Solving Inverse Source Problems for linear PDEs using Sparse Sensor Measurements

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***Abstract*—Many physical phenomena across several applications can be described by partial differential equations (PDEs). In these applications, sensors collect sparse samples of the resulting phenomena with the aim of detecting its cause/source, using some intelligent data analysis tools on the samples. These problems are commonly referred to as inverse source problems. This work presents a novel framework for solving such inverse source problem for linear PDEs by drawing from certain recent results in modern sampling theory. Under the new framework, we study the well-known diffusion PDE and present numerical results that highlight the validity and robustness of the approach.**

I. Introduction

Partial differential equations (PDEs) are a ubiquitous mathematical tool used to describe a wide spectrum of physical phenomena across many application domains. In majority of these applications, a collection of sensor nodes are deployed over a region of interest to collect sparse spatiotemporal measurements of the phenomena (signal), such as: neuronal currents in brain source imaging, thermal fields in server clusters/multi-core processors and the concentration of toxic substances/releases into the environment. Hence, given these sensor measurements of the phenomena, one might be interested in locating cortical avalanches for epilepsy management/diagnosis [1], localizing hot spots in servers/processors for load balancing [2] or plume sources/leakages [3], [4] respectively. This class of estimation problems are more commonly referred to as *inverse source problems* (ISPs).

For these ISPs, there are generally two central tasks: data collection and data processing. Although the fabrication of sophisticated sensors for data collection in several application scenarios is well-established, the art of processing and analyzing the data remains an area of intensive research. In this work, we will focus on the latter, specifically we introduce a new framework for designing reliable sampling and reconstruction schemes that exploit the underlying PDE model of the measured signals to solve the ISP. In the example of plumes/leakages, the emitted substance comprises microscopic particles that propagate, over time, from the source thereby spreading throughout the monitored region. Therein, the underlying field is the concentration of the released substance over space and time, whilst the corresponding PDE is the *diffusion equation* [5], [6].

Other prime examples of commonly occurring PDEs include the wave and Poisson PDEs [7]. The wave equation is pervasive in the modeling of signals prevalent in speech recognition [8], acoustic tomography [9], speech and sound enhancement [10], sound/wave source localization [10], [11], whilst the Poisson equation is of huge importance in biomedical engineering applications, such as the localization of sources of neuronal activity (also known as brain source imaging (BSI)) from electroencephalographic (EEG) signals [1], [12], [13].

Over the years, several techniques have been proposed to solve these ISPs. For example, [14] proposes a technique that utilizes the structure of euclidean distance matrices to solve the ISP for acoustic wave fields. Other proposed schemes have been largely based on compressed sensing [15]–[17], statistical estimation techniques [18]–[20] and finite/boundary element methods [21]. In most of these techniques however the underlying PDE model isn't exploited, and even when it is, the resulting schemes are usually *ad hoc* in that they do not generalize easily to other PDEs. To this end, we propose a universal approach that still exploits the underlying PDE model of the signal obtained. Our approach shows that we can solve the ISP—for the class of linear PDEs—given access to a sequence of generalized measurements. We further demonstrate that these generalized measurements can be obtained, with relative ease, by evaluating a linearly weighted sum of the sensor data. Finally, we show how to compute the proper weights of this sum; interestingly, we will see that it they directly depend on the Green's function of the underlying PDE.

The rest of the paper is organized as follows. The class of ISP of interest is defined in Section II. We then present the main results of our framework and summarize the proposed approach in Section III. Next we investigate the validity and performance of the resulting scheme, in the diffusion field setting, in Section IV and finally the paper is concluded in Section V.

II. The Inverse Source Problem

Let $\Omega \subset \mathbb{R}^d$ be a d-dimensional homogeneous and isotropic medium in which the PDE-driven signal $u(\mathbf{x}, t)$ is embedded, then the continuous field (signal) $u(\mathbf{x}, t)$ induced by an unknown source distribution $f(\mathbf{x}, t)$, compactly supported on $\Omega \times \mathbb{R}_+$ can be written in the form:

$$u(\mathbf{x}, t) = (g * f)(\mathbf{x}, t), \tag{1}$$

where $g(\mathbf{x}, t)$ is the Green's function of the PDE model of the induced field. For some well known physical phenomena, such as diffusion and wave fields, analytic expressions exist for the Green's function $g(\mathbf{x}, t)$, otherwise they can be computed numerically. Our aim is to reconstruct $f(\mathbf{x},t)$ from sparse measurements, say $\varphi_{n,l} = u(\mathbf{x}_n, t_l)$, of the field $u(\mathbf{x},t)$ obtained at sensor locations $\{\mathbf{x}_n\}_{n=1}^N$ and at sampling instants $\{t_l\}_{l=0}^L$. However due to the possibly infinite dimensionality of $f(\mathbf{x},t)$ this problem as it stands is ill-posed, nonetheless we can regularize it by imposing a structure on $f(\mathbf{x},t)$. For instance we may assume that the sources are sparse in both space and time and so can be accurately represented by the source parametrization:

$$f(\mathbf{x},t) = \sum_{m=1}^{M} c_m \delta(\mathbf{x} - \boldsymbol{\xi}_m, t - \tau_m), \tag{2}$$

where $c_m, \tau_m \in \mathbb{R}$ are the intensity and activation time of the m-th source respectively, situated at $\boldsymbol{\xi}_m = (\xi_{i,m})_{i=1}^d \in \mathbb{R}^d$. Under this assumption and with knowledge of $g(\mathbf{x},t)$ for the underlying PDE, our aim is to recover the unknowns $\{c_m, \tau_m, \boldsymbol{\xi}_m\}_{m=1}^M$ from the sparse field samples $\{\varphi_{n,l}\}_{n,l}$.

Although the results presented in this paper focus on recovering instantaneous point source distribution (with $d = 2$), we remark that it is possible to extend them to other equally interesting source distributions, as well as fields in three spatial dimensions (i.e. $d = 3$). In addition, certain multidimensional results are stated without proof for brevity. Detailed proofs appear in our paper [22].

III. Solving the Inverse Problem

In [4, Proposition 1], the authors demonstrated that the unknown source parameters $\{c_m, \tau_m, \boldsymbol{\xi}_m\}_{m=1}^M$ can be recovered simultaneously from a sequence of, so called, *generalized measurement*s $\{\mathcal{R}(k)\}_{k=0}^K$, using such algebraic techniques as Prony's method [23]. In particular, the generalized measurements we seek are assumed to be of the form:

$$\begin{aligned}\mathcal{R}(k) &= \langle f(\mathbf{x},t), \Psi_k(\mathbf{x})\Gamma(t)\rangle \\ &\stackrel{\text{def}}{=} \int_{\mathbf{x}}\int_t f(\mathbf{x},t)\Psi_k(\mathbf{x})\Gamma(t)\mathrm{d}V\mathrm{d}t,\end{aligned} \tag{3}$$

where $\Psi_k(\mathbf{x})$ and $\Gamma(t)$ are functions chosen such that $\Psi_k(\mathbf{x}) = e^{-k(x_1+\mathrm{j}x_2)}$, $\Gamma(t) = e^{-\mathrm{j}t/T}$ and $k = 0, 1, \ldots, K$, whilst $\mathrm{d}V = \mathrm{d}x_1\mathrm{d}x_2$. Furthermore, notice that substituting the source parametrization (2) into the right hand side of (3) gives the following sum of exponentials:

$$\mathcal{R}(k) = \sum_{m=1}^{M} c_m e^{-\mathrm{j}\tau_m/T} e^{-k(\xi_{1,m}+\mathrm{j}\xi_{2,m})}. \tag{4}$$

The problem of solving (4) to find $\{(c_m, \tau_m, \boldsymbol{\xi}_m)\}_{m=1}^M$ is well-studied in the signal processing community; specifically, Prony's method can be used to recover the unknowns, given $\{\mathcal{R}(k)\}_{k=0}^K$ where $K \geq 2M - 1$. See for example, [4], [24]. Therefore, given such a sequence $\mathcal{R}(k)$ applying Prony's method to it solves the ISP of interest.

Consequently, our focus now turns to obtaining the sequence $\mathcal{R}(k)$. The approach of [4] achieves this by evaluating numerically a particular family of integrals that depend on the continuous field measurements. In contrast to that approach however, the current one is based on computing $\mathcal{R}(k)$ by using a linearly weighted sum of the sparse sensor measurements, i.e.:

$$\mathcal{R}(k) = \sum_{n,l} w_{n,l}(k)\varphi_{n,l}. \tag{5}$$

In fact, it can be shown that:

$$\begin{aligned}&\mathcal{R}(k) = \sum_{n,l} w_{n,l}(k)\varphi_{n,l} \\ \Longleftrightarrow &\sum_{n,l} w_{n,l}(k) g(\mathbf{x} - \mathbf{x}_n, t - t_l) = \Psi_k(\mathbf{x})\Gamma(t),\end{aligned}$$

where $\{w_{n,l}(k)\}_{n,l}$ are weights to be found, for each $k = 0, 1, \ldots, K$. In fact, this equivalence possesses an interesting interpretation, namely: the sequence of weights $\{w_{n,l}(k)\}$ that reproduces the family of functions $\{\Psi_k(\mathbf{x})\Gamma(t)\}_k$, using linear translates of $g(\mathbf{x},t)$ coincide with those that map the sensor measurements to the desired generalized measurements via a linearly weighted sum (5).

One straightforward way to proceed, to find the weights, is by discretizing $\mathbf{x}$ and t in

$$\sum_{n,l} w_{n,l}(k) g(\mathbf{x} - \mathbf{x}_n, t - t_l) = e^{-k(x_1+\mathrm{j}x_2)} e^{-\mathrm{j}t/T}, \tag{6}$$

to form an overdetermined system of linear equations; where (6) above results from choosing $\Psi_k(\mathbf{x}) = e^{-k(x_1+\mathrm{j}x_2)}$ and $\Gamma(t) = e^{-\mathrm{j}t/T}$. The resulting linear system can be shown to admit a least-squares solution for the desired weights $\{w_{n,l}(k)\}$. However, we remark that the resulting system may sometimes be ill-conditioned, hence inverting it can lead to numerical instabilities. An interesting alternative is to then consider (6) and notice that finding the desired $w_{n,l}(k)$ is an exponential reproduction problem – on that is rather common in the finite rate of innovation and function approximation theory literature [25], [26]. Consequently, by leveraging techniques from these domains we can show that the desired weights in this sum are directly related to the Green's function of the underlying field, when the sensors are arranged uniformly.

To be precise, let us consider the case when the spatiotemporal samples at $(\mathbf{x}_n, t_l)$ are on a uniform grid (i.e. $\mathbf{x}_n = (n_1\Delta_{x_1}, n_2\Delta_{x_2})$ and $t_l = l\Delta_t$ for any $n_1, n_2, l \in \mathbb{N}$), then the corresponding exponential reproduction problem is:

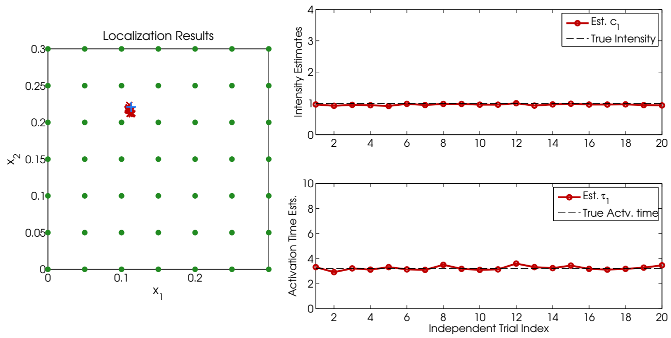
$$\begin{aligned}\sum_{n_1,n_2,l} &w_{n_1,n_2,l}(k) g(x_1 - n_1\Delta_{x_1}, x_2 - n_2\Delta_{x_2}, t - l\Delta_t) \\ &= e^{-\mathrm{j}(\omega_{1,k}x_1+\omega_{2,k}x_2+\omega_t t)},\end{aligned} \tag{7}$$

where $\omega_{1,k} = -\mathrm{j}k$, $\omega_{2,k} = k$ and $\omega_t = 1/T$. Moreover let us define $G(\omega_{x_1}, \omega_{x_2}, \omega_t)$ to be the multi-dimensional Fourier transform of $g(\mathbf{x},t)$, given by

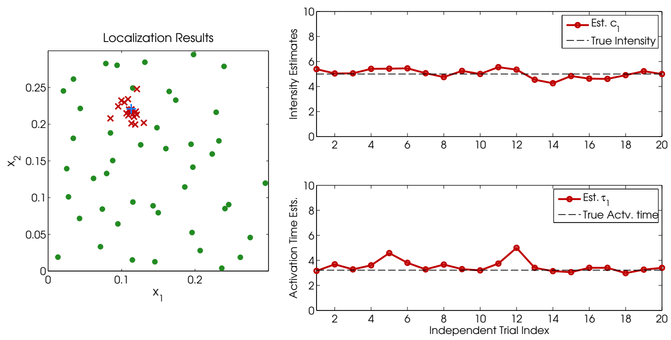
$$G(\omega_{x_1}, \omega_{x_2}, \omega_t) = \int_{t\in\mathbb{R}}\int_{\mathbf{x}\in\mathbb{R}^2} g(\mathbf{x},t) e^{-\mathrm{j}(\omega_{x_1}x_1+\omega_{x_2}x_2+\omega_t t)}\mathrm{d}V\mathrm{d}t. \tag{8}$$

Then it can be shown that, if the generator $g(\mathbf{x},t)$ (i.e. the Green's function of the underlying PDE) satisfies the generalized Strang-Fix conditions [26], [27]:

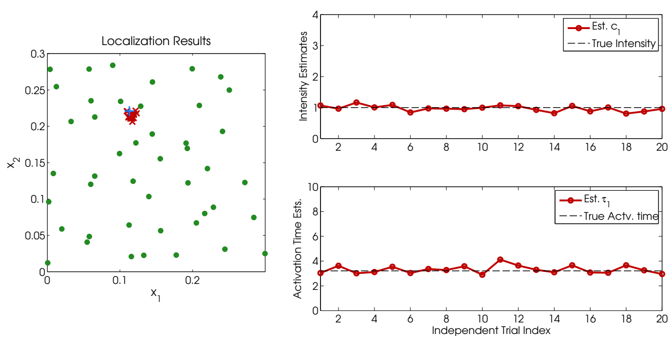
$$\begin{aligned}&G(\omega_{k,1}, \omega_{k,2}, \omega_t) \neq 0 \text{ and} \\ &G(\omega_{k,1}+2\pi\ell_1, \omega_{k,2}+2\pi\ell_2, \omega_t+2\pi\ell_3) = 0 \; \forall \ell_i \in \mathbb{Z}\setminus\{0\}\end{aligned} \tag{9}$$



(a) Uniform sampling using coefficients (10).



(b) Non-uniform sampling (linear systems approach).



(c) Non-uniform sampling (with interpolation).

Fig. 1. Single diffusion source estimation given noisy sensor data sampled at 1Hz for T = 20s (SNR = 10dB). Spatial sensing function is $\{\Psi_k(\mathbf{x})\}_{k=0}^{K}$ with $K = 1$. In each subfigure, the scatterplot shows the sensor locations (green •), as well as, the true (blue +) and estimated (red ×) source locations, whilst the line plots show the source intensities (top) and activation times (bottom).

for $i = 1, 2, 3$, the exact exponential reproducing weights are

$$w_{n_1,n_2,l}(k) = \frac{\Delta_{x_1}\Delta_{x_2}\Delta_t e^{-k(n_1\Delta_{x_1}+\mathrm{j}n_2\Delta_{x_2})}e^{-\mathrm{j}l\Delta_t/T}}{G(\mathrm{j}k,-k,-1/T)}. \quad (10)$$

Hence given access to uniform samples of a physical field driven by a linear PDE with constant coefficients, our proposed approach for solving the ISP can be summarized as follows. First, compute the multidimensional Fourier transform $G(\omega_{x_1},\omega_{x_2},\omega_t)$ of the field using (8). Second, given $G(\omega_{x_1},\omega_{x_2},\omega_t)$, we evaluate $w_{n_1,n_2,l}(k)$ using (10) for $k = 0,\dots,K$, $n_1 = 0,\dots,N_1$, $n_2 = 0,\dots,N_2$ and $l = 0,1\dots,L$. Third, with the sequence of weights $\{w_{n_1,n_2,l}(k)\}$ and the samples $\{\varphi_{n,l}\}_{n,l}$, we can retrieve the desired generalized

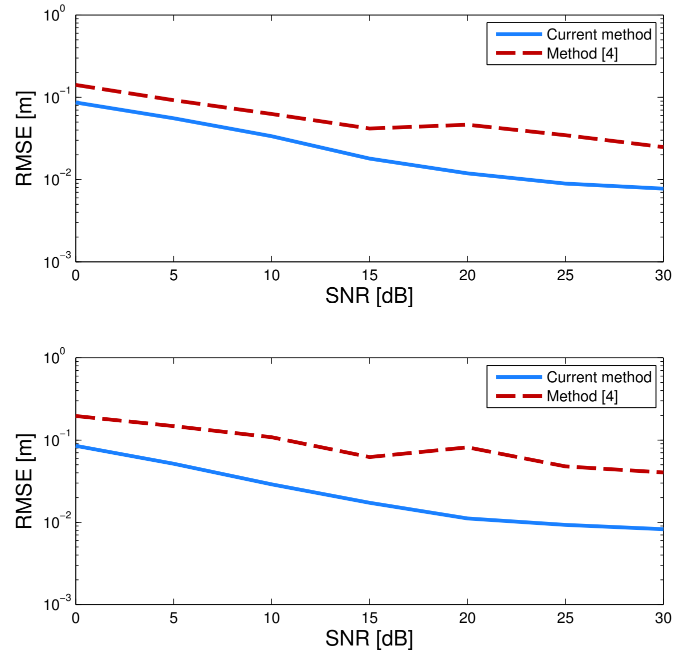


Fig. 2. Effect of measurement noise (SNR) on the single diffusion source localization accuracy. The single source field is sampled with a uniform array of 49 sensors at 1Hz for T = 20s, at varying noise levels. We use $K = 5$ for the current approach and compare it to the approach of [4] with $K = 5$ and $r = 0$. Given the estimated location $\hat{\boldsymbol{\xi}} = (\hat{\xi}_1, \hat{\xi}_2)$, the top plot shows the RMSE of $\hat{\xi}_{1,1}$, whilst the bottom one gives the RMSE of $\hat{\xi}_{2,1}$.

measurements $\mathcal{R}(k)$ by using (5). Finally, we then apply Prony's method to annihilate $\{\mathcal{R}(k)\}_{k=0}^{K}$ to recover the unknown source parameters $\{(c_m, \tau_m, \boldsymbol{\xi}_m)\}_{m=1}^{M}$ simultaneously given that $K \geq 2M - 1$.

In the case of non-uniform sensor placement, we may resort to discretizing the expression (6) in order to compute the weights provided the obtained linear system is well-conditioned. An alternative, and more effective approach as we will see in the simulation results, is to first of all interpolate the field using a spline and then resample it uniformly such that the coefficients (10) above can be used.

In addition, for PDEs with a Green's function that does not satisfy the Strang-Fix conditions (9), we can demonstrate that the coefficients (10) are approximately able to reproduce the desired exponentials, providing that the Fourier transform $G(\omega_{k,1},\omega_{k,2},\omega_t)$ has a sufficiently fast decay rate.

IV. Numerical Simulations

We present results for the 2D diffusion field, which has $g(\mathbf{x},t) = \frac{1}{4\pi\mu t} e^{-\frac{\|\mathbf{x}\|^2}{4\mu t}} H(t)$ where $H(t)$ is the unit step function, in Figure 1. The field samples are simulated numerically in MATLAB and then corrupted with additive white Gaussian noise and estimation results for all three approaches of computing $\{w_{n,l}(k)\}$ discussed in Section III are presented.

A. Single source diffusion field

We first show in Figure 1 that the proposed approach is able solve the ISP for the single source diffusion field setting,

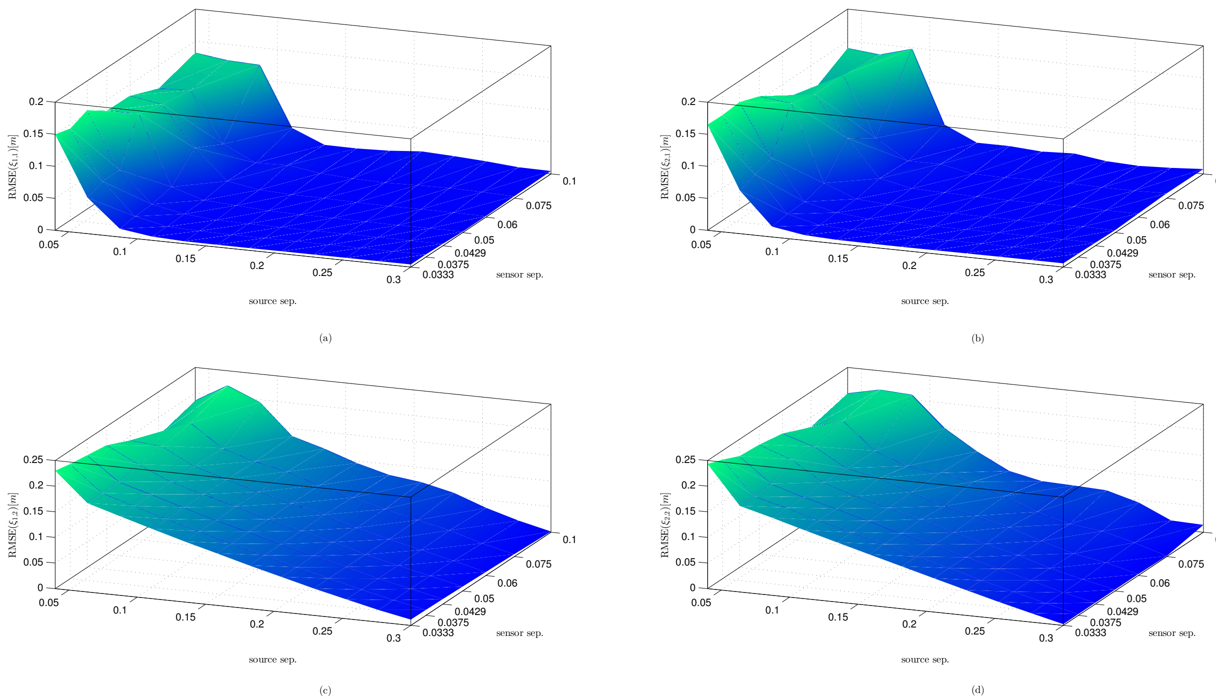


Fig. 3. **Effects of sensor density and source separation on the performance of the localization algorithm.** The field induced by two sources is sampled at 1Hz for $T = 20s$ with varying sensor density and source separation. For each value sensor separation and source separation the obtained noisy measurements, with SNR= 20dB, are used to estimate the locations of the unknown diffusion sources. The RMSE of the estimates obtained using $K = 5$ over 1000 independent trials are shown: i.e. in (a) RMSE of $\hat{\xi}_{1,1}$, in (b) RMSE of $\hat{\xi}_{2,1}$, in (c) RMSE of $\hat{\xi}_{1,2}$ and in (d) RMSE of $\hat{\xi}_{2,2}$.

in both the uniform and non-uniform spatial sampling regimes even in the presence of noise. The obtained results demonstrate that uniform sampling out-performs non-uniform sampling, since the variance of its estimates about the true value is much smaller. In addition, with access to non-uniform samples, our results suggest that it is better to interpolate on a uniform grid and then use the coefficients (10) instead of formulating and solving a linear system.

Furthermore, in Figure 2, we compare the noise resilience of the present approach against that of [4]. To achieve this result, we perform 5000 independent trials at each SNR level and then compute the RMSE for each coordinate of the estimated location $\hat{\boldsymbol{\xi}} = (\xi_1, \xi_2)$. The obtained results suggest that, in this setting, the estimation accuracy of the present approach is higher; moreover, the performance of both algorithms improve with increasing SNR.

B. Effects of sensor and source separation on the location estimates

In this experiment, we consider the diffusion field induced by two point sources with $c_1 = c_2 = 1$, $\tau_1 = \tau_2 = 1$ and fix the location of the first source $\boldsymbol{\xi}_1 = (\xi_{1,1}, \xi_{2,1})$ whilst the location of the second source $\boldsymbol{\xi}_2 = (\xi_{1,2}, \xi_{2,2})$ is allowed to vary such that the separation between them $S_{\text{source}} = \|\boldsymbol{\xi}_2 - \boldsymbol{\xi}_1\| \in \{0.04 + \frac{0.26i}{11}\}_{i=0}^{11}$. Furthermore, the sensor density also varies such that the uniform spatial sampling interval $\boldsymbol{\Delta}_{\mathbf{x}} = (\Delta_{x_1}, \Delta_{x_2})$ where $\Delta_{x_1} = \Delta_{x_2} \in \{0.0333, 0.0375, 0.0429, 0.05, 0.06, 0.075, 0.01\}$.[1] The field measurements, sampled at 1Hz for $T = 20$ seconds by the sensor network, are assumed to be noisy with fixed SNR= 20dB. Consequently, for each fixed value of S_{source} and $\boldsymbol{\Delta}_{\mathbf{x}}$ we recover the estimates $\hat{\boldsymbol{\xi}}_1$ and $\hat{\boldsymbol{\xi}}_2$ of the true source locations $\boldsymbol{\xi}_1$ and $\boldsymbol{\xi}_2$, respectively, using $K = 4$. The pairing $(\hat{\boldsymbol{\xi}}_m, \boldsymbol{\xi}_m)$ of the true value and its estimate is chosen to minimize the overall error (with respect to the Euclidean distance). The RMSE for each of the estimates is computed and provided in Figure 3, using 1000 independent trials of the experiment.

Observe in Figure 3 that, in line with expectation, the performance of our estimation algorithm improves as the sensor density and separation between the two sources increases. In particular, the RMSE of the estimates for the first source—i.e. RMSE($\hat{\boldsymbol{\xi}}_1$) as shown in Figure 3(a) and (b)—decreases when the sensor density increases. This is a consequence of the reduction in the approximation error obtained in the exponential reproduction step, as $\boldsymbol{\Delta}_{\mathbf{x}}$ decreases. Furthermore, the effect of the source separation on the localization performance becomes more noticeable as the sensor spacing decreases. For instance, when $\boldsymbol{\Delta}_{\mathbf{x}} = (0.05, 0.05)$ we notice a a gradual but steady decrease in the RMSE of $\hat{\boldsymbol{\xi}}_1$ in Figure 3(a) and (b). This improvement in estimation performance is even higher

[1]This yields uniform 2−D sensor arrays of size $\{10 \times 10, 9 \times 9, \ldots, 4 \times 4\}$ respectively.

for the first source $\hat{\boldsymbol{\xi}}_1$ compared to the second $\hat{\boldsymbol{\xi}}_2$.

V. Conclusion

We have demonstrated how to solve the inverse source problem for a class of PDE-driven phenomena given sparse observations of the phenomena. In our approach we extend non-trivially results of modern sampling theory, turning the problem to one of fitting an exponential with a certain prototype function specific to the PDE of interest (i.e. its Green's function). The coefficients that produces this fitting are exactly the weights that when applied to the sparse samples gives a sequence that is governed by a specific sum of exponentials. The solution of this system then coincides with the desired source parameters. Finally we have also presented simulation results to reinforce our approach.

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