SOLVING PHYSICS-DRIVEN INVERSE PROBLEMS VIA STRUCTURED LEAST SQUARES

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ABSTRACT

Numerous physical phenomena are well modeled by partial differential equations (PDEs); they describe a wide range of phenomena across many application domains, from modeling EEG signals in electroencephalography to, modeling the release and propagation of toxic substances in environmental monitoring. In these applications it is often of interest to find the sources of the resulting phenomena, given some sparse sensor measurements of it. This will be the main task of this work. Specifically, we will show that finding the sources of such PDE-driven fields can be turned into solving a class of well-known multi-dimensional structured least squares problems. This link is achieved by leveraging from recent results in modern sampling theory – in particular, the approximate Strang-Fix theory. Subsequently, numerical simulation results are provided in order to demonstrate the validity and robustness of the proposed framework.

Index Terms— Spatiotemporal sampling, sensor networks, inverse source problems, structured least squares, Prony's method, finite rate of innovation (FRI)

1. INTRODUCTION

Often, one encounters the problem of determining a cause of some measured effect in numerous natural science, medical and engineering applications; such a problem is called an inverse problem. A classical example being Computerized Tomography [1,2]. In these problems, the measurements (effect) and the desired parameters (cause) are linked by a mathematical model. When the model is linear/nonlinear the problem is known as a linear/nonlinear inverse problem.

In this paper we consider the inverse source problem [3] and propose a framework for solving it when the measured data is linked to the sources to be estimated through a partial differential equation. This class of inverse problems continues to receive considerable research interests from a range of communities, including the signal processing community, due to their ubiquity across many applications involving, for example, sound/wave source localization [4], brain source localization [5], and plume/leakage detection [6,7]. Our proposed framework relies on the premise that the unknown sources of the multidimensional field is sparse. With this assumption we demonstrate how to reformulate our problem in the form of a Prony-like system, that can be solved efficiently using structured least squares methods assuming we have access to a set of proper measurements which we call *generalized measurements*. Next, leveraging from the universal sampling paradigm and approximate Strang-Fix theory [8], we show how these *generalized measurements* may be obtained from the sensor data and the Green's function of the underlying field alone, under either uniform or nonuniform spatial sampling. Finally we present numerical simulation results to further reinforce the validity of our proposed scheme for Laplace and diffusion fields, in two and three spatial dimensions.

The rest of the paper is arranged as follows: we formally outline the inverse problem of interest in Section 2. In Section 3 we argue that the unknown sources can be recovered from the so called *generalized measurements* using structured least squares methods, whilst Section 4 discusses how to obtain these generalized measurements from uniform or nonuniform sensor data. We present numerical simulation results in Section 5 and then conclude the paper in Section 6.

2. PROBLEM FORMULATION

We consider the inverse problems of physical fields governed by well-known PDEs. In particular, given access to spatiotemporal samples of a physical field we seek a unifying framework for recovering the unknown sources inducing the field. Consider the d-dimensional homogeneous and isotropic region $\Omega \subset \mathbb{R}^d$. Using the method of Green's functions, the field $u(\mathbf{x}, t)$ induced by some source distribution $f(\mathbf{x}, t)$, propagating through Ω can be written in the form:

$$u(\mathbf{x}, t) = (g * f)(\mathbf{x}, t), \tag{1}$$

where $g(\mathbf{x}, t)$ is the Green's function of the physical field. For some well known physical phenomena, we can obtain analytic expressions for the Green's function.

2.1. Examples of Physical Fields and their PDE Models

1. **Laplace's Equation:** is such that $\nabla^2 u(\mathbf{x}) = f(\mathbf{x})$. The Green's function for this PDE in 2D (i.e. $d = 2$) is:

$$g(\mathbf{x}) = \frac{1}{2\pi} \log(\|\mathbf{x}\|) . \tag{2}$$

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Whilst for $d = 3$ the Green's function becomes:

$$g(\mathbf{x}) = -\frac{1}{4\pi\|\mathbf{x}\|}. \quad (3)$$

2. **Diffusion Equation:** is $\frac{\partial}{\partial t}u(\mathbf{x},t)=\mu\nabla^2 u(\mathbf{x},t)+f(\mathbf{x},t)$, where μ is the diffusivity of the medium. The corresponding Green's function is,

$$g(\mathbf{x}, t) = \frac{1}{(4\pi\mu t)^{d/2}} e^{-\frac{\|\mathbf{x}\|^2}{4\mu t}} H(t), \quad (4)$$

where $d = \{2, 3\}$ and $H(t)$ is the unit step function.

For the Green's functions stated, a Sommerfeld radiation condition, i.e. a *quiescent condition* at an initial time $u(\mathbf{x}, t)\big|_{t=0} = \frac{\partial}{\partial t}u(\mathbf{x}, t)\big|_{t=0} = 0$ and a *convergence condition* at infinity $u(\mathbf{x}, t)\big|_{\|\mathbf{x}\|\to\infty} = \frac{\partial}{\partial x_1}u(\mathbf{x}, t)\Big|_{\|\mathbf{x}\|\to\infty} = \frac{\partial}{\partial x_2}u(\mathbf{x}, t)\Big|_{\|\mathbf{x}\|\to\infty} = 0$, is assumed. See, for example, [9] for the derivation of these expressions.

Now (1) implies that the entire field $u(\mathbf{x}, t)$ may be perfectly reconstructed provided the source distribution $f(\mathbf{x}, t)$ is known exactly. We can now precisely state the class of inverse problems considered in this paper:

Problem 1 *Let $\mathcal{S} = \{\mathbf{x}_n\}_{n=1}^N$ denote a network of N sensors, such that the n-th sensor situated at $\mathbf{x}_n$ collects samples $\varphi_{n,l} = u(\mathbf{x}_n, t_l)$ of the field u, at times t_l for $l = 0, 1, \ldots, L$. Given these spatiotemporal samples, and knowledge of the Green's function of the field, we intend to estimate the unknown source distribution $f(\mathbf{x}, t)$.*

3. SOURCE ESTIMATION FROM GENERALIZED MEASUREMENTS

Our proposed scheme for solving the inverse source problem can be split into two steps. The initial step involves estimating a sequence of *generalized measurements* of the form:

$$\mathcal{Q}(\mathbf{k}, r) = \langle f(\mathbf{x}, t), \Psi_{\mathbf{k}}(\mathbf{x})\Gamma_r(t)\rangle_{\mathbf{x},t}\,, \quad (5)$$

where $\Psi_{\mathbf{k}}(\mathbf{x})$ and $\Gamma_r(t)$, for each $\mathbf{k} \in \mathbb{Z}^2$, $r \in \mathbb{Z}$, are a family of properly chosen functions which we call *spatial* and *temporal sensing functions* respectively, due to their argument. In the second step, given a sequence (over $\mathbf{k} \in \mathbb{Z}^2$) of the generalized measurements, we show that structured least squares and its variations can be used to recover the unknown sources depending on our choice of sensing functions. The latter step will be the topic of this section. Hence, we first discuss how to choose the sensing functions below and consider how to obtain the generalized measurements from the sensor measurements in Section 4.

Herein, we focus on fields due to localized sources which is suitable when the sources of the field are many times smaller than the monitored region within which the field travels. A typical example in environmental monitoring is a plume source. We describe M such sources using

$$f(\mathbf{x}, t) = \sum_{m=1}^{M} c_m \delta(\mathbf{x} - \boldsymbol{\xi}_m, t - \tau_m), \quad (6)$$

where $c_m, \tau_m \in \mathbb{R}$ are the intensity and activation time of the m-th source respectively, situated at $\boldsymbol{\xi}_m = (\xi_{i,m})_{i=1}^d \in \mathbb{R}^d$, where d is the number of spatial dimensions. The problem of recovering such sources now becomes one of estimating all M triples $\{c_m, \tau_m, \boldsymbol{\xi}_m\}_{m=1}^M$.

Actually, under this source distribution, observe that the inner product (5) reduces to:

$$\mathcal{Q}(\mathbf{k}, r) = \sum_{m=1}^{M} c_m \Psi_{\mathbf{k}}(\boldsymbol{\xi}_m)\Gamma_r(\tau_m). \quad (7)$$

Thus, our task is now to choose $\Psi_{\mathbf{k}}(\mathbf{x})$ and $\Gamma_r(t)$ such that we are able to recover $\{c_m, \tau_m, \boldsymbol{\xi}_m\}_{m=1}^M$ from $\{\mathcal{Q}(\mathbf{k}, r)\}_{\mathbf{k},r}$. Our proposition is to choose $\Psi_{\mathbf{k}}(\mathbf{x})$ and $\Gamma_r(t)$ to be exponentials. This choice results in an algebraically coupled power-sum series, which can be solved efficiently using Prony's method and its variations [10, 11]. Explicitly,

1. **Sensing in Time and 2-D Space:** In this case $t \in \mathbb{R}_+$ and $\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2$. A valid temporal sensing function is $\Gamma_r(t) = e^{\mathrm{j}rt/T}$, where $T = t_L$ i.e. the instant at which the sensors measure the last sample of the field. Whereas for $\mathbf{k} \overset{\text{def}}{=} (k_1, k_2) \in \mathbb{Z}^2$, we choose the spatial sensing function $\Psi_{\mathbf{k}}(\mathbf{x}) = e^{\mathrm{j}k_1 x_1 + \mathrm{j}k_2 x_2}$. This choice turns (7) into:

$$\mathcal{Q}(\mathbf{k}, r) = \sum_{m=1}^{M} c_m e^{\mathrm{j}r\tau_m/T} e^{\mathrm{j}k_1\xi_{1,m} + \mathrm{j}k_2\xi_{2,m}}. \quad (8)$$

2. **Sensing in Time and 3-D Space:** In this case $\mathbf{x} = (x_1, x_2, x_3) \in \mathbb{R}^3$, $\Gamma_r(t) = e^{\mathrm{j}rt/T}$ as before, but $\Psi_{\mathbf{k}}(\mathbf{x}) = e^{k_1(x_1+\mathrm{j}x_2)+\mathrm{j}k_2 x_3}$. Given this choice,

$$\mathcal{Q}(\mathbf{k}, r) = \sum_{m=1}^{M} c_m e^{\mathrm{j}r\tau_m/T} e^{k_1(\xi_{1,m}+\mathrm{j}\xi_{2,m})+\mathrm{j}k_2\xi_{3,m}}. \quad (9)$$

We remark that $\Psi_{\mathbf{k}}(\mathbf{x}) = e^{\mathrm{j}k_1 x_1 + k_2(x_3+\mathrm{j}x_2)}$, $\Psi_{\mathbf{k}}(\mathbf{x}) = e^{k_1(\xi_{1,m}+\mathrm{j}\xi_{3,m})+\mathrm{j}k_2\xi_{2,m}}$, $\Psi_{\mathbf{k}}(\mathbf{x}) = e^{k_1(x_1+\mathrm{j}x_2)+k_2 x_3}$ (and so on) would also be valid choices.

Notice now that, for some fixed $r \neq 0$, in particular $r = 1$, (8) and (9) are of the form:

$$\mathcal{Q}(\mathbf{k}, 1) \overset{\text{def}}{=} \mathcal{Q}(k_1, k_2, 1) = \sum_{m=1}^{M} a_m u_m^{k_1} v_m^{k_2}.$$

This is a coupled Prony system, which can be solved using Algebraically Coupled Matrix Pencil (ACMP) method [12, 13] to find jointly $\{c_m, \tau_m, \boldsymbol{\xi}_m\}_{m=1}^M$ from $\{\mathcal{Q}(\mathbf{k}, 1)\}_{\mathbf{k}}$ with $k_1 = 0, 1, \ldots, K_1$ and $k_2 = 0, 1, \ldots, K_2$.

4. SPATIOTEMPORAL SENSING: FROM SENSOR DATA TO GENERALIZED MEASUREMENTS

Equation (1) implies that,

$$u(\mathbf{x}, t) = \int_{\mathbf{x}'\in\mathbb{R}^2}\int_{t'\in\mathbb{R}} g(\mathbf{x}', t') f(\mathbf{x}-\mathbf{x}', t-t')\,\mathrm{d}t'\mathrm{d}\mathbf{x}' \quad (10)$$

where $\mathbf{x}' = (x_1, x_2)$ and $\mathrm{d}\mathbf{x}' = \mathrm{d}x_1'\mathrm{d}x_2'$. Equivalently, we can rewrite the above equation as $u(\mathbf{x}, t) = \langle f(\mathbf{x}', t'), g(\mathbf{x} - \mathbf{x}', t - t')\rangle_{\mathbf{x}',t'}$. Thus, the measurement obtained by the n-th sensor (situated at $\mathbf{x}_n$) at some time instant $t = t_l$, is:

$$\varphi_{n,l} = u(\mathbf{x}_n, t_l) = \langle f(\mathbf{x}', t_l'), g(\mathbf{x}_n - \mathbf{x}', t_l - t')\rangle_{\mathbf{x}',t'}. \quad (11)$$

Let $\mathcal{N} = \{n\}_{n=1}^N$ be the index set of the sensor locations $\mathcal{S}$ and consider the weighted sum of the samples, $\{\varphi_{n,l}\}_{n,l}$ below:

$$\sum_{n\in\mathcal{N}}\sum_{l=0}^{L} w_{n,l}\varphi_{n,l} = \sum_{n\in\mathcal{N}}\sum_{l=0}^{L} w_{n,l}\,\langle f(\mathbf{x},t), g(\mathbf{x}_n-\mathbf{x}, t_l-t)\rangle_{\mathbf{x},t}$$
$$= \left\langle f(\mathbf{x},t), \sum_{n\in\mathcal{N}}\sum_{l=0}^{L} w_{n,l}\, g(\mathbf{x}_n-\mathbf{x}, t_l-t)\right\rangle_{\mathbf{x},t}, \quad (12)$$

where $w_{n,l} \in \mathbb{C}$ are some arbitrary weights we wish to compute. In particular, if we want to obtain (5) from (12), then it follows that the specific sequence of weights $\{w_{n,l}\}$ we desire here are those that reproduce $\Psi_{\mathbf{k}}(\mathbf{x})\Gamma_r(t)$ from $\{g(\mathbf{x}_n - \mathbf{x}, t_l - t)\}_{n,l}$. Mathematically, we desire the equality:

$$\sum_{n\in\mathcal{N}}\sum_{l=0}^{L} w_{n,l}\, g(\mathbf{x}_n - \mathbf{x}, t_l - t) = \Psi_{\mathbf{k}}(\mathbf{x})\Gamma_r(t), \quad (13)$$

where $\Gamma_r(t) = e^{-\mathrm{j}rt/T}$, whilst $\Psi_{\mathbf{k}}(\mathbf{x}) = e^{\mathrm{j}k_1x_1+\mathrm{j}k_2x_2}$ or $\Psi_{\mathbf{k}}(\mathbf{x}) = e^{k_1(x_1+\mathrm{j}x_2)+\mathrm{j}k_2x_3}$ according to whether the inverse source problem is 2D or 3D respectively. We can now discuss how to compute the desired coefficients.

4.1. Computing the coefficients $(w_{n,l})_{n\in\mathcal{N},l\in\mathbb{N}_0}$

For each member, i.e. $\Gamma_r(t)\Psi_{\mathbf{k}}(\mathbf{x})$, of the *spatiotemporal* sensing function (STSF) family we want to reconstruct, a different set of weights must be found. More explicitly, we note that the weights to be found actually depend on the indices $\mathbf{k}$ and r; in order to emphasize this dependence, we will henceforth use $w_{n,l}(\mathbf{k}, r)$ in place of $w_{n,l}$. To compute the desired coefficients $w_{n,l}(\mathbf{k}, r)$, we can leverage from certain results in modern sampling theory. We consider separately the cases of uniform and nonuniform spatial samples.

4.1.1. Uniform Sensor Placement: Approximate Strang-Fix

To reduce the notational load we develop the theory for the 2D case, noting that a 3D extension follows similarly. For uniform spatial sampling in $\mathbf{x} \in \mathbb{R}^2$, we assume access to the samples $\{\varphi_{n_1,n_2,l} = u(n_1\Delta_{x_1}, n_2\Delta_{x_2}, l\Delta_t)\}_{n_1,n_2,l}$, where $n_1 = 0, 1, 2, \ldots, N_1$, $n_2 = 1, 2, \ldots, N_2$ and $l = 0, 1, \ldots, L$. Whilst $\Delta_{x_1}, \Delta_{x_2}$ and Δ_t are understood to be the sampling intervals in each dimension.

Remark 1 *Note that $(n_1\Delta_{x_1}, n_2\Delta_{x_2})$ is a sensor location and the lexicographic ordering of $\{(n_1\Delta_{x_1}, n_2\Delta_{x_2})\}_{n_1,n_2} = \{\mathbf{x}_n\}_{n=1}^N$ gives the usual $n = 1, \ldots, N$, where $N = N_1N_2$.*

Consequently, we may rewrite (13) more clearly as:

$$\sum_{n_1,n_2,l} w_{n_1,n_2,l}(\mathbf{k}, r)\, g(n_1\Delta_{x_1}-x_1, n_2\Delta_{x_2}-x_2, l\Delta_t-t)$$
$$= \Psi_{k_1,k_2}(\mathbf{x})\Gamma_r(t). \quad (14)$$

Recall that $\Psi_{\mathbf{k}}(\mathbf{x}) = e^{\mathrm{j}k_1x_1+\mathrm{j}k_2x_2}$ and $\Gamma_r(t) = e^{\mathrm{j}rt/T}$ for the 2D case. Thus the resulting problem is to find the coefficients that reproduces exponentials (in space and time) using shifted versions of the Green's function.

Consider the typical exponential reproduction problem

$$\sum_{n\in\mathbb{Z}} w_n(k)\, g(x - n) = e^{\mathrm{j}\omega_k x} \quad (15)$$

for $k \in \mathbb{Z}$, commonly encountered in the finite rate of innovation (FRI) framework [11, 14]. The class of functions that satisfy (15) are known as exponential reproducing kernels. These class of functions satisfy the generalized Strang-Fix conditions [15]:

$$G(\omega_k) \neq 0 \text{ and } G(\omega_k + 2\pi\ell) = 0 \;\; \forall\ell \in \mathbb{Z}\backslash\{0\}, \quad (16)$$

where $G = \mathcal{F}(g)$ is the continuous Fourier transform of g. For physical fields which are of interest here, the kernel g corresponds to the Green's function of the field; whilst these will generally not satisfy the Strang-Fix condition (16), we still wish to approximately reproduce exponentials with them. Fortunately, we can apply the so called approximate Strang-Fix method introduced in [8], which relaxes the assumptions on g, such that we are now after the best set of coefficients that leads to approximate exponential reproduction given any kernel g. Mathematically we desire

$$\sum_{n\in\mathbb{Z}} w_n(k)\, g(x - n) \approx e^{\mathrm{j}\omega_k x}, \quad (17)$$

where g does not necessarily satisfy the Strang-Fix conditions. There are a few possible choices one may make for the "best" approximation coefficients (see [8] for details), but we focus on the constant least squares coefficients of the form:

$$w_n(k) = \frac{1}{G(\omega_k)} e^{\mathrm{j}\omega_k n}, \quad (18)$$

for their simplicity and accuracy. Given these coefficients, the approximation $\hat{\psi}_k(x)$ of the exponential $\psi_k(x) = e^{\mathrm{j}\omega_k x}$ is $\hat{\psi}_k(x) \overset{\text{def}}{=} \sum_{n\in\mathbb{Z}} w_n(k)g(x - n)$. Which becomes $\hat{\psi}_k(x) = e^{\mathrm{j}\omega_k x}\frac{1}{G(\omega_k)}\sum_{\ell\in\mathbb{Z}} G(\omega_k + 2\pi\ell)e^{\mathrm{j}2\pi\ell x}$ when we substitute in (18) and apply Poisson's summation formula. We can show that the error $\varepsilon(x) = \psi_k(x) - \hat{\psi}_k(x)$ for this approximation is:

$$\varepsilon(x) = e^{\mathrm{j}\omega_k x}\left(1 - \frac{1}{G(\omega_k)}\sum_{\ell\in\mathbb{Z}} G(\omega_k + 2\pi\ell)e^{\mathrm{j}2\pi\ell x}\right), \quad (19)$$

which will be small if $G(\omega_k + 2\pi\ell)$ decays quickly enough to zero as $|\ell|$ increases.

For our multidimensional setup, we can re-derive a similar (multidimensional) expression using the linearity of the multidimensional Fourier transform and the Poisson summation formula (for lattices). Doing this allows us to show that the desired coefficients for our exponential reproduction problem in (2D space and $r = 0$), i.e. (14), are given by:

$$w_{n_1,n_2,l}(\mathbf{k}, r) = \frac{1}{G(k_1, k_2, t_l - t)} e^{\mathrm{j}k_1n_1}e^{\mathrm{j}k_2n_2} \quad (20)$$

where $G(\omega_{x_1}, \omega_{x_2}, \omega_t)$ is defined to be the multi-dimensional Fourier transform of $g(\mathbf{x}, t)$:

$$G(\omega_{x_1}, \omega_{x_2}, t-t_l) = \int_{\mathbf{x}\in\mathbb{R}^2} g(\mathbf{x}, t-t_l)e^{-\mathrm{j}(\omega_{x_1}x_1+\omega_{x_2}x_2)}\mathrm{d}\mathbf{x}. \quad (21)$$

The weighted-sum of the sensor measurements, using these coefficients, gives the desired generalized measurements (8) in 2D (or (9) in 3D) from which we obtain the unknown source parameters as described in Section 3.

4.1.2. Non-uniform Sensor Placement

For non-uniformly placed sensors, it is not possible to find a closed expression for the desired weights $\{w_{n,l}(\mathbf{k}, r)\}_{n,l}$. However we can resort to formulating the following linear system to find $\{w_{n,l}(\mathbf{k}, r)\}_{n,l}$:

$$\begin{bmatrix} g(\mathbf{x}_1-\mathbf{x}_1', t_l-t_j) \cdots g(\mathbf{x}_N-\mathbf{x}_1', t_l-t_j) \\ g(\mathbf{x}_1-\mathbf{x}_2', t_l-t_j) \cdots g(\mathbf{x}_N-\mathbf{x}_2', t_l-t_j) \\ \vdots \quad \ddots \quad \vdots \\ g(\mathbf{x}_1-\mathbf{x}_I', t_l-t_j) \cdots g(\mathbf{x}_N-\mathbf{x}_I', t_l-t_j) \end{bmatrix} \begin{bmatrix} w_{1,l}(\mathbf{k}, r) \\ w_{2,l}(\mathbf{k}, r) \\ \vdots \\ w_{N,l}(\mathbf{k}, r) \end{bmatrix} = \begin{bmatrix} \Psi_{\mathbf{k}}(\mathbf{x}_1')\Gamma_r(t_j) \\ \Psi_{\mathbf{k}}(\mathbf{x}_2')\Gamma_r(t_j) \\ \vdots \\ \Psi_{\mathbf{k}}(\mathbf{x}_I')\Gamma_r(t_j) \end{bmatrix}$$

$$\Rightarrow \mathbf{G}_{l,j}\mathbf{w}_l(\mathbf{k}, r) = \mathbf{p}_j(\mathbf{k}, r). \qquad (22)$$

Moreover by stacking (22) for all l, j we can finally get

$$\begin{bmatrix} \mathbf{G}_{0,1}\, \mathbf{G}_{1,1} \cdots \mathbf{G}_{L,1} \\ \mathbf{G}_{0,2}\, \mathbf{G}_{1,2} \cdots \mathbf{G}_{L,2} \\ \vdots \qquad \vdots \\ \mathbf{G}_{0,J}\, \mathbf{G}_{1,J} \cdots \mathbf{G}_{L,J} \end{bmatrix} \begin{bmatrix} \mathbf{w}_0(\mathbf{k}, r) \\ \mathbf{w}_1(\mathbf{k}, r) \\ \vdots \\ \mathbf{w}_L(\mathbf{k}, r) \end{bmatrix} = \begin{bmatrix} \mathbf{p}_1(\mathbf{k}, r) \\ \mathbf{p}_2(\mathbf{k}, r) \\ \vdots \\ \mathbf{p}_J(\mathbf{k}, r) \end{bmatrix}$$

$$\mathbf{G}\mathbf{w}(\mathbf{k}, r) = \mathbf{p}(\mathbf{k}, r), \qquad (23)$$

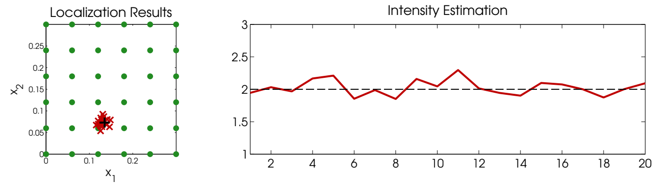
where $\mathbf{G} \in \mathbb{R}^{IJ \times N(L+1)}$, $\mathbf{w}(\mathbf{k}, r) \in \mathbb{R}^{N(L+1)}$ are the desired weights and $\mathbf{p}(\mathbf{k}, r) \in \mathbb{R}^{IJ}$ for each $\mathbf{k} \in \mathbb{R}^2$ and $r \in \mathbb{R}$. Consequently, in order to recover the desired field analysis coefficients, we would need to solve the system (23). In general, this system admits a least-squares solution if $IJ \geq N(L+1)$, where the observation matrix $\mathbf{G}$ can be constructed from the Green's function of the problem at hand (i.e. (2), (3), or (4)).

5. NUMERICAL SIMULATIONS AND RESULTS

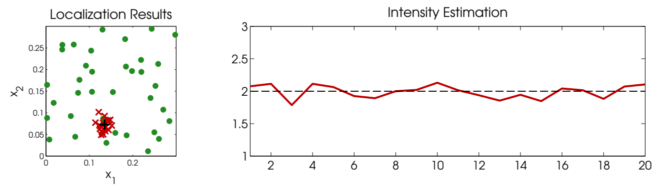
In this section, we present some numerical results to validate the proposed framework for solving PDE-driven inverse (source) problems. For illustrative purposes we present results for two underlying physical models for the sensor measurements, namely: Laplace's equation (in 2D and 3D) and the diffusion equation, for uniform and non-uniform spatial sampling cases. The sensor measurements are simulated numerically using Matlab with the sensors distributed over a square region in 2D (and equivalently a cubic region in 3D); the measurements are then corrupted by white Gaussian noise (SNR = 20dB) before applying our source estimation scheme. We perform 20 independent trials in each experiment, with each trial using a new noise realization, and a new arbitrary sensor placement for the nonuniform sampling experiments.

5.1. Laplace's Equation

The 2D (or 3D) Laplace field is obtained by evaluating the expression obtained when we substitute (2) (or (3)) and the point source distribution into (1). We summarize the estimation results for uniform sampling in Figure 1(a) and non-uniform sampling in Figure 1(b). In both cases the scatter plot shows the estimated source locations (red '×'), which are close to the true location (blue '+'); we also show one realization of the sensor locations (green '•'). In addition the intensity estimates, red curves as seen in rightmost plots, vary only marginally about the true intensity (black dashed line) with each independent trial.



(a) Uniform spatial sampling ($N = 25$)



(b) Non-uniform spatial sampling ($N = 36$)

Fig. 1. Single source recovery for 2D Laplace field from spatial samples. Here $K_1 = K_2 = 10$ for the spatial sensing function family $\{\Psi_{\mathbf{k}}(\mathbf{x}) = e^{\mathrm{j}k_1 x_1 + \mathrm{j}k_2 x_2}\}_{\mathbf{k}}$.

The estimation results for the 3D problem is summarized in Figure 2, where we can observe that all the source unknowns are recovered reliably for each independent trial.

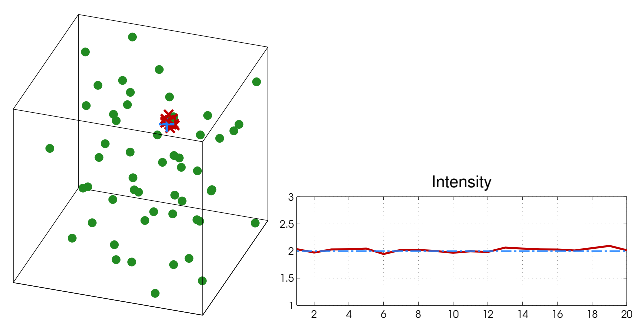
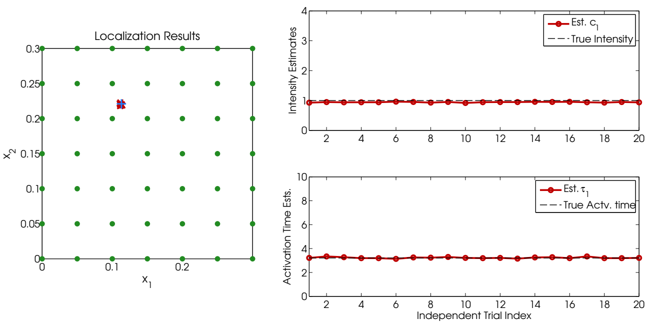


Fig. 2. Single point source recovery in 3D using samples obtained by $N = 57$ sensors with $K_1 = K_2 = 1$ for the spatial sensing function family $\{\Psi_{\mathbf{k}}(\mathbf{x}) = e^{k_1(x_1 + \mathrm{j}x_2) + \mathrm{j}k_2 x_3}\}_{\mathbf{k}}$.

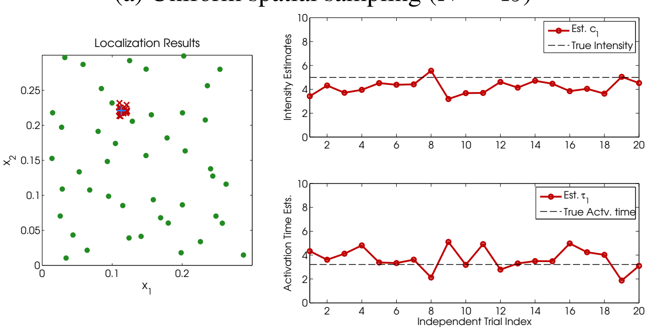
5.2. Diffusion Equation

The diffusion field is simulated by substituting (4) and the point source distribution (6) into (1). Given the sensor measurements we apply the proposed estimation algorithm and perform 20 independent trials for both the uniform and nonuniform sensor distribution. For this PDE we retrieve the coefficients $\{w_{n,l}(\mathbf{k}, 0)\}_{n,l}$, i.e. we set $r = 0$ and only reproduce spatially varying exponentials $\Psi_{\mathbf{k}}(\mathbf{x})$ at fixed time snapshots, this allows us to recover $\{c_1, \boldsymbol{\xi}_1\}$. Given these estimates the activation time τ_1 is then recovered by performing a line search. For multiple sources we can recover each source sequentially as described in [7], for example.

The results are summarized in Figure 3, as expected the unknown source location, intensity and activation times are recovered reliably. Although the estimates resulting from the nonuniform sampling case have a larger variance compared to uniform sampling estimates.



(a) Uniform spatial sampling ($N = 49$)



(b) Non-uniform spatial sampling ($N = 45$)

Fig. 3. Single source recovery from 2D diffusion field samples. Here $K_1 = K_2 = 10$ for the spatial sensing function family $\left\{\Psi_{\mathbf{k}}(\mathbf{x}) = e^{\mathrm{j}k_1 x_1 + \mathrm{j}k_2 x_2}\right\}_{\mathbf{k}}$ with $k_2 = \mathrm{j}k_1$.

6. CONCLUSION

In this paper we demonstrate how to solve the inverse source problem for a class of PDE-driven fields. Our proposed approach extends non-trivially results of modern sampling theory, allowing us to reduce the problem to solving a multidimensional structured least squares problem. Finally simulation results presented further corroborates the validity of the proposed framework.

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