RECONSTRUCTING NON-POINT SOURCES OF DIFFUSION FIELDS USING SENSOR MEASUREMENTS

John Murray-Bruce and Pier Luigi Dragotti

Imperial College London
Electrical and Electronic Engineering Department
e-mail:{john.murray-bruce07, p.dragotti}@imperial.ac.uk

ABSTRACT

We present a framework for estimating non-localized sources of diffusion fields using spatiotemporal measurements of the field. Specifically in this contribution, we consider two non-localized source types: straight line and polygonal sources and assume that the induced field is monitored using a sensor network. Given the sensor measurements, we demonstrate, for each non-point source parameterization, how to reduce the source estimation problem to a system governed by a power series expansion that can then be efficiently solved using Prony's method, in order to reconstruct the source. We then evaluate the proposed algorithms by performing some numerical simulations using both noiseless and noisy spatiotemporal sensor measurements of the field.

Index Terms— Diffusion fields, finite rate of innovation (FRI), sensor networks, spatiotemporal sampling, source estimation.

1. INTRODUCTION

The sensing of physical phenomena driven by mathematical models is an important application of sensor networks [1, 2]. Diffusion fields, for example, model accurately several real life phenomena encountered in biology, physics and engineering. Consequently the analysis of such fields from spatiotemporal sensor network measurements has received considerable research efforts in recent times [3–5]. In these contributions, the main focus has been on the centralized [6, 7] and distributed estimation [8–11] of point diffusion sources with instantaneous [7] or time-varying [5, 12] temporal evolutions. The point source model however is suitable for the diffusion source estimation problem when the size of the sources are several orders of magnitude smaller than the monitored region. This assumption however is otherwise invalid, for example, in temperature monitoring of multicore processors for load balancing, the sources may be better approximated by planar polygons (see [13]).

As a result we are concerned with the problem of reconstructing non-localized sources of diffusion fields given spatiotemporal sensor measurements of the induced field. In particular we will consider two typical spatial distributions for diffusion sources – i.e. straight line sources and convex polygonal sources. Then we will propose suitable, noise robust, source reconstruction schemes from spatiotemporal samples of the field. To achieve this, we will assume we have access to some *generalized measurements* of the field. In the point source distribution case, it has been shown in [5] that, these generalized measurements are governed by a power series expansion in the unknown point source parameters as such can be solved efficiently using Prony's method. When the unknown sources are spatially non-localized however, the power series expansion of the generalized measurements no longer holds true. In this paper we demonstrate using techniques from complex analysis that it is possible to properly modify the generalized measurements in such a way that we obtain a new power series expansion. This new series expansion can then be solved using Prony's method to recover simultaneously the endpoints (for a line source) or the vertices (for a convex polygonal source) as well as the activation time of the diffusion source.

The remainder of this paper is organized as follows. In Section 2 we provide useful parameterizations of the non-localized sources of interest and hence formally state the diffusion source reconstruction problem. Section 3 presents a derivation of the proposed method. Numerical simulations are given in Section 4 to validate the performance of the proposed scheme. We finally conclude the article in Section 5.

2. PROBLEM FORMULATION

We consider the problem of estimating the sources of diffusion fields using spatiotemporal samples of the field. In this contribution we are concerned with non-localized sources of diffusion fields, with the intention of fully estimating their geometry/shape and the activation times. Recall that a diffusion field $u(\mathbf{x}, t)$ at location $\mathbf{x} \in \mathbb{R}^2$ and time t, induced by some unknown source distribution $f(\mathbf{x}, t)$ within a two-

dimensional region Ω will propagate through Ω according to the diffusion equation,

$$\frac{\partial}{\partial t}u(\mathbf{x},t) = \mu\nabla^2 u(\mathbf{x},t) + f(\mathbf{x},t), \qquad (1)$$

where μ is the diffusivity of the medium Ω. The theory of Green's functions allows us to obtain solutions of the PDE (1) according to:

$$u(\mathbf{x},t) = (g * f)(\mathbf{x},t), \qquad (2)$$

where $g(\mathbf{x},t) = \frac{1}{4\pi\mu t}e^{-\frac{\|\mathbf{x}\|^2}{4\mu t}}H(t)$ is the Green's function of the two-dimensional diffusion field, and $H(t)$ is the unit step function. A consequence of (2) is that the entire field $u(\mathbf{x},t)$ may be perfectly reconstructed provided the source distribution $f(\mathbf{x},t)$ is known exactly. This is why we focus, herein, on recovering $f(\mathbf{x},t)$ using spatiotemporal samples of its induced field.
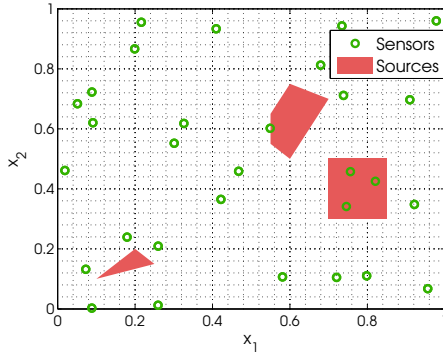

Fig. 1: Non Localized Sources in Ω.

We parameterize the sources we are interested in estimating. Specifically,

1. *Straight Line Source:* parameterized as follows

$$f(\mathbf{x},t) = cL(\mathbf{x})\delta(t-\tau), \qquad (3)$$

where $c,\tau \in \mathbb{R}$ are the intensity and activation time respectively, and $L(\mathbf{x}) \in \Omega$ describes a line coinciding with the position of the straight line source. In this case, clearly the geometry of the source is uniquely defined by its endpoints, i.e. the pair $\boldsymbol{\xi}_1, \boldsymbol{\xi}_2 \in \Omega$, with $\boldsymbol{\xi}_1 = (\xi_{1,1}, \xi_{2,1})$ and $\boldsymbol{\xi}_2 = (\xi_{1,2}, \xi_{2,2})$.

2. *Convex Polygonal Source:* these are characterized by their spatial and temporal characteristics as follows:

$$f(\mathbf{x},t) = cF(\mathbf{x})\delta(t-\tau), \qquad (4)$$

where $c,\tau \in \mathbb{R}$ are the intensity and activation time respectively, and $F(\mathbf{x}) \in \Omega$ is the region describing the location and shape of the convex polygonal diffusion source. Such a convex region $F(\mathbf{x})$ is uniquely specified by its vertices, that is, the collection $\{\boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \ldots, \boldsymbol{\xi}_M\}$, with $\boldsymbol{\xi}_m = (\xi_{1,m}, \xi_{2,m}) \in \Omega$ (see also Figure 1).

3. NON-LOCALIZED SOURCE RECOVERY

We now outline our proposed schemes for recovering non-localized source distributions (3) and (4) respectively, assuming access to the following *generalized measurements*:

$$\mathcal{Q}(k,r) = \langle \Psi_k(\mathbf{x})\Gamma_r(t), f \rangle = \int_\Omega \int_t \Psi_k(\mathbf{x})\Gamma_r(t)f(\mathbf{x},t)\mathrm{d}t\mathrm{d}V, \qquad (5)$$

where $\Psi_k(\mathbf{x}) = e^{-k(x_1+\mathrm{j}x_2)}$ and $\Gamma_r(t) = e^{-\mathrm{j}rt/T}$, with $k,r \in \mathbb{N}$; the reason for this choice will become apparent in what follows. Specifically, we show that the above sequence of integral measurements may be appropriately modified to obtain a new sequence that is governed by a power sum series, and as such can be efficiently solved using Prony's method [14–16], to recover the unknown source parameters. We then demonstrate in Section 3.3, how to obtain such generalized measurements $\{\mathcal{Q}(k,r)\}_{k,r}$ from the spatiotemporal field samples.

3.1. Analytic Recovery of Line Sources

In what follows we demonstrate how to recover the unknown straight line source parameters $(c,\tau,\boldsymbol{\xi}_1,\boldsymbol{\xi}_2)$ from the generalized measurements $\mathcal{Q}(k,r)$.

Proposition 1. *Let $\Psi_k(\mathbf{x})$ be the analytic function $\Psi_k(\mathbf{x}) = e^{-k(x_1+jx_2)}$, where $k = 1,2,\ldots,K$ with $K \geq 4$ and let $\Gamma(t) = e^{-\mathrm{j}rt/T}$, where $r = 0,1,\ldots,R$ and $R \geq 1$, then the generalized measurements $\mathcal{Q}(k,r)$ in (5) can be used to recover jointly, the unknown line source intensity, location and activation time.*

Proof. We begin the proof by considering the expression (5) and substitute the source parameterization (3) as follows:

$$\mathcal{Q}(k,r) = \langle \Psi_k(\mathbf{x})\Gamma_r(t), f \rangle = \int_\Omega \int_t \Psi_k(\mathbf{x})\Gamma_r(t)f(\mathbf{x},t)\mathrm{d}t\mathrm{d}V$$

$$= c\int_t \Gamma_r(t)\delta(t-\tau)\mathrm{d}t \int_\Omega \Psi_k(\mathbf{x})L(\mathbf{x})\mathrm{d}V$$

$$= c\Gamma_r(\tau) \int_{L(\mathbf{x})} \Psi_k(\mathbf{x})\mathrm{d}S, \qquad (6)$$

the last equality follows from the fact that $L(\mathbf{x})$ is only non-zero along the shortest line joining the endpoints $\boldsymbol{\xi}_1$ and $\boldsymbol{\xi}_2$. Next given a parametric representation of the line segment such as

$$L(\mathbf{x}(\theta)) : \begin{cases} x_1(\theta) = (1-\theta)\xi_{1,1} + \theta\xi_{1,2} \\ x_2(\theta) = (1-\theta)\xi_{2,1} + \theta\xi_{2,2} \end{cases}, \theta \in [0,1], \quad (7)$$

we have that

$$\int_{L(\mathbf{x})} \Psi_k(\mathbf{x})\mathrm{d}S = \int_0^1 \Psi_k(\mathbf{x}(\theta))\sqrt{\left(\frac{\mathrm{d}x_1}{\mathrm{d}\theta}\right)^2 + \left(\frac{\mathrm{d}x_2}{\mathrm{d}\theta}\right)^2}\mathrm{d}\theta$$

$$= \sqrt{(\xi_{1,2}-\xi_{1,1})^2 + (\xi_{2,2}-\xi_{2,1})^2} \int_0^1 \Psi_k(\mathbf{x}(\theta))\mathrm{d}\theta$$

$$= \ell(\boldsymbol{\xi}_1,\boldsymbol{\xi}_2)\left(e^{-k(\xi_{1,2}+\mathrm{j}\xi_{2,2})} - e^{-k(\xi_{1,1}+\mathrm{j}\xi_{2,1})}\right).$$

Thus

$$\int_{L(\mathbf{x})} \Psi_k(\mathbf{x})\mathrm{d}S = -\frac{1}{k}\ell(\boldsymbol{\xi}_1,\boldsymbol{\xi}_2)\sum_{m=1}^{2}(-1)^m e^{-k(\xi_{1,m}+\mathrm{j}\xi_{2,m})}, \tag{8}$$

where $\ell(\boldsymbol{\xi}_1,\boldsymbol{\xi}_2) = \frac{\sqrt{(\xi_{1,2}-\xi_{1,1})^2+(\xi_{2,2}-\xi_{2,1})^2}}{(\xi_{1,2}-\xi_{1,1})+j(\xi_{2,2}-\xi_{2,1})}$. Substituting (8) into (6) and recalling that $\Gamma_r(t) = e^{-\mathrm{j}rt/T}$ yields the following power sum series,

$$-k\mathcal{Q}(k,r) = \ell(\boldsymbol{\xi}_1,\boldsymbol{\xi}_2)ce^{-\mathrm{j}r\tau/T}\sum_{m=1}^{2}(-1)^m e^{-k(\xi_{1,m}+\mathrm{j}\xi_{2,m})}. \tag{9}$$

Thus the unknowns $(c,\boldsymbol{\xi}_1,\boldsymbol{\xi}_2)$ can be recovered from the sequence $\{-k\mathcal{Q}(k,r)\}_{k=1}^{K}$ using Prony's method provided $K \geq 4$, whilst τ can be recovered from $\{-k\mathcal{Q}(k,r)\}_{r=1}^{R}$ providing $R \geq 1$. □

Notice that the choice $\Psi_k(\mathbf{x}) = e^{-k(x_1+\mathrm{j}x_2)}$ and $\Gamma_r(t) = e^{-\mathrm{j}rt/T}$ is key to obtaining the desired power sum series (9).

3.2. Analytic Recovery of Polygonal Sources

We now outline how to estimate the unknown parameters c, τ and $\{\boldsymbol{\xi}_m\}_{m=1}^{M}$ for an M-sided convex polygonal diffusion source given access to the generalized measurements $\mathcal{Q}(k,r)$. We firstly state the following lemmas that will be useful in proving the scheme for recovering polygonal sources.

Lemma 1. *Let* $\Psi_k(\mathbf{x}) = e^{-k(x_1+\mathrm{j}x_2)}$, *then*

$$\Psi_k(\mathbf{x}) = \frac{1}{k^2}\Psi_k''(\mathbf{x}), \tag{10}$$

where $(\cdot)'$ *is used to denote the derivative with respect to the complex variable '$x_1 + \mathrm{j}x_2$'.*

Proof. Follows from two applications of the complex derivative (*w.r.t* the complex variable '$x_1 + \mathrm{j}x_2$'). □

Lemma 2. *Let* $\Psi_k(\mathbf{x})$ *be analytic inside the convex polygon* $F(\mathbf{x})$, *with vertices* $\{\boldsymbol{\xi}_m\}_{m=1}^{M}$, *then [17]*

$$\int_{F(\mathbf{x})} \Psi_k''(\mathbf{x})\mathrm{d}V = \sum_{m=1}^{M} a_m \Psi_k(\boldsymbol{\xi}_m). \tag{11}$$

Proof. For a proof of this lemma, see [17–19] □

Proposition 2. *Let* $\Psi_k(\mathbf{x})$ *be the analytic function* $\Psi_k(\mathbf{x}) = e^{-k(x_1+jx_2)}$, *where* $k = 1, 2, \ldots, K$ *with* $K \geq 2M$ *and let* $\Gamma(t) = e^{-\mathrm{j}rt/T}$, *where* $r = 0, 1, \ldots, R$ *and* $R \geq 1$, *then the generalized measurements* $\mathcal{Q}(k,r)$ *in (5) can be used to recover jointly, the unknown intensity, vertices and activation time of an* M-sided polygonal diffusion source.

Proof. Consider now the expression (5) and substitute the source parameterization (4) as follows:

$$\mathcal{Q}(k,r) = \langle \Psi_k(\mathbf{x})\Gamma_r(t), f \rangle = \int_\Omega \int_t \Psi_k(\mathbf{x})\Gamma_r(t)f(\mathbf{x},t)\mathrm{d}t\mathrm{d}V$$

$$= c\int_t \Gamma_r(t)\delta(t-\tau)\mathrm{d}t \int_\Omega \Psi_k(\mathbf{x})F(\mathbf{x})\mathrm{d}V$$

$$= c\Gamma_r(\tau)\int_{F(\mathbf{x})} \Psi_k(\mathbf{x})\mathrm{d}V$$

$$\overset{(i)}{=} c\Gamma_r(\tau)\int_{F(\mathbf{x})} \frac{1}{k^2}\Psi_k''(\mathbf{x})\mathrm{d}V$$

$$\overset{(ii)}{=} \frac{1}{k^2}c\Gamma_r(\tau)\sum_{m=1}^{M} a_m \Psi_k(\boldsymbol{\xi}_m)$$

where the equality (i) follows from Lemma 1 and (ii) from Lemma 2. Multiplying through by $k^2, k \neq 0$ and substituting the expressions for $\Psi_k(\mathbf{x})$ and $\Gamma_r(t)$ gives

$$k^2\mathcal{Q}(k,r) = ce^{-\mathrm{j}r\tau/T}\sum_{m=1}^{M} a_m e^{-k(\xi_{1,m}+\mathrm{j}\xi_{2,m})} \tag{12}$$

which is again a coupled power sum series. Thus the unknowns of the M-sided polygonal diffusion source can be recovered from $\{\mathcal{Q}(k,r) : k = 1, \ldots, K, \ r = 0, \ldots, R\}_{k,r}$ using Prony's method provided $K \geq 2M$ and $R \geq 1$. □

3.3. Computing the Generalized Measurements $\mathcal{Q}(k,r)$

In the section we discuss how to stably obtain the desired generalized measurements $\mathcal{Q}(k,r)$ from measurements of the field. This process has been outlined in [5], however we provide a summary herein for completeness. To begin we relate, using Green's second theorem, measurements of the diffusion field $u(\mathbf{x},t)$ along an arbitrary contour $\partial\Omega$ to the measurements of the field within the contour as follows:

$$\oint_{\partial\Omega} (\Psi_k\nabla u - u\nabla\Psi_k)\cdot\hat{\mathbf{n}}_{\partial\Omega}\mathrm{d}S = \int_\Omega (\Psi_k\nabla^2 u - u\nabla^2\Psi_k)\mathrm{d}V, \tag{13}$$

where $\hat{\mathbf{n}}_{\partial\Omega}$ is the outward pointing unit normal vector to the boundary $\partial\Omega$, Ω is the region enclosed by the contour $\partial\Omega$ and Ψ_k is a function chosen to satisfy $\frac{\partial\Psi_k}{\partial t} + \mu\nabla^2\Psi_k = 0$. Specifically, this choice of $\Psi_k(\mathbf{x})$ reduces (13) to,

$$\int_\Omega \frac{\partial}{\partial t}(u\Psi_k)\mathrm{d}V - \mu\oint_{\partial\Omega} (\Psi_k\nabla u - u\nabla\Psi_k)\cdot\hat{\mathbf{n}}_{\partial\Omega}\mathrm{d}S = \int_\Omega \Psi_k f\mathrm{d}V, \tag{14}$$

when we substitute (1) and $\nabla^2\Psi_k = -\frac{1}{\mu}\frac{\partial\Psi_k}{\partial t}$ into (13) and rearrange. Moreover this PDE is satisfied by any time-independent analytic function, as a result and for stability purposes we choose $\Psi_k(\mathbf{x}) = e^{-k(x_1+\mathrm{j}x_2)}$. Next multiply both sides of (14) by a *temporal sensing function* $\Gamma(t)$ and then time-integrate over $t \in [0, T]$ to obtain,

$$\int_\Omega \int_t \Psi_k(\mathbf{x})\Gamma_r(t)f(\mathbf{x},t)\mathrm{d}t\mathrm{d}V = \int_\Omega \Psi_k(\mathbf{x})\dot{U}(\mathbf{x},T)\mathrm{d}V$$
$$-\mu\oint_{\partial\Omega} \big(\Psi_k(\mathbf{x})\nabla U(\mathbf{x},T) - U(\mathbf{x},T)\nabla\Psi_k(\mathbf{x})\big)\cdot\hat{\mathbf{n}}_{\partial\Omega}\mathrm{d}S, \tag{15}$$

where $\dot{U}(\mathbf{x},T) = \Gamma(T)u(\mathbf{x},T) - \int_0^T \frac{\partial\Gamma}{\partial t}u(\mathbf{x},t)\mathrm{d}t$ and $U(\mathbf{x},T) = \int_0^T \Gamma(t)u(\mathbf{x},t)\mathrm{d}t$. By comparing equation (15)

with (5), we observe that $\mathcal{Q}(k,r) = \int_\Omega \Psi_k(\mathbf{x})\dot{U}(\mathbf{x},T)\,\mathrm{d}V - \mu \oint_{\partial\Omega}\left(\Psi_k(\mathbf{x})\nabla U(\mathbf{x},T) - U(\mathbf{x},T)\nabla\Psi_k(\mathbf{x})\right)\cdot\hat{\mathbf{n}}_{\partial\Omega}\,\mathrm{d}S$. Consequently, the generalized measurements can be obtained from the diffusion field $u(\mathbf{x},t)$. However given only spatiotemporal samples $\varphi_{n,l}$, the generalized measurements have to be approximated numerically from them. In particular, $\Phi_n(t_L) \stackrel{\text{def}}{=} U(\mathbf{x}_n,t_L)$ and similarly $\dot{\Phi}_n(t_L) \stackrel{\text{def}}{=} \dot{U}(\mathbf{x}_n,t_L)$ can be numerically computed from the temporal measurements $\{\varphi_{n,l}\}_{l=0}^L$ of the n-th sensor. The spatial integrals can then be approximated from the quantities $\{\Phi_n(t_L)\}_{n=1}^N$ and $\{\dot{\Phi}_n(t_L)\}_{n=1}^N$ using standard quadrature techniques as discussed in [5].
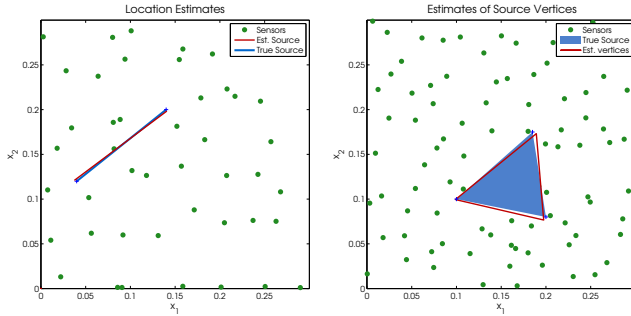
(a) Line source estimation using 45 sensors. (b) Triangular source estimation using 90 sensors.

Fig. 2: Non-point source estimation using noiseless spatiotemporal measurements obtained by arbitrarily placed sensors. Field sampled at $10Hz$ for $T = 7s$. For the temporal sensing function family, $R = 5$; (a) $K = 4$, and (b) $K = 6$.

4. SIMULATION RESULTS

The field due to the non-localized sources have been simulated using COMSOL Multiphysics and spatiotemporal samples are obtained at arbitrary spatial locations. Specifically we simulate the field due to a straight line diffusion source in Figure 3 and a triangular source in Figure 4; in both cases the sources are assumed to be instantaneous in time – we use a COMSOL's built-in Gaussian pulse function 'gp1()' with a standard deviation of $\sigma = 0.0025$ to simulate a delta.

Having obtained the measurements using COMSOL, on this data we apply the proposed algorithms to reconstruct the non-point diffusion source spatially. The results of these simulations are summarized in Figure 2, we can see that both source types have been accurately reconstructed from the noiseless field measurements. Furthermore, in order to investigate the robustness of the proposed algorithms to noise, we artificially corrupt the measurements with additive white Gaussian noise and employ the proposed non-point source reconstruction algorithms. We perform 10 independent trials using noisy data. For each indpendent trial we use a new realization of sensor placement and a new sensor noise process. As can be seen in Figures 3 and 4, the unknown source

parameters of interest (specifically the vertices and source activation times) are recovered fairly reliably in this noisy setting. Given noiseless field measurements, the non-point sources are reconstructed almost perfectly (see Figure 2).
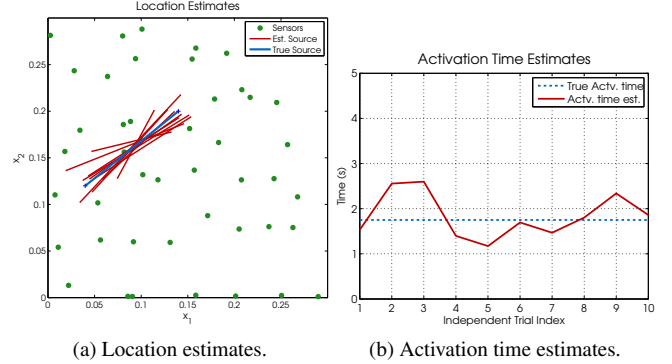

(a) Location estimates. (b) Activation time estimates.

Fig. 3: Line source estimation using noisy spatiotemporal measurements obtained by 45 arbitrarily placed sensors. Field sampled at $10Hz$ for $T = 10s$, measurement SNR$= 20$dB. For the spatial and temporal sensing functions family, $K = 6$ and $R = 5$ respectively.

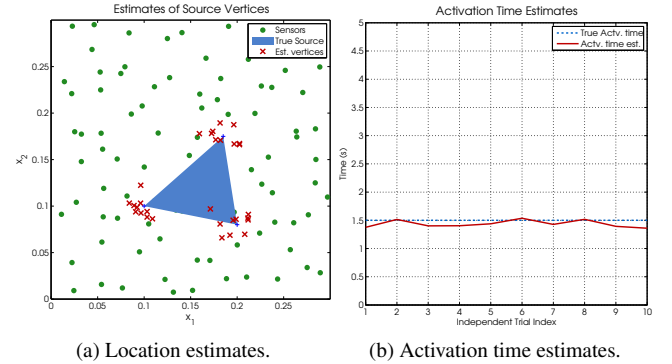(a) Location estimates. (b) Activation time estimates.

Fig. 4: Triangular source estimation using noisy spatiotemporal measurements obtained by 90 arbitrarily placed sensors. Field sampled at $10Hz$ for $T = 10s$, measurement SNR$= 35$dB. For the spatial and temporal sensing functions, $K = 9$ and $R = 8$ respectively.

5. CONCLUSIONS

In this paper we have presented algorithms for the recovery of non-localized sources of diffusion fields from spatiotemporal samples of the field observed through sensor networks. In particular, we have shown how this estimation problem can be reformulated as a non-linear system that can be efficiently solved for the unknown source parameters using Prony's method. In so doing, we were able to devise suitable recovery algorithms depending on the source shape. We also verified the proposed schemes through numerical simulations, our simulations confirm that we are able to recover the unknown parameters from the noisy field samples.

6. REFERENCES

[1] I. F. Akyildiz, W. Su, Y. Sankarasubramaniam, and E. Cayirci, "A survey on sensor networks," *IEEE Communications Magazine*, vol. 40, no. 8, pp. 102–114, Aug 2002.

[2] J. Yick, B. Mukherjee, and D. Ghosal, "Wireless sensor network survey," *Computer networks*, vol. 52, no. 12, pp. 2292–2330, 2008.

[3] Juri Ranieri and Martin Vetterli, "Sampling and reconstructing diffusion fields in presence of aliasing," in *Proc. IEEE International Conference on Acoustics, Speech, And Signal Processing (ICASSP'13)*, Vancouver, Canada, May 2013.

[4] S. Maranò, D. Fäh, and H.-A. Loeliger, "A state-space approach for the analysis of wave and diffusion fields," in *Proc. 40th IEEE International Conference on Acoustics, Speech and Signal Processing (ICASSP'15)*, Brisbane, Australia, apr 2015.

[5] J. Murray-Bruce and P. L. Dragotti, "Estimating localized sources of diffusion fields using spatiotemporal sensor measurements," *IEEE Transactions on Signal Processing*, vol. 63, no. 12, pp. 3018–3031, June 2015.

[6] J. Murray-Bruce and P. L. Dragotti, "Spatio-Temporal sampling and reconstruction of diffusion fields induced by point sources," in *Proc. 39th IEEE International Conference on Acoustics, Speech and Signal Processing (ICASSP'14)*, Florence, Italy, May 2014, pp. 31–35.

[7] Y. M. Lu, P.-L. Dragotti, and M. Vetterli, "Localizing Point Sources in Diffusion Fields from Spatiotemporal Samples," in *International Conference on Sampling Theory and Applications (SampTA)*, Singapore, May 2011.

[8] Y. M. Lu and M. Vetterli, "Distributed spatio-temporal sampling of diffusion fields from sparse instantaneous sources," in *3rd IEEE International Workshop on Computational Advances in Multi-Sensor Adaptive Processing (CAMSAP'09)*, Aruba, Dutch Antilles, 2009, pp. 205–208, IEEE.

[9] T. van Waterschoot and G. Leus, "Distributed estimation of static fields in wireless sensor networks using the finite element method," in *IEEE International Conference on Acoustics, Speech and Signal Processing (ICASSP'12)*, Kyoto, Japan, March 2012, pp. 2853–2856.

[10] Günter Reise, Gerald Matz, and Karlheinz Grochenig, "Distributed field reconstruction in wireless sensor networks based on hybrid shift-invariant spaces," *IEEE Transactions on Signal Processing*, vol. 60, no. 10, pp. 5426–5439, 2012.

[11] J. Murray-Bruce and P. L. Dragotti, "Consensus for the distributed estimation of point diffusion sources in sensor networks," in *Proc. 40th IEEE International Conference on Acoustics, Speech and Signal Processing (ICASSP'15)*, Brisbane, Australia, Apr. 2015.

[12] Günter Reise and Gerald Matz, "Reconstruction of time-varying fields in wireless sensor networks using shift-invariant spaces: Iterative algorithms and impact of sensor localization errors," in *IEEE 11th International Workshop on Signal Processing Advances in Wireless Communications (SPAWC'10)*. IEEE, 2010, pp. 1–5.

[13] J. Ranieri, A. Vincenzi, A. Chebira, D. A. Alonso, and M. Vetterli, "EigenMaps: Algorithms for Optimal Thermal Maps Extraction and Sensor Placement on Multicore Processors," in *Proceedings of the 49th Design Automation Conference (DAC'12)*, San Francisco, 2012, Design Automation Conference (DAC'12), pp. 636–641, ACM.

[14] G. R. de Prony, "Essai expérimental et analytique: sur les lois de la dilatabilité des fluides élastiques et sur celles de la force expansive de la vapeur de l'alkool, à différentes températures," *Journal de l'école Polytechnique*, vol. 1, 1795.

[15] T. Blu, P.-L. Dragotti, M. Vetterli, P. Marziliano, and L. Coulot, "Sparse sampling of signal innovations," *Signal Processing Magazine, IEEE*, vol. 25, no. 2, pp. 31–40, March 2008.

[16] P. Stoica and R.L. Moses, *Introduction to spectral analysis*, Prentice Hall, Englewood Cliffs, NJ, 1997.

[17] Philip J. Davis, "Triangle formulas in the complex plane," *Mathematics of Computation*, vol. 18, no. 88, pp. pp. 569–577, 1964.

[18] Philip J Davis, "Plane regions determined by complex moments," *Journal of Approximation Theory*, vol. 19, no. 2, pp. 148–153, 1977.

[19] M. Elad, P. Milanfar, and G.H. Golub, "Shape from moments - an estimation theory perspective," *IEEE Transactions on Signal Processing*, vol. 52, no. 7, pp. 1814–1829, July 2004.