RECONSTRUCTING DIFFUSION FIELDS SAMPLED WITH A NETWORK OF ARBITRARILY DISTRIBUTED SENSORS

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ABSTRACT

Sensor networks are becoming increasingly prevalent for monitoring physical phenomena of interest. For such wireless sensor network applications, knowledge of node location is important. Although a uniform sensor distribution is common in the literature, it is normally difficult to achieve in reality. Thus we propose a robust algorithm for reconstructing two-dimensional diffusion fields, sampled with a network of arbitrarily placed sensors. The two-step method proposed here is based on source parameter estimation: in the first step, by properly combining the field sensed through well-chosen test functions, we show how Prony's method can reveal locations and intensities of the sources inducing the field. The second step then uses a modification of the Cauchy-Schwarz inequality to estimate the activation time in the single source field. We combine these steps to give a multi-source field estimation algorithm and carry out extensive numerical simulations to evaluate its performance.

Index Terms— Spatio-temporal sampling, sensor networks, diffusion process, reciprocity gap, Prony's method

1. INTRODUCTION

The diffusion equation is the underlying model for numerous biological and physical phenomena such as, temperature variation in fluids, disease epidemic dynamics, nuclear and bio-chemical substance releases. Indeed the use of sensor networks to obtain spatio-temporal samples of such physical fields is common. However, the space-time dimensions of these diffusion processes are generally inhomogeneous, thus regular multidimensional sampling theory [1] no longer applies. Consequently, a robust and efficient solution to this sampling and reconstruction problem will strongly impact several real-life applications such as, pollution detection [2], environmental monitoring [3] and energy efficiency monitoring in large data center clusters [4].

In this paper, we demonstrate that the diffusion field sampling and reconstruction problem can be reformulated as a parametric source estimation problem. Indeed works in the area have focused mainly on source localization. For example, Lu and Vetterli proposed two methods for source localization, namely spatial super-resolution [5] and an adaptive sampling scheme [6]. A localization method based on L_1 constrained optimization is introduced in [7]. Ranieri *et al* proposed a compressed sensing approach [8], whilst Auffray *et al* proposed a method based on the reciprocity gap [9]. Ranieri and Vetterli [10] have also considered uniform spatial sampling and reconstruction using classical interpolation techniques. Moreover, Rostami *et al* [11] solved the reconstruction problem using diffusive compressive sensing (DCS), whereas Lu *et al* proposed a method to fully estimate a single source field by solving a set of linear equations [12]. Unfortunately some of these state of the art techniques are either: unstable in the presence of noise, unable to fully reconstruct the entire field, or require uniform spatial sampling, which is often difficult to achieve in practice [13].

Therefore, we propose a simple, noise robust algorithm that properly operates on arbitrary spatiotemporal samples of the field—obtained by an arbitrary network of sensors—in order to fully infer the diffusion field. The method proposed herein is two-step: *(i)* relying on the use of a family of proper analytic test functions to sense the field, we reformulate the source location and intensity estimation problem in the traditional finite rate of innovation setting [14] which can be solved using Prony's method; a similar technique is applied in [15] for static fields governed by *Poisson's equation*, then, *(ii)* the single source activation time is retrieved by performing a simple linear search. Furthermore, we propose the use of a damped exponential as the test function for increased stability in the localization step. We also provide simulation results to test and validate the proposed algorithm.

The paper is organized as follows. Section 2 formally presents the sampling and reconstruction problem in the source estimation setting. In Section 3, we use Green's second identity for multiple source intensity and location retrieval, along with the linear search method for single source activation time estimation. This section is concluded by combining these solutions to give a single source estimation algorithm which is then generalized to a multiple source field. Numerical simulations are given in Section 4 and concluding remarks in Section 5.

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2. RECONSTRUCTION PROBLEM FORMULATION

We consider the problem of reconstructing two-dimensional diffusion fields. Specifically, we focus on the case where the spatiotemporal samples of the field are obtained by a network of randomly deployed sensors. Denote by $u(\mathbf{x}, t)$ the diffusion field at location $\mathbf{x} \in \mathbb{R}^2$ and time t, induced by some unknown source distribution $f(\mathbf{x}, t)$ within a two-dimensional region Ω. In such a setting the field will propagate Ω according to the diffusion equation,

$$\frac{\partial}{\partial t} u(\mathbf{x}, t) = \mu \nabla^2 u(\mathbf{x}, t) + f(\mathbf{x}, t), \tag{1}$$

where μ is the diffusivity of the medium through which the field propagates. Moreover, from the theory of Green's functions this PDE has solution:

$$u(\mathbf{x}, t) = (g * f)(\mathbf{x}, t), \tag{2}$$

where $g(\mathbf{x}, t) = \frac{1}{4\pi\mu t} e^{-\frac{\|\mathbf{x}\|^2}{4\mu t}} H(t)$ is the Green's function of the two-dimensional diffusion field, and $H(t)$ is the unit step function. In fact Equation (2) implies that the entire field $u(\mathbf{x}, t)$ can be perfectly reconstructed provided the source distribution $f(\mathbf{x}, t)$ is known exactly. Therefore, this paper will concentrate on estimating the source distribution f given discrete measurements of the field. In particular, we will restrict our discussion to fields induced by M sources localized in both space and time, such a distribution is characterised by:

$$f(\mathbf{x}, t) = \sum_{m=1}^{M} c_m \delta(\mathbf{x} - \boldsymbol{\xi}_m, t - t_m), \tag{3}$$

where $c_m, t_m \in \mathbb{R}$ are the intensity and activation time of the m-th source respectively and $\boldsymbol{\xi}_m \in \Omega$ is the source location.

In this context, the field reconstruction problem is equivalent to estimating the parameters $\{c_m, \boldsymbol{\xi}_m, t_m : m = 1, \ldots, M\}$ using arbitrary spatiotemporal samples of the field u. For clarity, the problem can be stated as follows;

$\mathcal{P}$: Given spatiotemporal samples $\varphi_n(t_l) = u(\mathbf{x}_n, t_l)$, of the field u, at times t_l for $l = 0, 1, \ldots, L$ and at arbitrary spatial locations $\mathbf{x}_n \in \Omega$ for $n = 1, 2, \ldots, N$, we intend to estimate $\{c_m, \boldsymbol{\xi}_m, t_m : m = 1, \ldots, M\}$ from $\{\varphi_n(t_l) : n = 1, \ldots, N;\ l = 0, \ldots, L\}$.

In the following section, it is shown that finding the intensities and locations (in the problem $\mathcal{P}$) can be mapped to the typical finite rate of innovation (FRI) [14] setting. In addition for $M = 1$, we also show how the activation time may be retrieved using a simple *line search* algorithm.

3. DIFFUSION SOURCE ESTIMATION

In this section we use Green's second identity to relate the field measurements within the domain Ω to the locations and intensities of the sources inside Ω. This relation yields a Vandermonde system, which can be solved under certain conditions. Next we propose a method for estimating the activation time for the single source case, given its location and intensity.

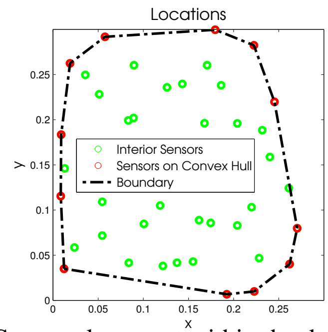


Fig. 1. Sensor placement within the domain Ω.

3.1. Multi-Source Localization and Intensity Estimation

We begin by relating the field measurements in Ω to the source parameters. Let ψ and u be twice differentiable functions in Ω, then Green's second identity relates the boundary integral and the integral over the bounded region as follows:

$$\oint_{\partial\Omega} (\psi \nabla u - u \nabla \psi) \cdot \hat{\mathbf{n}}_{\partial\Omega}\, \mathrm{d}S = \int_{\Omega} \psi \nabla^2 u - u \nabla^2 \psi \,\mathrm{d}V, \tag{4}$$

where $\hat{\mathbf{n}}_{\partial\Omega}$ is the outward pointing unit normal vector to the boundary $\partial\Omega$ of Ω. Moreover, if ψ satisfies $\frac{\partial \psi}{\partial t} + \mu \nabla^2 \psi = 0$ in Ω, and $u(\mathbf{x}, t)$ satisfies (1), then Equation (4) is such that $\oint_{\partial\Omega} (\psi \nabla u - u \nabla \psi) \cdot \hat{\mathbf{n}}_{\partial\Omega}\, \mathrm{d}S = \frac{1}{\mu} \int_{\Omega} \psi \left(\frac{\partial u}{\partial t} - f\right) + u \frac{\partial \psi}{\partial t} \,\mathrm{d}V = \frac{1}{\mu} \int_{\Omega} \frac{\partial}{\partial t}(u\psi) - \psi f \,\mathrm{d}V$. Furthermore, multiplying through by μ and time-integrating over $t \in [0, T]$ yields:

$$\begin{aligned} \int_{\Omega} (\psi u)(\mathbf{x}, T)\, \mathrm{d}V - \mu \oint_{\partial\Omega} (\psi \nabla U - U \nabla \psi) \cdot \hat{\mathbf{n}}_{\partial\Omega}\, \mathrm{d}S \\ = \int_0^T \int_{\Omega} \psi f \,\mathrm{d}V \mathrm{d}t, \end{aligned} \tag{5}$$

where $U(\mathbf{x}) = \int_0^T u(\mathbf{x}, t) \mathrm{d}t$. For convenience, we will denote the left-hand side of Equation (5) by $\mathcal{R}(\psi)$. Hence,

$$\mathcal{R}(\psi) = \int_0^T \int_{\Omega} \psi f \,\mathrm{d}V \mathrm{d}t. \tag{6}$$

Setting $\psi \to \Psi_k(\mathbf{x}) = e^{-k(x_1 + j x_2)}$, where $k \in \mathbb{Z}$ and given the instantaneous source parameterisation of Equation (3), Equation (6) results in the following Vandermonde system:

$$\mathcal{R}(k) = \sum_{m=1}^{M} c_m e^{-k(\xi_{1,m} + j\xi_{2,m})},\ k = 0, 1, \ldots, K \tag{7}$$

where $\mathcal{R}(k) = \mathcal{R}(\Psi_k)$. Such a system is well studied and can be solved using Prony's method [14, 16] provided $K \geq 2M - 1$. The choice of Ψ_k here is important for numerical stability of the Vandermonde system, hence we choose the damped complex exponential specifically for this reason.

Notice that the sequence $\mathcal{R}(k)$ is estimated by properly combining measurements $\{\varphi_n(t_l)\}$ of the field, through the use of sensing functions Ψ_k. Specifically in this 2D setup, a surface and boundary integral both need to be estimated. For the surface integral, we retrieve the Delaunay triangulation of all the sensor locations to obtain a set $\{\Delta_i\}_{i=1}^{I}$ of I triangular elements such that $\cup_{i=1}^{I}\Delta_i = \Omega$, and $\Delta_i \cap \Delta_j = \varnothing$ for $i \neq j$. Then according to Georg [17], $\int_\Omega h(\mathbf{x})\mathrm{d}V \approx \sum_i \frac{1}{3}\sum_{j=1}^{3} h(\mathbf{v}_{ij})\mathrm{Area}(\Delta_i)$, where $\mathbf{v}_{ij}$ are the vertices of Δ_i. To estimate the line integral, measurements obtained by the convex hull sensors are combined, in a similar manner, using standard quadrature methods [18] with a piecewise linear approximation. Consequently we are able to obtain an approximation for the sequence $\{\mathcal{R}(k)\}_{k=0}^{K}$. Then provided $K \geq 2M-1$, Prony's method can be used to reveal the source locations and concentrations by annihilating the sequence $\{\mathcal{R}(k)\}$. Since this sequence also captures information on the number of active sources over a measurement horizon, it is possible to exploit this property in the multi-source estimation case. In particular, by examining the rank of the Hankel matrix constructed from $\{\mathcal{R}(k)\}$ we can accurately infer the number of sources inducing the field.

3.2. Single Source Activation Time Retrieval

Let $M = 1$ such that the 2-D field of interest is induced by a single source having intensity c, location ξ and activation time τ. We propose a simple *line search* algorithm to estimate τ, provided ξ and c have already been retrieved using Equation (7). Consider the samples $\{\varphi_n(t_l) : l = 0, \dots, L\}$ collected by the n-th sensor (located at $\mathbf{x}_n$); the aim is to choose a $\hat{\tau}$ that produces a reconstructed sequence $\hat{\varphi}_n(t_l) = \hat{u}(\mathbf{x}_n, t_l) = \frac{c}{4\pi\mu(t_l-\hat{\tau})} e^{-\frac{\|\mathbf{x}_n-\xi\|^2}{4\mu(t_l-\hat{\tau})}} H(t-\hat{\tau})$ that is most similar to the measured sequence $\{\varphi_n(t_l) : l = 0, \dots, L\}$. To this end, we vary $\hat{\tau}$ between $(0, T]$ and reconstruct the sequence $\hat{\varphi}_n(t_l)$ for each value of $\hat{\tau}$. Comparing the normalized inner product between the reconstructed sequence and the measured sequence, we choose the $\hat{\tau}$ that maximizes this normalized inner product – a modification of the Cauchy-Schwarz inequality for vectors.

3.3. Single Source Estimation Algorithm

The single source estimation algorithm from sensor network measurements is presented in Algorithm 1. The suggested approach is two-step, in the first step the sensor measurements are used collaboratively to infer the source's location and intensity. In the second step, a selection of the nearest sensors to the estimated source are each independently used to estimate the activation time of the source; their average is taken to give an improved estimate of the activation time.

3.4. Multi-Source Estimation Algorithm

Algorithm 1 is easily extended to the multiple source case provided the diffusion sources have distinct activation times; such that the sampling interval is small enough to resolve the activation of two consecutive sources. The modification is based on finding a time interval over which a single source is active—by examining the rank of the Hankel matrix constructed from the sequence $\mathcal{R}(k)$, for example—estimating the source and then removing its contribution to the field measurements, to give the *adjusted measurements*. This process is then repeated on the adjusted measurements until it falls below a predefined threshold.

Algorithm 1 Single Diffusion Source Estimation

Require: $\{\varphi_n(t_l)\}$, sensor locations $\mathbf{x}_n$, sampling interval ΔT

Retrieve the *convex hull* of the set of points $\mathbf{x}_n$, these sensors define the boundary $\partial\Omega$ in Equation (5).
Initialize $K \geq 1$ and set window length $T = \alpha\Delta T$ (where $\alpha \in \mathbb{Z}, \alpha >> 1$).
Estimate sequence $\{\mathcal{R}(k) : k = 0, \dots, K\}$ for $t \in [0, T]$.
Annihilate the sequence $\{\mathcal{R}(k) : k = 0, 1, \dots, K\}$ to find concentration-location pair $(\sigma, \boldsymbol{\xi})$. For multiple pairs $(\sigma_i, \boldsymbol{\xi}_i)$, select the pair with largest σ_i.
Select the $\beta \in \mathbb{N}$ nearest sensors to $\boldsymbol{\xi}$. For each of the β sensors, retrieve $\hat{\tau}_1, \dots, \hat{\tau}_\beta$ as described in Section 3.2.
Then $c \leftarrow \sigma$, $\tau \leftarrow$average$\{\hat{\tau}_1, \hat{\tau}_2, \dots, \hat{\tau}_\beta\}$.
Return concentration c, location $\boldsymbol{\xi}$ and activation time τ.

4. SIMULATIONS AND RESULTS

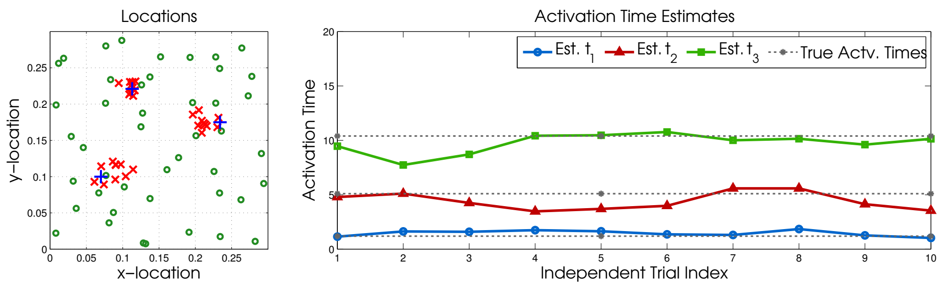
The 2-D diffusion field is simulated numerically on MATLAB using Equation (2) and samples of the field collected by sensors randomly deployed over a square region. In the results presented, each trial utilizes both a new arbitrary placement of sensors and realization of white Gaussian noise (with specified SNR).

4.1. Source Estimation Results

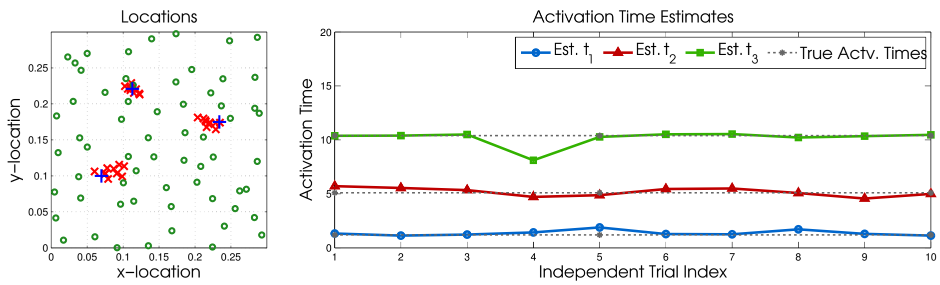
Trial	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
Intensity	1.21	0.81	1.07	0.94	1.00	0.91	0.99	1.20	0.94	1.02

Table 1. Corresponding estimated concentrations.

In the single source estimation results summarized in Figure 2 and table 1 the algorithm successfully recovers the source parameters with good accuracy even in the presence of noise, given only a few sensors. Furthermore, Figure 3 shows the performance of the algorithm in the multi-source case. In Figure 3(a) recovery is attempted given a few sensors, estimation accuracy is satisfactory but can be improved by increasing the number of sensors as shown in Figure 3(b). In fact, increasing the number of sensors improves the robustness of the algorithm to sensor noise. As we will discuss in Section 4.2, this claim if further reinforced by the results given in Figure 4.



(a) 45 Randomly distributed sensors.



(b) 63 Randomly distributed sensors.

Fig. 3. Estimation of $M = 3$ diffusion sources using randomly distributed sensors in 20dB of measurement noise. Intensities $c_1 = c_2 = c_3 = 1$; locations $\boldsymbol{\xi}_1 = (0.113, 0.221), \boldsymbol{\xi}_2 = (0.234, 0.175), \boldsymbol{\xi}_3 = (0.070, 0.100)$; and activation times $t_1 = 1.2s, t_2 = 5.1s, t_3 = 10.4s$. Field is sampled for $T_{end} = 15$seconds at a frequency $\frac{1}{\Delta T} = 1Hz$ and $K = 5$ i.e. $k = 0, 1, \ldots, 5$ for the test function family $\Psi_k(\mathbf{x}) = e^{-k(x_1+jx_2)}$. The scatter-plot shows the true source locations (blue '$+$'), the estimated locations (red '$\times$') and one realization of the sensor distribution (green '$\circ$').

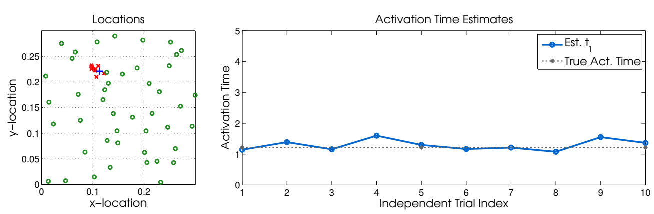


Fig. 2. Single $(M = 1)$ diffusion source estimation using 45 randomly distributed sensors in 15dB of measurement noise. Intensity $c_1 = 1$, location $\boldsymbol{\xi}_1 = (0.113, 0.221)$ and activation time $t_1 = 1.213s$. Field is sampled for $T_{end} = 10$seconds at a frequency $\frac{1}{\Delta T} = 1Hz$ and $K = 3$ i.e. $k = 0, 1, \ldots, 3$ for the test function family $\Psi_k(\mathbf{x}) = e^{-k(x_1+jx_2)}$. The scatter-plot shows the true source location (blue '$+$'), the estimated locations (red '$\times$') and one realization of the sensor distribution (green '$\circ$').

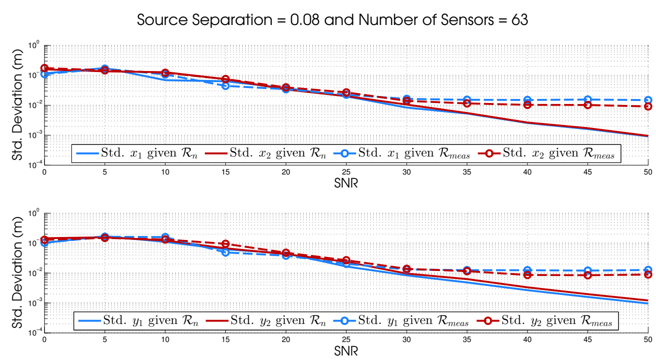


Fig. 4. Standard deviation of location estimates (500 trials) for simultaneous double $(M = 2)$ source localization using 63 randomly distributed sensors. $c_1 = c_2 = 1$; $\boldsymbol{\xi}_1 = (0.23, 0.15), \boldsymbol{\xi}_2 = (0.15, 0.15)$; and $t_1 = t_2 = 1.2s$. $T_{end} = 10$seconds, $\frac{1}{\Delta T} = 1Hz$ and $K = 5$.

4.2. Approximation Errors: Discretization of Integrals

The results provided in this section aim to demonstrate that the effects of sensor noise outweigh the "errors" from approximating the integrals in (5) given discrete measurements of the field. To achieve this, we compare the localization results from applying Prony's method to two sequences which we denote as $\{\mathcal{R}_{meas}(k)\}$ and $\{\mathcal{R}_n(k)\}$ for $k = 0, 1, \ldots, K$. $\{\mathcal{R}_{meas}(k)\}$ is constructed from noisy sensor measurements and represents the real discretized case where the integrals in (5) are approximated by weighted sums of the field. Conversely, $\{\mathcal{R}_n(k)\}$ is obtained by adding an equivalent noise process to the exact power-sum series, and thus represents the non-discretized case where we are able to compute the integrals in (5) exactly. Figure 4 shows the standard deviation of the estimated xy-locations for the double source field given $\{\mathcal{R}_{meas}(k)\}$ (dashed lines) and $\{\mathcal{R}_n(k)\}$ (solid lines). Observe that for realistic sensor network SNRs of interest, i.e.

30dB or less, the performance of the location recovery coincides with that of the ideal case – that is $\{\mathcal{R}_n(k)\}$ which represents a sequence constructed from continuous but noisy field measurements. These simulations also suggest that the estimation algorithm remains unbiased despite the discrete approximation of the integrals in Equation (5).

5. CONCLUSION

An algorithm for reconstructing a 2-D diffusion field from its arbitrary spatiotemporal samples is presented. The method proposed herein solves the source estimation problem when the sources are localized and instantaneous. Moreover simulations show that the proposed algorithm is robust to noise, even in the multiple source setting, thanks to the averaging effects of the time integrated field and the averaging of multiple activation time estimates from the nearest sensors to the source. In addition we demonstrate through simulations that the effects of discretization is negligible for realistic measurement noise levels, since the performance of the localization step is similar for the approximate sequence $\mathcal{R}_{meas}(k)$ constructed from noisy sensor measurements, when compared to the exact sequence $\mathcal{R}_n(k)$ corrupted with noise.

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