THE GRAPH FRI FRAMEWORK- SPLINE WAVELET THEORY AND SAMPLING ON CIRCULANT GRAPHS

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ABSTRACT

The objective of this work is to consider sparse representations of certain classes of signals on circulant graphs, by introducing families of graph wavelets which possess vanishing (exponential) moment properties. In light of this, we propose a novel framework of sampling and perfect reconstruction of sparse and wavelet-sparse signals on circulant graphs, which we denote as the Graph FRI framework, as an extension to the traditional discrete case. Given the dimensionality-reduced GFT of a sparse signal on a graph G, we can perfectly reconstruct the latter, while inferring a distinct downsampling pattern and the structure of the associated coarsened graph through decomposition of the GFT-basis as the product between a coefficient matrix $\mathbf{C}$ and the multiresolution filtering operation with a low-pass graph e-spline filter. Hereby, we demonstrate that for a sufficiently banded adjacency matrix $\mathbf{A}$ of G, the obtained coarse graph preserves the original generating set S of G in a scheme of spectral sampling with respect to the original eigenbasis of $\mathbf{A}$.

Index Terms— Graph signal processing, graph wavelet, circulant graph, sparsity, finite rate of innovation

1. INTRODUCTION

The emerging need for superior representation and processing of large complex data through the higher-dimensional dependency structures of a graph as well as the appeal of developing an associated mathematical framework, which goes beyond traditional signal processing theory, have given rise to the field of graph signal processing, which in recent years has experienced a breadth of contributions. With the overall aim to establish equivalent concepts to traditional signal processing with respect to newly arising data dependencies, the collective of theoretical frameworks and corresponding applications are divided into two main approaches: one with focus on the graph Laplacian as a positive semi-definite matrix, derived from i.a. spectral graph theory ([1], [2]), and another with focus on the adjacency matrix [3]. The study of wavelets on graphs in particular poses a promising new venue considering the associated potential to operate with respect to the inherent geometry of given data, whose higher-dimensional dependencies are represented by a graph and/or graph signal, and a number of designs have been proposed ([4], [5], [6],[7]).

One of the challenges in graph signal processing theory is to determine a suitable set of vertices for sampling given that the high connectivity of graphs provides multiple directions along which one can downsample [1], so that existing filterbank constructions have been favourably modelled for specific types of graphs, which facilitate more intuitive operations. Graph wavelets on circulant graphs have been studied by Ekambaram et al. ([8], [9], [10]), including the critically-sampled perfect reconstruction spline-like graph wavelet filterbank, which provided the inspiration for our graph wavelet constructions, along with downsampling rules based on the regularity of the graphs. The class of circulant graphs proves as particularly convenient due to their Linear Shift Invariance (LSI) property [10] in addition to the fact that corresponding circulant graph Laplacian matrices give rise to a Graph Fourier Transform (GFT), which can be represented as a permutation of the classical DFT. This facilitates the link to the classical domain, thereby making an intuition for higher-dimensional extensions more concrete.

Contributions: In this paper, we present a set of novel wavelet filterbanks on circulant graphs, which have been tailored to the annihilation of certain classes of graph signals. We are interested in the study of sparsity and sparsity-inducing wavelet constructions on circulant graphs, and in prior work [11] have introduced a higher-order graph wavelet filterbank, which, based on the vanishing moments associated with the circulant symmetric graph Laplacian, extends the spline property to the graph domain. Following a similar line of derivation, we introduce the notion of a graph e-spline wavelet on circulant graphs; hereby we define a degree-parameterised e-graph Laplacian, which can annihilate complex exponential graph signals, giving rise to novel families of graph e-spline wavelets and filterbanks. Equipped with a set of sparsifying wavelet transforms, we proceed to propose a framework of sparse and wavelet-sparse signal sampling and reconstruction on circulant graphs as an extension of the traditional Finite Rate of Innovation (FRI) framework [12] to the graph domain. In particular, we show that sparse graph signals on circulant graphs can be perfectly reconstructed based on their dimensionality-reduced GFT representation and additionally extract the associated coarsened graph through a scheme of spectral sampling, utilising the theory of exponential reproducing kernels [12]. We took a first look at this problem in [13]. The developed theory can be applied to arbitrary graphs through suitable approximation schemes by using circulant graphs as building blocks, whose discussion we defer to a longer version due to space constraints.

Related Work: Sampling theory for signals on graphs has been studied for bandlimited graph signals using different approaches ([14], [15]). In particular, our graph coarsening scheme is comparable to the one introduced in [15], with the difference that we have a fixed downsampling pattern, and focus exclusively on its implications for circulant graphs and the associated property preservation. In addition, our reconstruction framework does not depend on the locations of the chosen samples, which are rather used to identify a suitable coarse graph.

This paper is organised as follows: we begin by outlining background theory in Section 2. In Section 3, we first review our prior

This work is in part supported by the European Research Council (ERC) starting investigator award Nr. 277800 (RecoSamp).

work on graph spline wavelets, before introducing the novel graph e-spline wavelets. In Section 4, we present our developed Graph FRI (GFRI) framework, before making concluding remarks and discussing future directions in Section 5.

2. PRELIMINARIES

2.1. Background: Graph Signal Processing

We consider graphs, which are undirected, weighted, connected and circulant, without self-loops. A graph $G = (V, E)$, of cardinality $|V| = N$, is defined by a vertex set V and an edge set E, whose connectivity is reflected in the adjacency matrix $\mathbf{A}$, with $A_{i,j} > 0$ if there is an edge between nodes $i, j \in V$, and $A_{i,j} = 0$ otherwise. The degree matrix $\mathbf{D}$, with diagonal entries $D_{i,i} = \sum_j A_{i,j}$ reflects the degree per node, while the non-normalized graph Laplacian is defined as $\mathbf{L} = \mathbf{D} - \mathbf{A}$, which we focus on throughout this work. In particular, $\mathbf{L}$ is positive semi-definite, with a complete set of orthonormal eigenvectors $\{\mathbf{u}\}_{i=0}^{N-1}$ and corresponding nonnegative eigenvalues $0 = \lambda_0 < \lambda_1 \leq ... \leq \lambda_{N-1}$. A circulant graph G is defined via a generating set $S = \{s_k\}_{k=1}^{M}$, with $0 \leq s_k \leq N/2$, whose elements indicate edges between the node pairs $(i, (i \pm s_k)_{modN}), \forall s_k \in S$; in particular, there exists a labelling on G such that its adjacency matrix is circulant. The symmetric circulant matrix $\mathbf{L}$, with first row $[l_0 \; l_1 \; l_2 \quad ... \quad l_2 \; l_1]$ and bandwidth M, can be defined via its representer polynomial $l(z) = l_0 + \sum_{i=1}^{M} l_i(z^i + z^{-i})$.

A graph signal on G can be represented as a vector $\mathbf{x} \in \mathbb{C}^N$ with sample value $x(i)$ at node i. While traditionally real-valued, we generalise the definition of $\mathbf{x}$ for illustration purposes in our discussion, and still require the weights on G to be real. Analogously to the traditional DFT, the Graph Fourier Transform (GFT) $\mathbf{X}^G$ of $\mathbf{x}$ on G is the projection onto the basis $\mathbf{U} = [\mathbf{u}_0|...|\mathbf{u}_{N-1}]$ such that $\mathbf{X}^G = \mathbf{U}^H\mathbf{x}$, where H denotes the Hermitian transpose. The GFT of a circulant graph can be expressed as a permutation of the DFT.

We can downsample a signal on a circulant graph G by 2 with respect to the outmost cycle, i.e. by skipping every other labelled node, requiring $s_1 = 1 \in S$ (i.e. G is connected) [8]. For simplicity, we focus on this case here, and restrict the number of nodes to be $N = 2^n, n \in \mathbb{Z}^+$. The spline-like critically sampled graph wavelet transform (GWT) on undirected, connected circulant graphs, which inspired our current designs, is composed of a set of 1-hop localized low- and high-pass filters, $\mathbf{H}_{LP} = \frac{1}{2}\left(\mathbf{I}_N + \frac{\mathbf{A}}{d}\right)$ and $\mathbf{H}_{HP} = \frac{1}{2}\left(\mathbf{I}_N - \frac{\mathbf{A}}{d}\right)$ (see [9], Thm. 1), where d is the degree per node. The GWT can be iterated on the low-pass branch for a multiresolution representation, whereby the downsampled nodes can be reconnected using various schemes, such as Kron reduction [10], depending on the application of interest. For this work, we choose to reconnect such that the coarsened graph retains its original generating set S, assuming $\mathbf{A}$ is banded of bandwidth M.

3. SPLINE WAVELETS ON CIRCULANT GRAPHS

3.1. Prior Work: Higher-Order Graph Spline Wavelets

In order to formulate our sampling framework for sparse and wavelet-sparse graph signals, we need to state the concept of the graph spline (wavelet) [11]:

Definition 3.1: *We define a graph signal $\mathbf{y} \in \mathbb{R}^N$ on the vertices of G to be (piecewise) polynomial if its sequence of sample values, with $y(i)$ at node i, is a discretized (piecewise) polynomial.*

Lemma 3.1: *For an undirected, circulant graph G of dimension N, the representer polynomial $l(z)$ of graph Laplacian $\mathbf{L}$ has two vanishing moments, i.e. $\mathbf{L}$ annihilates up to linear polynomial graph signals, subject to a border effect determined by the bandwidth M of $\mathbf{L}$, whereby $2M << N$.*

Based on the aforementioned inherent annihilation property of $\mathbf{L}$ for undirected circulant graphs, we derived a variety of graph wavelet filterbanks which inherit and extend this property to higher-order through the high-pass filter, as captured in the following [11]:

Theorem 3.1: *Given an undirected, and connected circulant graph G of dimension N, with adjacency matrix $\mathbf{A}$ and degree d per node, the higher-order graph-spline wavelet transform (HGSWT) is composed of the filters*

$$\mathbf{H}_{LP} = \frac{1}{2^k}\left(\mathbf{I}_N + \frac{\mathbf{A}}{d}\right)^k \tag{1}$$

$$\mathbf{H}_{HP} = \frac{1}{2^k}\left(\mathbf{I}_N - \frac{\mathbf{A}}{d}\right)^k = \left(\frac{\mathbf{L}}{2d}\right)^k \tag{2}$$

whereby the high-pass polynomial function $H_{HP}(z)$ has $2k$ vanishing moments. This filterbank is invertible for any downsampling pattern, as long as at least one node retains the low-pass component.

Furthermore, we can tailor the design of such filterbanks to incorporate a well-defined synthesis branch, with the ability to also reproduce polynomial graph signals by performing spectral factorization. This leads to a modified analysis low-pass filter $H_{LP,an}(z)$, which can be linked to (1) via a coefficient matrix $\mathbf{C}$, given the graph Laplacian-based analysis high-pass filter $H_{HP}(z)$, within a biorthogonal perfect reconstruction filterbank [11]:

Theorem 3.2: *Given an undirected, and connected circulant graph G of dimension N, with adjacency matrix $\mathbf{A}$ and degree d per node, we define the higher-order 'complementary' graph-spline wavelet transform (HCGSWT) via the set of analysis filters:*

$$\mathbf{H}_{LP,an} = \mathbf{C}\mathbf{H}_{LP} = \frac{1}{2^k}\mathbf{C}\left(\mathbf{I}_N + \frac{\mathbf{A}}{d}\right)^k \tag{3}$$

$$\mathbf{H}_{HP,an} = \frac{1}{2^k}\left(\mathbf{I}_N - \frac{\mathbf{A}}{d}\right)^k \tag{4}$$

and the set of synthesis filters:

$$\mathbf{H}_{LP,syn} = c_1\mathbf{H}_{HP,an} \circ I_{HP} \tag{5}$$

$$\mathbf{H}_{HP,syn} = c_2\mathbf{H}_{LP,an} \circ I_{LP} \tag{6}$$

where $\circ$ is the Hadamard product, for constants $c_{1/2}$, and circulant indicator matrices $I_{LP/HP}$, whose entries $\{1, -1\}$ coincide with those of $\mathbf{H}_{LP/HP,an}$.

The filter $\mathbf{H}_{LP,an}$ is localized within a sub-region of the $\tilde{k}$-hop neighborhood of each node, whereby $\tilde{k}$ depends on the constraints we impose on $H_{LP,an}(z)$, such as the ability to reproduce polynomials.

3.2. Graph E-spline Wavelets

We discover that the previously described vanishing moment property of the graph Laplacian, can be further extended for the annihilation of a wider range of graph signal classes, by introducing the exponential-graph Laplacian.

Definition 3.2: *A complex exponential graph signal* $\mathbf{y} \in \mathbb{C}^N$ *is defined such that its sequence of sample values, with* $y(j) = e^{i\alpha t_j}$ *at node* j*, is complex exponential, with* $\alpha \in \mathbb{R}$ *and* $t_j = j-1, j \in \mathbb{Z}^+$.

Definition 3.3: *Let* G *be an undirected, circulant graph with adjacency matrix* $\mathbf{A}$ *and degree* $d = \sum_{j=1}^{M} 2d_j$ *per node with symmetric weights* $d_j = A_{i,j+i}$. *The parameterised e-graph Laplacian of* G *is given by* $\tilde{\mathbf{L}}_\alpha = \tilde{\mathbf{D}}_\alpha - \mathbf{A}$*, where* $\tilde{d}_\alpha = \sum_{j=1}^{M} 2d_j \cos(\alpha j)$ *is the exponential degree*.

In particular, we note that the e-graph Laplacian $\tilde{\mathbf{L}}_\alpha$ represents a generalisation of the traditional graph Laplacian $\mathbf{L}$ with respect to the exponential parameter α, whereby we have that $\tilde{\mathbf{L}}_\alpha = \mathbf{L}$ when $\alpha = 0$. This is further reinforced by the following result:

Lemma 3.2: *For an undirected, circulant graph* G *of dimension* N*, the representer polynomial* $\tilde{l}_\alpha(z)$ *of e-graph Laplacian* $\tilde{\mathbf{L}}_\alpha$ *has two vanishing exponential moments; i.e.* $\tilde{\mathbf{L}}_\alpha$ *annihilates complex exponential graph signals with* $\pm\alpha \in \mathbb{R}$. *Unless the graph signal at hand is periodic, for certain* α*, this is subject to a border effect determined by the bandwidth* M *of* $\tilde{\mathbf{L}}_\alpha$*, whereby* $2M << N$.

Proof. Consider the representer polynomial $\tilde{l}_\alpha(z)$ of $\tilde{\mathbf{L}}_\alpha$:

$$\begin{aligned}\tilde{l}_\alpha(z) &= \sum_{j=1}^{M} (2d_j \cos(\alpha j) - d_j(z^j + z^{-j})) \\ &= \sum_{j=1}^{M} d_j(1 - e^{i\alpha j} z^j)(1 - e^{-i\alpha j} z^j)(-z^{-j}).\end{aligned}$$

We note that $(1 - e^{\mp i\alpha} z^{-1})$ is a factor of $(1 - e^{\pm i\alpha j} z^j)$, which corresponds to two vanishing exponential moments [16]. □

By proceeding in a similar manner as in the case of the graph spline, we can now model graph e-spline wavelet filterbanks originating from the notion of the e-graph Laplacian as a high-pass filter, which we define as $\mathbf{H}_{HP_\alpha} = \frac{1}{2}\left(\beta \mathbf{I}_N - \frac{\mathbf{A}}{d}\right)$, for parameter $\beta = \frac{\tilde{d}_\alpha}{d}$ with $\tilde{d}_\alpha = \sum_{j=1}^{M} 2d_j \cos(\alpha j)$, and propose novel families of wavelets on circulant graphs.
According to ([17], Thm. 1), for scaling function $H_j(z)$ at level j to reproduce a function of the form $e^{\gamma_m t}$, it is necessary and sufficient that the latter is divisible by the term $R_{2^j \vec{\gamma}}(z), \forall j \leq j_0 - 1$, where $R_{\vec{\gamma}}(z) = \prod_{m=1}^{M}(1 + e^{\gamma_m} z^{-1})$, with $\vec{\gamma} = (\gamma_1, ..., \gamma_M)^T \in \mathbb{C}^M$, i.e. $H_j(z)$ satisfies the generalized Strang-Fix conditions, having no opposite or zero roots, for suitable $\vec{\gamma}$.

In our ensuing discussion, we are interested in creating wavelet filters on graphs which, together with their duals, are symmetric real valued, of compact support, and possess vanishing exponential moments, mirroring the complementary constructions in Thm. 3.2. In particular, given the analysis high-pass filter polynomial $H_{HP_\alpha}(z)$ with 2 vanishing exponential moments, we can determine the analysis lowpass filter $H_{LP_{\alpha,an}}(z)$, via spectral factorization following the same reasoning as for Thm. 3.2, as an extension of the adjacency-matrix based low-pass construction $\mathbf{H}_{LP_\alpha} = \frac{1}{2}\left(\beta \mathbf{I}_N + \frac{\mathbf{A}}{d}\right)$ via a coefficient matrix $\mathbf{C}$. In addition, for the analysis and synthesis filters to have (an equal number of) vanishing moments, we set $H_{LP_{\alpha,an}}(z) = (z + e^{i\alpha})(1 + e^{-i\alpha} z^{-1})R(z)$, where $R(z)$ is the polynomial to be determined. This gives rise to the following:

Theorem 3.3: *Given the undirected, and connected circulant graph* G *of dimension* N*, with adjacency matrix* $\mathbf{A}$ *and degree* d *per node, we define the 'complementary' graph e-spline wavelet transform (CGESWT) via the set of analysis filters:*

$$\mathbf{H}_{LP_\alpha,an} = \mathbf{C}\mathbf{H}_{LP_\alpha} = \frac{1}{2}\mathbf{C}\left(\beta \mathbf{I}_N + \frac{\mathbf{A}}{d}\right) \quad (7)$$

$$\mathbf{H}_{HP_\alpha,an} = \frac{1}{2}\left(\beta \mathbf{I}_N - \frac{\mathbf{A}}{d}\right) \quad (8)$$

and the set of synthesis filters:

$$\mathbf{H}_{LP_\alpha,syn} = c_1 \mathbf{H}_{HP_\alpha,an} \circ I_{HP_\alpha} \quad (9)$$

$$\mathbf{H}_{HP_\alpha,syn} = c_2 \mathbf{H}_{LP_\alpha,an} \circ I_{LP_\alpha} \quad (10)$$

for some constants $c_{1/2}$*, and circulant indicator matrices* I_{LP_α/HP_α}*, whose entries* $\{1, -1\}$ *coincide with those of* $\mathbf{H}_{LP/HP_\alpha,an}$.

For a multiresolution decomposition, we need to ensure that the filters $H_j(z)$ satisfy the root constraints at each level j, and modify the parameter $2^j\alpha$ in β accordingly due to the non-stationarity of the filterbank; see [17] for a detailed discussion. In addition, we note that the filterbank can be generalised to annihilate and reproduce multiple signals with $\pm\alpha_n$ by selecting $H_{HP_{\vec{\alpha}},an}(z) = \prod_n \frac{1}{(2d)} \tilde{l}_{\alpha_n}(z)$ and $H_{LP_{\vec{\alpha}},an}(z) = R(z)\prod_n (z + e^{i\alpha_n})(1 + e^{-i\alpha_n} z^{-1})$ respectively, however, a polynomial solution $R(z)$ exists only if the remaining factor does not contain zero and opposing roots [17].

Remark 1: The existence of a coefficient matrix $\mathbf{C}$ in Thms 3.2 and 3.3 is based on invertibility of the low-pass filter matrix (1) (and $\mathbf{H}_{LP_\alpha}$ in (7)), which is satisfied when G is non-bipartite.

Remark 2: The introduced graph (e-)spline property can be directly related to the traditional case for a simple cycle graph, and further extended when the graph at hand is bipartite circulant. In this case, the filters (1) and $\mathbf{H}_{LP_\alpha}$ in (7) can also reproduce polynomials and complex exponentials respectively. We omit an in-depth discussion of this phenomenon here for brevity.

4. THE FRI FRAMEWORK ON CIRCULANT GRAPHS

4.1. The classical FRI framework

In the discrete-time domain, consider a K-sparse signal $\mathbf{x} \in \mathbb{R}^N$, $||\mathbf{x}||_0 = K$, and define the measurement vector $\mathbf{y}$ in the Fourier domain, such that $\mathbf{y} = \mathbf{F}\mathbf{x}$, where $\mathbf{F} \in \mathbb{C}^{N \times N}$ is the DFT-matrix. Using Prony's method, we can perfectly reconstruct $\mathbf{x}$ with $M \geq 2K$ consecutive sample values of $\mathbf{y}$ [12].
Furthermore, in the traditional FRI-framework, a signal $x(t)$ can be sampled with a general exponential reproducing kernel $\phi(t)$ and its shifted versions in continuous-time

$$\sum_{n \in \mathbb{Z}} c_{m,n}\phi(t-n) = e^{\alpha_m t} \quad \text{for} \quad \alpha_m \in \mathbb{C}, m = 0, ..., P \quad (11)$$

for a proper choice of coefficients $c_{m,n} = c_{m,0}e^{\alpha_m n}$ ([18], [19]).

4.2. The graph FRI framework

Through the breadth of sparsifying wavelet-transforms, introduced in the previous section, we can facilitate a sparse multiresolution representation for certain classes of graph signals. In particular, we perform graph wavelet analysis of e.g. (piecewise) polynomial and/or complex exponential graph signals $\mathbf{y} \in \mathbb{C}^N$ on a given graph G using a suitable GWT $\mathbf{W}_j$ at level j, such that $\mathbf{w} = \mathbf{PWy}$, where $\mathbf{W}$ is the resulting GWT matrix product obtained via iteration on the low-pass branch,

$$\mathbf{W} = \begin{bmatrix} \mathbf{W}_j & \\ & \mathbf{I}_{N-\frac{N}{2^j}} \end{bmatrix} \cdots \begin{bmatrix} \mathbf{W}_1 & \\ & \mathbf{I}_{\frac{N}{2}} \end{bmatrix} \mathbf{W}_0$$

and $\mathbf{P}$ a permutation matrix. Hereby, the multiresolution representation $\mathbf{w}$ is a projection from the corresponding coarse graphs G_j onto the original G, with an appropriate (permutation) relabelling $\mathbf{P}$.

Definition 4.1: *On a circulant, undirected graph G, we define the class X of K-sparse graph signals, with $\mathbf{x} \in X$ of the form $\mathbf{x} \in \mathbb{C}^N$, $||\mathbf{x}||_0 = K$, and the class of wavelet-K-sparse graph signals W, with $\mathbf{y} \in W$, such that $\mathbf{y} \in \mathbb{C}^N$, and multiresolution representation $\mathbf{w} \in \mathbb{C}^N$, $||\mathbf{w}||_0 = K$, via a suitable GWT.*

In light of this, we define the FRI framework for circulant graphs:

Theorem 4.1 (Graph-FRI): *Define the permuted GFT basis $\mathbf{U}$ of a given circulant graph G such that $\mathbf{U}^H$ is the DFT-matrix. We can sample and perfectly reconstruct a (wavelet-)K-sparse graph signal (with multiresolution) $\mathbf{x} \in \mathbb{C}^N$, on the vertices of G using the dimensionality-reduced GFT representation $\mathbf{y} = \mathbf{U}_M^H \mathbf{x}, \mathbf{y} \in \mathbb{C}^M$, whereby $\mathbf{U}_M^H$ are the first M rows of $\mathbf{U}^H$, as long as $M \geq 2K$.*

4.3. Graph Coarsening

We wish to identify the coarse graph $\tilde{G}$ associated with the dimensionality reduced spectral graph signal $\mathbf{y}$, for which we require a distinct sampling pattern in the vertex domain. Therefore, we resort to representing the matrix $\mathbf{U}_M^H$ as the product between a fat coefficient matrix $\mathbf{C}$ and a low-pass GWT filter on G which can reproduce complex exponential graph signals, à la Thm. 3.3.
We begin by noting that the graph Laplacian eigenvector $\mathbf{u}_k$ of G is interpreted as having sample $u_k(i)$ at node i [1], thereby suggesting that a coarsened graph may be obtained via sampling in the spectral domain of the eigenbasis, given a pattern. Hence, we give the following result pertaining to circulant graphs (illustrated in Fig. 1):

Lemma 4.1: *Consider an undirected circulant graph G with generating set S, and adjacency matrix $\mathbf{A} = \frac{1}{N}\mathbf{U\Lambda U}^H \in \mathbb{R}^{N\times N}$ with bandwidth M, where $\frac{1}{\sqrt{N}}\mathbf{U}^H$ is the DFT matrix. We downsample by 2 via the binary matrix $\mathbf{\Psi}_{\downarrow 2} \in \mathbb{R}^{N/2\times N}$ on the first $N/2$ rows in $\mathbf{U}^H$ and eigenvalues $\mathbf{\Lambda}$, such that $\tilde{\mathbf{U}}^H = \mathbf{U}^H_{1:N/2}\mathbf{\Psi}^T_{\downarrow 2}$, and $\tilde{\mathbf{\Lambda}} = \mathbf{\Psi}_{\downarrow 2}\mathbf{\Lambda}\mathbf{\Psi}^T_{\downarrow 2}$. The resulting adjacency matrix $\tilde{\mathbf{A}} = \frac{1}{N/2}\tilde{\mathbf{U}}\tilde{\mathbf{\Lambda}}\tilde{\mathbf{U}}^H \in \mathbb{R}^{N/2\times N/2}$ is circulant with the same generating set S as G.*

Proof. The eigenvalues of $\mathbf{A}$ with first row $[0 \quad a_1 ... a_1]$ are $\lambda_j = \sum_{k=1}^{M} 2a_k cos\left(\frac{2\pi jk}{N}\right)$, $j = 0, ..., N$-1. Thus the eigenvalues of $\tilde{\mathbf{A}}$ with the same entries a_i and bandwidth $M < N/4$, are $\tilde{\lambda}_j = \sum_{k=1}^{M} 2a_k cos\left(\frac{2\pi(2j)k}{N}\right) = \lambda_{2j}$, $j = 0, ..., N/2$-1. We can similarly show the preservation of the downsampled DFT-eigenbasis, omitted for brevity. □

We reformulate the GFRI framework as the filtering of sparse graph signal $\mathbf{x}$ with exponential reproducing GWT filter $\mathbf{E}$, which produces a dimensionality-reduced signal $\tilde{\mathbf{y}}$ on a coarse graph $\tilde{G}$, with subsequent projection onto the lower-dimensional GFT domain via a coefficient matrix, to facilitate perfect reconstruction (see Fig. 2):

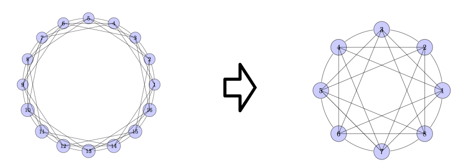

Fig. 1. Graph Coarsening for a Circulant Graph with $S = \{1, 2, 3\}$

Theorem 4.2: *Consider the decomposition of spectral graph signal $\mathbf{y} \in \mathbb{C}^M$, as defined in Thm. 4.1,*

$$\mathbf{y} = \mathbf{U}_M^H \mathbf{x} = \mathbf{C}\tilde{\mathbf{y}} = \mathbf{C}\prod_{j=0}^{J-1}(\mathbf{\Psi}_{j\downarrow 2}\mathbf{E}_{2^j\vec{\alpha}})\mathbf{x} \qquad (12)$$

whereby $\mathbf{C} \in \mathbb{C}^{M\times\tilde{M}}$ is a coefficient matrix, $\mathbf{\Psi}_{j\downarrow 2} \in \mathbb{R}^{N/2^{j+1}\times N/2^j}$ the binary sampling matrix, and $\mathbf{E}_{2^j\vec{\alpha}} \in \mathbb{R}^{N/2^j\times N/2^j}$ a graph e-spline low-pass filter on $\tilde{G}_j$, with $\vec{\alpha} = (\alpha_1, .., \alpha_M) = \left(0, ..., \frac{2\pi(M-1)}{N}\right)$. At each level $j \leq J$, we define eigenbasis $(\tilde{\mathbf{U}}_j, \tilde{\mathbf{\Lambda}}_j) \in \mathbb{C}^{N/2^j\times N/2^j}$ through the projection of $\mathbf{\Psi}_{j-1\downarrow 2}$ on $(\tilde{\mathbf{U}}_{j-1}, \tilde{\mathbf{\Lambda}}_{j-1})$ (see Lemma 4.1), with $\tilde{M} = \frac{N}{2^J}$, given M. Thus the coarser graph $\tilde{G}_j$, with graph signal $\tilde{\mathbf{y}}_j = \prod_{k=0}^{j-1}(\mathbf{\Psi}_{k\downarrow 2}\mathbf{E}_{2^k\vec{\alpha}})\mathbf{x}$, has adjacency matrix

$$\mathbf{A}_j = (2^j/N)\tilde{\mathbf{U}}_j\tilde{\mathbf{\Lambda}}_j\tilde{\mathbf{U}}_j^H$$

which preserves the generating set S of G for a sufficiently small bandwidth.

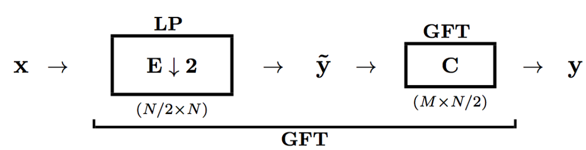

Fig. 2. Sampling Scheme with One Level of Coarsening

Hereby, we iterate for a multiresolution representation, where $\mathbf{E}_{2^j\vec{\alpha}}$ is designed to reproduce complex exponential graph signals with $\pm 2^j\vec{\alpha}$ (based on Thm. 3.3), ensuring that the necessary conditions are satisfied at each level; this restricts the number of levels J and reproducible rows M of $\mathbf{U}^H$, whose discussion we omit for brevity. In addition, we note that $\mathbf{C}$ is the product between a diagonal coefficient matrix and the reduced DFT (GFT) $\tilde{\mathbf{U}}_M^H \in \mathbb{C}^{M\times N/2^J}$, justifying the interpretation that the respective filtering of $\mathbf{x}$ and $\tilde{\mathbf{y}}$ with reduced GFT bases of different dimensions is equivalent.

5. CONCLUSION

We have introduced novel families of graph e-spline wavelets and associated filterbanks which reveal e-spline-like functions on circulant graphs and incorporate annihilation and reproduction properties for complex exponential graph signals, complementing our previous work on graph spline wavelets. Based on the derived constructions, we have established the graph-FRI framework of sampling and reconstruction with graph coarsening on circulant graphs for classes of (wavelet-)sparse graph signals. In future work, we aim to investigate the existence of broader classes of graph signals which can be annihilated by existing and/or evolved GWT constructions.

6. REFERENCES

[1] D. I Shuman, S. K. Narang, P. Frossard, A. Ortega, and P. Vandergheynst, "The emerging field of signal processing on graphs: Extending high-dimensional data analysis to networks and other irregular domains," *IEEE Signal Process. Mag.*, vol. 30, no. 3, pp. 83–98, 2013.

[2] F. R. K. Chung, *Spectral Graph Theory*, American Mathematical Society, 1997.

[3] A. Sandryhaila and J.M.F. Moura, "Discrete signal processing on graphs," *IEEE Transactions on Signal Processing*, vol. 61, no. 7, pp. 1644–1656, April 2013.

[4] R. R. Coifman and M. Maggioni, "Diffusion wavelets," *Applied and Computational Harmonic Analysis*, vol. 21, no. 1, pp. 53–94, 2006.

[5] D. K. Hammond, P. Vandergheynst, and R. Gribonval, "Wavelets on graphs via spectral graph theory," *Applied and Computational Harmonic Analysis*, vol. 30, no. 2, pp. 129 – 150, 2011.

[6] S.K. Narang and A. Ortega, "Perfect reconstruction two-channel wavelet filter banks for graph structured data," *Signal Processing, IEEE Transactions on*, vol. 60, no. 6, pp. 2786–2799, June 2012.

[7] S.K. Narang and A. Ortega, "Compact support biorthogonal wavelet filterbanks for arbitrary undirected graphs," *IEEE Transactions on Signal Processing*, vol. 61, no. 19, pp. 4673–4685, Oct 2013.

[8] V. N. Ekambaram, G. C. Fanti, B. Ayazifar, and K. Ramchandran, "Circulant structures and graph signal processing," in *IEEE International Conference on Image Processing, ICIP 2013*, Sept 2013, pp. 834–838.

[9] V. N. Ekambaram, G. Fanti, B. Ayazifar, and K. Ramchandran, "Critically-sampled perfect-reconstruction spline-wavelet filterbanks for graph signals," in *IEEE Global Conference on Signal and Information Processing, GlobalSIP 2013*, Dec 2013, pp. 475–478.

[10] V. Ekambaram, *Graph Structured Data Viewed Through a Fourier Lens*, Ph.D. thesis, EECS Department, University of California, Berkeley, Dec 2013.

[11] M. S. Kotzagiannidis and P. L. Dragotti, "Higher-order graph wavelets and sparsity on circulant graphs," in *Proc. SPIE, Wavelets and Sparsity XV*, 2015, to appear.

[12] P. L. Dragotti, M. Vetterli, and T. Blu, "Sampling moments and reconstructing signals of finite rate of innovation: Shannon meets strang-fix," *IEEE Trans. Signal Process.*, vol. 55, no. 5, pp. 1741–1757, May 2007.

[13] M. S. Kotzagiannidis and P. L. Dragotti, "Sparse graph signal reconstruction and image processing on circulant graphs," in *IEEE GlobalSIP*, 2014, pp. 923–927.

[14] A. Anis, A. Gadde, and A. Ortega, "Towards a sampling theorem for signals on arbitrary graphs," in *Acoustics, Speech and Signal Processing (ICASSP), 2014 IEEE International Conference on*, May 2014, pp. 3864–3868.

[15] A. Sandryhaila S. Chen, R. Varma and J. Kovacevic, "Discrete signal processing on graphs: Sampling theory," *CoRR*, vol. abs/1503.05432, 2015.

[16] I. Khalidov and M. Unser, "From differential equations to the construction of new wavelet-like bases," *Signal Processing, IEEE Transactions on*, vol. 54, no. 4, pp. 1256–1267, April 2006.

[17] C. Vonesch, T. Blu, and M. Unser, "Generalized daubechies wavelet families," *Signal Processing, IEEE Transactions on*, vol. 55, no. 9, pp. 4415–4429, Sept 2007.

[18] J. A. Urigüen, P. L. Dragotti, and T. Blu, "On the exponential reproducing kernels for sampling signals with finite rate of innovation," in *Proceedings of the 9th International Workshop on SampTA*, 2011.

[19] J.A. Uriguen, T. Blu, and P.L. Dragotti, "Fri sampling with arbitrary kernels," *IEEE Transactions on Signal Processing*, vol. 61, no. 21, pp. 5310–5323, Nov 2013.