{\theta_1,\theta_2}(x,y)\right\rangle \quad (6)$$

where $\zeta_{\theta_1,\theta_2}(x,y) = \frac{\beta^0_{\theta_1,\theta_2}(x,y)*\varphi_{xy}(x,y)}{|\sin(\theta_2-\theta_1)|}$. Notice that $\beta^0_{\theta_1,\theta_2}(x,y) = \beta^0_{\theta_1}(x,y) * \beta^0_{\theta_2}(x,y)$, and $\beta^0_{\theta_1}$, $\beta^0_{\theta_2}$ are the 2-D box B-Splines of order zero along the orientations θ_1 and θ_2 respectively.

The modification of $\varphi_{xy}(x,y)$ into $\zeta_{\theta_1,\theta_2}(x,y)$ around a given corner point reclines in the directional differences over $S_{j,k} \in \mathbb{Z}^2$ directed by the sampling matrix V_Λ. The kernel $\zeta_{\theta_1,\theta_2}(x,y)$ has a support $(|v_{1,1}|+|v_{2,1}|+L_x)\times(|v_{1,2}|+|v_{2,2}|+L_y)$, where $L_x \times L_y$ is the support of the original sampling kernel $\varphi_{xy}(x,y)$. A simple illustration of both kernels is given in Figure 4. The presence of amplitude scaling factors $\frac{1}{|\det(V_\Lambda)|}$ and $\frac{1}{|\sin(\theta_2-\theta_1)|}$ in equation (6) is due to subsampling over lattices and to structural properties of the modified kernel.

Given a valid sampling kernel $\varphi_{xy}(x,y)$, the modified kernel $\zeta_{\theta_1,\theta_2}(x,y)$ always satisfies partition of unity (2) and reproduces polynomials of degree one along both x and y directions (3). These properties of the modified kernel $\zeta_{\theta_1,\theta_2}(x,y)$ enable us to determine the amplitude $a_{p,q}$ and coordinate positions x_p, y_q of the resultant 2-D Dirac at a given corner point from the set of samples $S_{j,k}$. We need only a finite number of samples, i.e. $(|v_{1,1}|+|v_{2,1}|+L_x)\times(|v_{1,2}|+|v_{2,2}|+L_y)$ in the vicinity of the corner point. The reconstruction scheme of (4) and (5) generalizes as follows

$$a_{p,q} = \frac{\sum_j\sum_k \mathcal{D}_{\theta_2}\left[\mathcal{D}_{\theta_1}\left[S_{j,k}\right]\right]}{|\det(V_\Lambda)|} \quad (7)$$

$$x_p = \frac{\sum_j\sum_k C^x_{\gamma,j}\,\mathcal{D}_{\theta_2}\left[\mathcal{D}_{\theta_1}\left[S_{j,k}\right]\right]}{a_{p,q}\,|\det(V_\Lambda)|}$$

$$y_q = \frac{\sum_j\sum_k C^y_{\gamma,k}\,\mathcal{D}_{\theta_2}\left[\mathcal{D}_{\theta_1}\left[S_{j,k}\right]\right]}{a_{p,q}\,|\det(V_\Lambda)|} \quad (8)$$

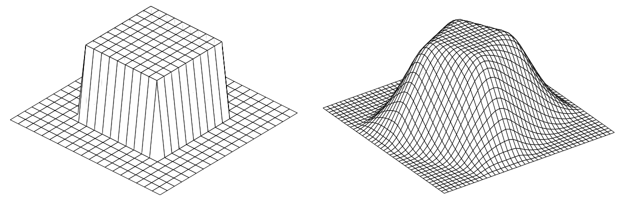
where both the sets of weighting coefficients $C^x_{\gamma,j}$ and $C^y_{\gamma,k}$ for the kernel $\zeta_{\theta_1,\theta_2}(x,y)$ follow equation (3).

By a finite number of iterations along the valid orientations (where $\tan(\theta) \in \mathbb{Q}$), N correct pairs of directional differences are identified. Using these pairs, a given polygon is decomposed into N corner points. The local reconstruction scheme of (7) and (8) equally applies to each of the N corner points individually.

This local sampling scheme is suitable for resolving planar polygons with varying amplitudes and with a large set of orientations of their sides using only a finite number of samples. Yet, there must be at most one corner point in the support of its associated modified kernel $\zeta_{\theta_1,\theta_2}(x,y)$. This scheme has a local complexity irrespective of the number of corner points in a given polygon.

6. CONCLUSION

In this paper, we have proposed several sampling schemes for the classes of 2-D nonbandlimited signals using sampling kernels that reproduce polynomials. Combining the tools like annihilating filters, complex-moments, and directional derivatives, we provide local and global sampling choices with varying degrees of complexity. Future work will explore cross-fertilization between the proposed sampling schemes and the potential extensions of footprints in 2-D for image resolution enhancement.



(a) (b)

Fig. 4. For example, (a) $\varphi_{xy}(x,y)$ is a Haar scaling function with support 1×1, (b) Modified kernel $\zeta_{\theta_1,\theta_2}(x,y)$ with support 4×4 for an arbitrary corner point of the polygon formed by the two sides with orientations $\tan(\theta_1) = 2/1$ and $\tan(\theta_2) = -1/2$.

7. ACKNOWLEDGEMENT

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