Solving Inverse Source Problems for Sources with Arbitrary Shapes using Sensor Networks

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Abstract. Recently, the use of wireless sensor networks for environmental monitoring has been a topic of intensive research. The sensor nodes obtain spatiotemporal samples of physical fields over the region of interest. For most cases these fields are driven by well-known partial differential equations—the diffusion and wave equations for example—and this prior knowledge can be used to solve such *physics-driven* inverse source problems (ISPs). In this work, we demonstrate how to estimate the unknown source shape inducing the field by assuming that it can be described by a model having a finite number of unknown parameters.

1 Introduction

Several naturally occurring phenomena or signals obey certain well known physical laws. For instance, the propagation of heat through a medium is governed by the well-known heat (or diffusion) equation, the propagation of sound can be accurately described through the acoustic wave equation, Bloch's equation is at the heart of Magnetic Resonance Imaging (MRI) and so on. Moreover sensor networks have emerged as a useful tool for monitoring such phenomena, with the aim of inferring some underlying properties of the measured field. For example, distinguishing hot from cold spots for load-balancing in the monitoring of large server clusters, detecting factory leakages and acoustic source localization and so on. The problems can, in general, be posed as inverse source problems (ISPs).

Although a variety of approaches based on compressed sensing [1, 2], statistical methods [3], finite/boundary element methods [4] and more, have been proposed to solve this problem, these approaches tend to be PDE-specific and not easily generalized. Furthermore, it is usual to assume spatially localized sources which is valid in scenarios where the spatial support of the sources are orders of magnitude smaller than the monitored region. Hence this work will consider the ISP for non-localized sources; specifically, we focus on a particular class of sources that are localized in time but their spatial support can be described using a modified version of the finite rate of innovation (FRI) curve model introduced by Pan *et al* in [5]. For these source types, we show how the recent approach of [6] can be generalized in order to estimate them.

The rest of this paper is organized as follows. We state the problem in Section 2. In Section 3, we present the proposed source recovery method, and then conclude the paper in Section 4.

*This work is supported by the European Research Council (ERC) starting investigator award Nr. 277800 (RecoSamp).

2 The physics-driven inverse source problem

In this paper we are concerned with physical fields $u(\mathbf{x}, t)$ governed by linear PDEs of the form:

$$\mathcal{D}u(\mathbf{x}, t) = f(\mathbf{x}, t), \quad (\mathbf{x}, t) \in \Omega \times \mathbb{R}_+, \tag{1}$$

where $u(\mathbf{x}, t)$ denotes the field induced by the source distribution $f(\mathbf{x}, t)$, $\mathcal{D} = \sum_{\boldsymbol{i} \leq I \in \mathbb{N}} a_{\boldsymbol{i}} \nabla^{\boldsymbol{i}}$, where $\boldsymbol{i} = (i_1, i_2, i_3)$ is a three-dimensional multi-index variable with $|\boldsymbol{i}| = i_1 + i_2 + i_3$, and $\nabla^{\boldsymbol{i}} = \frac{\partial^{|\boldsymbol{i}|}}{\partial x_1^{i_1} \partial x_2^{i_2} \partial t^{i_3}}$. For example, the diffusion equation [7] has $\mathcal{D} = \frac{\partial}{\partial t} - \mu \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} \right)$.

Then our problem of interest is stated as follows:

Problem 1 (Physics-driven inverse source problem (ISP))**.** *Let* $\varphi_{\boldsymbol{n},l} = u(\mathbf{x}_{\boldsymbol{n}}, t_l)$*, denote the uniform spatiotemporal samples of the field* $u(\mathbf{x}, t)$ *induced by a source distribution* $f(\mathbf{x}, t)$*, at discrete spatial locations* $\mathbf{x}_{\boldsymbol{n}} = \boldsymbol{\Delta}_{\mathbf{x}} \in \Omega$ *and* $\boldsymbol{n} = (n_1, n_2)$*, for* $n_1 = 0, 1, \ldots, N_1$*,* $n_2 = 0, 1, \ldots, N_2$ *and time instants* $t_l \in \mathbb{R}_+$ *with* $l = 0, 1, \ldots, L$*. Then the physics-driven ISP is to recover the unknown source distribution* f *from the samples* $\{\varphi_{\boldsymbol{n},l}\}_{\boldsymbol{n},l}$*.*

In its present form this problem is ill-posed [8], however it can be regularized by assuming a structure on $f(\mathbf{x}, t)$. We consider non-localized sources can be described parametrically using a finite number of parameters per unit space and time, i.e. sources (signals) with a *finite rate of innovation* (FRI) [9, 10].

Definition 1 (FRI source with arbitrary shape)**.** *This source* $f(\mathbf{x}, t)$ *is given by the integral:*

$$f(\mathbf{x}, t) = \frac{c\delta(t - \tau)}{2\pi \mathrm{j}} \int_F \frac{1}{\bar{z} - (x_1 + \mathrm{j}x_2)} \mathrm{d}\bar{z}, \tag{2}$$

where $\bar{z} = \bar{x}_1 + \mathrm{j}\bar{x}_2$ *and* F *is the contour of the integral described through the zeros of some mask function* (3)*. Specifically that the spatial distribution of the source is assumed to coincide with the contour,*

$$F: \sum_{l_1=-L_1}^{L_1} \sum_{l_2=-L_2}^{L_2} \alpha_{l_1,l_2} e^{\mathrm{j}\frac{2\pi l_1}{L_{x_1}} x_1 + \mathrm{j}\frac{2\pi l_2}{L_{x_2}} x_2} = \sum_{\boldsymbol{l}} \boldsymbol{\alpha}_{\boldsymbol{l}} e^{\mathrm{j}2\pi\left(\boldsymbol{l} \cdot \frac{\mathbf{x}}{\mathbf{L}_{\mathbf{x}}}\right)} = 0, \tag{3}$$

where the second equality results from utilizing a multi-index notation, with $\boldsymbol{l} = (l_1, l_2)$*,* $\mathbf{L}_{\mathbf{x}} = (L_{x_1}, L_{x_2})$ *and* $\frac{\mathbf{x}}{\mathbf{L}_{\mathbf{x}}} \stackrel{def}{=} \left(\frac{x_1}{L_{x_1}}, \frac{x_2}{L_{x_2}} \right)$*.*

Our choice is motivated by the fact that this model is potentially very rich, as it gives rise to very diverse topologies when we impose $\alpha_{l_1,l_2} = \alpha^*_{-l_1,-l_2}$ [5].

3 Methodology

In what follows, we demonstrate how to recover the unknown source parameters for the FRI source distribution of interest. Our proposed approach is two step:

(a) For the source distribution we demonstrate that there exists a proper vector sequence of generalized measurements [11]:

$$\mathcal{R}(\boldsymbol{k}) = \langle f(\mathbf{x},t), \Psi_{\boldsymbol{k}}(\mathbf{x})\Gamma(t)\rangle, \tag{4}$$

that can be annihilated using multidimensional variations of Prony's method [9, 12], in order to recover the unknown source parameters. Here $\Psi_{\boldsymbol{k}}(\mathbf{x})$ and $\Gamma(t)$ are chosen to be imaginary exponential functions[1].

(b) Subsequently, we show how to compute the desired sequence of generalized measurements through linear combinations of the sensor data.

3.1 Source recovery from the generalized measurements

The approach proposed is based on the framework of [6], wherein the focus is on point sources. However, in this work we describe how the framework of [6] can be generalized, albeit non-trivially, to FRI sources with arbitrary shapes.

3.1.1 Recovering the arbitrary FRI source shape

We now demonstrate that for the FRI source distribution (2), it is possible to recover the unknown source parameters c, τ and $\{\boldsymbol{\alpha_l}\}_{\boldsymbol{l}}$ by annihilating a properly scaled version of the sequence of generalised measurements $\mathcal{R}(\boldsymbol{k})$. To see this, we substitute (2) into (4), and choose $\Psi_{\boldsymbol{k}}(\mathbf{x}) = e^{-2\pi(\mathrm{j}k_1x_1/L_{x_1}+\mathrm{j}k_2x_2/L_{x_2})}$ and $\Gamma(t) = e^{-\mathrm{j}2\pi t/T}$ to be the imaginary exponentials. Then it follows that

$$\begin{aligned}\mathcal{R}(\boldsymbol{k}) &= ce^{-\mathrm{j}2\pi\tau/T}\langle \mathbb{1}_F(\mathbf{x}), e^{-2\pi\mathrm{j}\left(\frac{k_1x_1}{L_{x_1}}+\frac{k_2x_2}{L_{x_2}}\right)}\rangle \\ &= ce^{-\mathrm{j}2\pi\tau/T}\widehat{\mathbb{1}_F}(\boldsymbol{\omega}_{\mathbf{x}})\Big|_{\boldsymbol{\omega}_{\mathbf{x}}=2\pi\boldsymbol{k}/\mathbf{L}_{\mathbf{x}}},\end{aligned} \tag{5}$$

where $\widehat{\mathbb{1}_F}(\boldsymbol{\omega}_{\mathbf{x}})$ is the multidimensional Fourier transform of the spatially varying component of the source distribution.

Proposition 1. *Let $\mathcal{R}'(\boldsymbol{k}) = \mathrm{j}(k_1+\mathrm{j}k_2)\mathcal{R}(\boldsymbol{k})$, for the arbitrary FRI source model* (2)*, then it follows that for any $\tilde{\boldsymbol{l}}$:*

$$\sum_{\boldsymbol{l}=(-L_1,-L_2)}^{(L_1,L_2)} \boldsymbol{\alpha_l}\mathcal{R}'\left(\frac{2\pi(\tilde{l_1}-l_1)}{L_{x_1}}, \frac{2\pi(\tilde{l_2}-l_2)}{L_{x_2}}\right) = \sum_{\boldsymbol{l}} \boldsymbol{\alpha_l}\mathcal{R}'\left(\frac{2\pi(\tilde{\boldsymbol{l}}-\boldsymbol{l})}{\mathbf{L}_{\mathbf{x}}}\right) = 0. \tag{6}$$

Proof. Consider the multidimensional Fourier transform

$$\begin{aligned}\widehat{R}(\boldsymbol{\omega}_{\mathbf{x}},\omega_t) &= \langle f(\mathbf{x},t), e^{-\mathrm{j}\boldsymbol{\omega}_{\mathbf{x}}\cdot\mathbf{x}}e^{-\mathrm{j}\omega_t t}\rangle \\ &= ce^{-\omega_t\tau}\langle \mathbb{1}_F(\mathbf{x}), e^{-\mathrm{j}\boldsymbol{\omega}_{\mathbf{x}}\cdot\mathbf{x}}\rangle = ce^{-\mathrm{j}\omega_t\tau}\widehat{\mathbb{1}_F}(\boldsymbol{\omega}_{\mathbf{x}}),\end{aligned} \tag{7}$$

of the source distribution (2). Due to (5), we can immediately realize that $\mathcal{R}(\boldsymbol{k}) \stackrel{\text{def}}{=} \widehat{R}(\boldsymbol{\omega}_{\mathbf{x}},\omega_t)\Big|_{(\boldsymbol{\omega}_{\mathbf{x}},\omega_t)=2\pi(\boldsymbol{k}/\mathbf{L}_{\mathbf{x}},1/T)}$.

Next consider the Wirtinger spatial derivative $R'(\mathbf{x},t) = \frac{\partial R}{\partial x_1}+\mathrm{j}\frac{\partial R}{\partial x_2}$ of $R(\mathbf{x},t)$, whose Fourier transform can be related to that of $R(\mathbf{x},t)$ as follows:

$$\widehat{R'}(\boldsymbol{\omega}_{\mathbf{x}},\omega_t) = \mathrm{j}(\omega_{x_1}+\mathrm{j}\omega_{x_2})\widehat{R}(\boldsymbol{\omega}_{\mathbf{x}},\omega_t) = \mathrm{j}(\omega_{x_1}+\mathrm{j}\omega_{x_2})ce^{-\mathrm{j}\omega_t\tau}\widehat{\mathbb{1}_F}(\boldsymbol{\omega}_{\mathbf{x}}),$$

[1]Note that this is in contrast to the to [11] wherein the choice for $\Psi_{\boldsymbol{k}}(\mathbf{x})$ is chosen to be an analytic family of complex exponentials.

the second equality here follows by substituting (7). Furthermore using $\widehat{\mathbb{1}_F}(\boldsymbol{\omega}_{\mathbf{x}}) = \frac{1}{\omega_{x_1}+\mathrm{j}\omega_{x_2}} \int_F e^{-\mathrm{j}\boldsymbol{\omega}_{\mathbf{x}}\cdot\mathbf{x}} \mathrm{d}z$, where $z = x_1 + \mathrm{j}x_2$, gives

$$\widehat{R'}(\boldsymbol{\omega}_{\mathbf{x}}, \omega_t) = \mathrm{j}ce^{-\mathrm{j}\omega_t\tau} \int_F e^{-\mathrm{j}\boldsymbol{\omega}_{\mathbf{x}}\cdot\mathbf{x}} \mathrm{d}z. \tag{8}$$

Substitute (8) into the LHS of (6) and set $\boldsymbol{\omega}_{\mathbf{x}} = 2\pi\frac{\tilde{\boldsymbol{l}}-\boldsymbol{l}}{\mathbf{L}_{\mathbf{x}}}$, as well as $\omega_t = 2\pi/T$; then pass the summation inside the integral to get: $\widehat{R'}(2\pi(\tilde{\boldsymbol{l}} - \boldsymbol{l})/\mathbf{L}_{\mathbf{x}}, 2\pi/T) = \mathcal{R}'(\boldsymbol{k}) = \mathrm{j}ce^{-\mathrm{j}2\pi\tau/T} \int_F e^{-\mathrm{j}\frac{2\pi\tilde{\boldsymbol{l}}\cdot\mathbf{x}}{\mathbf{L}_{\mathbf{x}}}} \sum_{\boldsymbol{l}} \boldsymbol{\alpha}_{\boldsymbol{l}} e^{\mathrm{j}\frac{2\pi\boldsymbol{l}\cdot\mathbf{x}}{\mathbf{L}_{\mathbf{x}}}} \mathrm{d}z = 0$, as required. □

As a consequence, given the generalized measurements $\mathcal{R}(\boldsymbol{k})$ at a few discrete frequencies $\boldsymbol{\omega}_{\mathbf{x}} = \frac{2\pi(\tilde{\boldsymbol{l}}-\boldsymbol{l})}{\mathbf{L}_{\mathbf{x}}}$, we may write the discrete convolution (6) as follows

$$\mathbf{H}\mathbf{a} = \mathbf{0},$$

where $\mathbf{H}$ is a block-circulant convolution matrix formed using

$$\mathcal{R}'(\boldsymbol{k}) = \mathrm{j}(k_1 + \mathrm{j}k_2)\mathcal{R}(\boldsymbol{k}) \tag{9}$$

and $\mathbf{a}$ is a column vector obtained through a lexicographic ordering of $\{\boldsymbol{\alpha}_{\boldsymbol{l}}\}$. It can be shown, see [5] for example, that such a linear system admits a least-square solution provided that to recover all $\{\boldsymbol{\alpha}_{\boldsymbol{l}}\}_{\boldsymbol{l}=-(L_1,L_2)}^{(L_1,L_2)}$ the number of samples of $\{\mathcal{R}(\boldsymbol{k})\}$ with $\boldsymbol{k} = \frac{2\pi(\tilde{\boldsymbol{l}}-\boldsymbol{l})}{\mathbf{L}_{\mathbf{x}}}$ exceeds $(4L_1 + 1)(4L_2 + 1)$. The key message here is that we can obtain $\mathcal{R}'(\boldsymbol{k})$ from the generalized measurements $\mathcal{R}(\boldsymbol{k})$ by carefully choosing the sensing function family.

Finally, the activation time of the source τ can be estimated directly from $\mathcal{R}(0)$ using $\tau = -\frac{T}{2\pi}\arg(\mathcal{R}(0))$.

3.2 From sensor data to the generalized measurements

In what follows we discuss how to compute these generalized measurements from the spatiotemporal samples of the field, by taking proper linearly weighted combinations of the sensor data $\{\varphi_{\boldsymbol{n},l}\}_{\boldsymbol{n},l}$. Specifically, we require the weights $\{w_{\boldsymbol{n},l}(\boldsymbol{k})\}$ such that:

$$\sum_{\boldsymbol{n}} \sum_{l} w_{\boldsymbol{n},l}(\boldsymbol{k})\varphi_{\boldsymbol{n},l} \equiv \mathcal{R}(\boldsymbol{k}) = \langle f(\mathbf{x}, t), \Psi_{\boldsymbol{k}}(\mathbf{x})\Gamma(t)\rangle \tag{10}$$

where the weights are to be found. To this end we recall that, according to the method of fundamental solutions [7], $u(\mathbf{x}, t) = (f * g)(\mathbf{x}, t)$ for linear PDEs of the form (1). Hence, by writing $u(\mathbf{x}', t') = \langle f(\mathbf{x}, t), g(\mathbf{x}' - \mathbf{x}, t' - t)\rangle$ we see that spatiotemporal samples: $\varphi_{\boldsymbol{n},l} = u(\mathbf{x}_n, t_l) = \langle f(\mathbf{x}, t), g(\mathbf{x}_n - \mathbf{x}, t_l - t)\rangle$. Thus we notice that the weighted sum coincides with the generalized measurements if,

$$\sum_{\boldsymbol{n}} \sum_{l} w_{\boldsymbol{n},l}(\boldsymbol{k}) g(\mathbf{x}_n - \mathbf{x}, t_l - t) = \Psi_{\boldsymbol{k}}(\mathbf{x})\Gamma(t), \tag{11}$$

where for our choices of $\Psi_{\boldsymbol{k}}(\mathbf{x})$ and $\Gamma(t)$, the problem of finding the $w_{\boldsymbol{n},l}(\boldsymbol{k})$ in (11) is known as the exponential reproduction problem in the sampling and approximation theory literature [13, 14]. Therein the 1-D exponential reproduction, i.e. $\sum_n w_n(k)g(x-n) = e^{-\mathrm{j}\beta_k x}$, is possible iff the generalized Strang-Fix conditions [15], i.e. $\widehat{g}(\beta_k) \neq 0$ and $\widehat{g}(\beta_k + 2\pi\ell) = 0 \, \forall \ell \in \mathbb{Z}\backslash\{0\}$, holds true for all. Furthermore, it can be shown that under these conditions,

$$w_n(k) = \left(\widehat{g}(\beta_k)\right)^{-1} e^{-\mathrm{j}\beta_k n}. \tag{12}$$

However for Green's functions that do not satisfy the generalized Strang-Fix conditions, it can be shown that (12) still provides a good approximation as long as $\widehat{g}$ decays quickly. These arguments can be extended to the multidimensional case, alternatively one could formulate a linear system that admits a least-square solution [16].

Once we have the correct sequence of weights $\{w_{\boldsymbol{n},l}(\boldsymbol{k})\}$ have been found, then the desired generalized measurements can be obtained by evaluating the sum in (10) and then $\mathcal{R}'(\boldsymbol{k})$ can be computed using (9).

3.3 Distributed source recovery

The proposed framework is readily distributed as follows. When the sensors are deployed, they simply need to precompute their respective weights $\{w_{n,l}(\boldsymbol{k})\}_l$; this is made possible under the assumption that the sensors in the network know the overall topology. Once this is done, they can start to monitor the region of interest. Furthermore, in order to compute the weighted sum (10), each sensor in the network can compute its local generalized measurement $\sum_l w_{n,l}\varphi_{n,l}$. Consequently any gossip scheme for average consensus (see [17, 18, 19] and references therein) can then be utilized to aggregate these local generalized measurements, in such a way that every sensor can converge to the desired $\{\mathcal{R}(\boldsymbol{k})\}_{\boldsymbol{k}}$. Upon convergence any sensor can then estimate the unknown source parameters as described in Section 3.1.

4 Conclusion

In this paper we showed how to solve the inverse source problem for phenomena driven by linear PDEs. In our approach we leverage from results in modern sampling theory to reduce the ISP relating to localized and arbitrary non-localized sources into a problem that can be solved using multidimensional extensions of Prony's method.

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