Inverse Problems Regularized by Sparsity

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Abstract: Modelling signals as sparse in a proper domain has proved useful in many signal processing tasks and, in this paper, we show how sparsity can be used to solve inverse problems. We first recall that many inverse problems involve the reconstruction of continuous-time or continuous-space signals from discrete measurements and show how to relate the discrete measurements to some properties of the original signal (e.g., its Fourier transform at specific frequencies or its first L moments). Given this partial knowledge of the original signal, we then solve the inverse problem using sparsity. We focus on two specific problems which have important practical implications: localization of diffusion sources from sensor measurements and reconstrcution of planar domains from samples. We localize diffusion sources using a variation of the 'reciprocity gap' method and use it also to estimate the activation time of the source. We validate the method by estimating the location and activation time of a heat source from real measurements. Finally, we show how to reconstruct specific planar domains which are driven by sparsity models and apply this approach to enhance the resolution of natural images.

Keywords: Prony's method, sparsity, analytic functions

1 Introduction

The notion of sparsity, namely the idea that signals can be modelled using a small number of free parameters has proved useful in many signal processing applications and recently sparsity has been successfully used in sampling. In these new sampling methods, the prior that the signal is sparse in a basis or in a parametric space is used to perfectly reconstruct classes of non-bandlimited signals from a set of suitable measurements. Depending on the set-up and reconstruction method involved, the above sparse sampling problem goes under different names like compressed sensing, compressive sampling [1, 2] or sampling signals with finite rate of innovation (FRI) [3, 4].

Sampling can be seen as a particular type of inverse problem where one tries to reconstruct a certain phenomenon or function from a set of discrete measurements. There are two types of inverse problem of this nature that we consider in this paper.

We first put ourselves in the typical sampling setup depicted in Fig. 1 where the original continuous-time signal $x(t)$ is filtered before being (uniformly) sampled with sampling period T. If we call $y(t) = h(t) * x(t)$ the filtered version of $x(t)$, the samples y_n are given by $y_n = \langle x(t), \varphi(t/T - n)\rangle$ where the sampling kernel $\varphi(t)$ is the scaled and time-reversed version of $h(t)$. In this paper we discuss extensions of this framework to the two-dimensional (2-D) case, we thus assume that the incoming signal is a 2-D function $f(x, y)$ and try to reconstruct it from the discrete measurements $y_{m,n} = \langle f(x, y), \varphi(x/T - m, y/T - n)\rangle$.

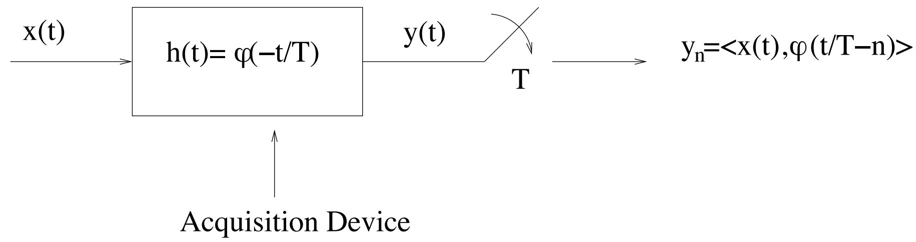


Figure 1. *Sampling set-up. Here, $x(t)$ is the continous-time signal, $h(t)$ the impulse response of the acquisition device and T the sampling period. The measured samples are $y_n = \langle x(t), \varphi(t/T - n)\rangle$.*

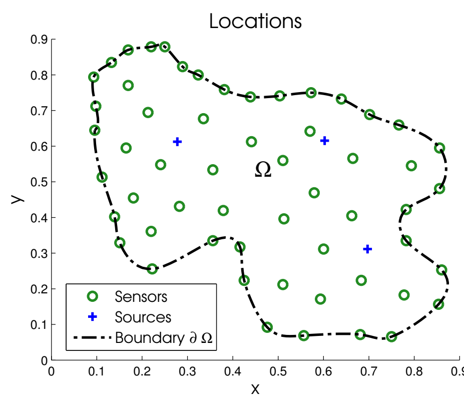


Figure 2. *Estimation of diffusion fields driven by localised sources using a sensor network.*

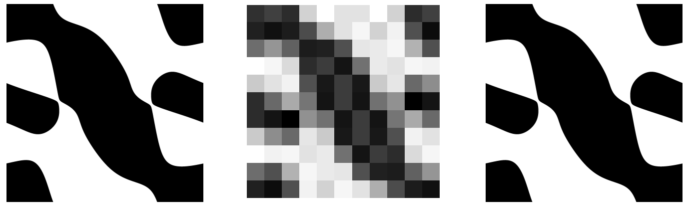
The second inverse problem we consider is depicted in Fig. 2. Here a sensor network is monitoring a diffusion field inside a region Ω and the task is to reconstruct the entire field from the spatio-temporal measurements given by the sensors under the assumption that the field is driven by M localised diffusion sources.

In both cases we solve the inverse problem by retrieving some continuous full-field information about the original signal/phenomenon and then reconstruct them using proper sparsity priors. Our methods are heavily influenced by the theory of sampling FRI signals introduced in [3, 4] and extended more recently in [5, 6, 7, 8, 9, 10, 11]. FRI sampling theory has also had impact in other applications such as image super-resolution [12], for depth sensing [13], for calcium transient detection [14] and in compression [15, 16, 17].

In what follows, we first discuss the problem of reconstructing 2-D domains from samples then in Section3 we provide an overview on how to reconstruct diffusion field from sensor measurements.

2 Reconstructing classes of 2-D domains from discrete measurements

For the sake of clarity, we begin by considering the 1-D case and the sampling set-up of Fig. 1. We want to show how we can retrieve some information about the Fourier transform of $x(t)$ from the samples y_n. The acquisition device or sampling kernel plays a central role in this context and a family of kernels that we will be considering is the family of exponential reproducing functions. A function $\varphi(t)$ is an exponential reproducing function of order P, if together with its shifted



(a) (b) (c)

Figure 3. *Sampling and reconstruction of classes of 2-D domains. Here, in (a) the original domain is acquired using exponential reproducing kernels. This leads to the samples of part (b). From these samples the Fourier transform of the original signal at specific frequencies is obtained and then the domain is perfectly reconstructed as shown in part (c). We refer to [19] for more details.*

versions, it is able to reproduce exponentials:

$$\sum_{n\in\mathbb{Z}} c_{m,n}\varphi(t-n) = \mathrm{e}^{j\omega_m t}, \tag{1}$$

for proper coefficients $c_{m,n}$, with $m = 0, \ldots, P$ and $\omega_m \in \mathbb{R}$. For the sake of argument, we are restricting our discussion to exponentials with purely imaginary exponents, however, the analysis can be extended to exponentials with arbitrary complex exponents. It is possible to show that a function satisfies (1) if and only if it meets the generalised Strang-Fix conditions [18]:

$$\hat{\varphi}(j\omega_m) \neq 0 \text{ and } \hat{\varphi}(j\omega_m + j2\pi l) = 0 \qquad l \in \mathbb{Z}\setminus\{0\}$$

where $\hat{\varphi}(\cdot)$ is the Fourier transform of $\varphi(t)$.

Exponential reproducing kernels are important because they allow us to map the samples y_n with the Fourier transform of $x(t)$ at $j\omega_m$ $m = 0, 1, .., P$ and this independently of the property of the incoming signal. For the sake of clarity, assume that the signal $x(t)$ has compact support such that it is characterised by only N non-zero samples. Moreover, assume that $T = 1$. We thus have that the N samples are of the form $y_n = \langle x(t), \varphi(t-n)\rangle$, $n = 0, 1, .., N-1$.

We now linearly combine the samples y_n using the coefficients $c_{n,m}$ of Eq. (1) to obtain:

$$\begin{aligned} s_m &= \textstyle\sum_{n=0}^{N-1} c_{m,n} y_n \\ &\overset{(a)}{=} \langle x(t), \textstyle\sum_{n=0}^{N-1} c_{m,n}\varphi(t-n)\rangle \\ &\overset{(b)}{=} \textstyle\int_{-\infty}^{\infty} x(t) e^{j\omega_m t} dt, \qquad m = 0, 1, .., P, \end{aligned}$$

where (a) follows from the linearity of the inner product and (b) is due to Eq. (1) and to the fact that $x(t)$ has compact support.

We note that $\int_{-\infty}^{\infty} x(t)e^{j\omega_m t}dt = \hat{x}(j\omega_m)$ is precisely the Fourier transform of $x(t)$ evaluated at $j\omega_m$, $m = 0, 1, .., P$.

The above derivation, therefore, shows that it is possible to obtain a partial knowledge of the continuous-time Fourier transform of the original signal from proper discrete samples. A similar derivation can be applied to the 2-D scenario showing that a partial knowledge of the Fourier transform of $f(x, y)$ can be obtained from the samples $y_{m,n}$.

We are now faced with the more traditional problem of estimating the entire signal from this partial knowledge. This can be achieved by assuming that the original signal is sparse in a proper domain. In [19], we introduced a class of 2-D domains whose contour can be described using a small number of parameters. These domains are therefore sparse and an example of how perfect reconstruction can be achieved from the samples is shown in Fig. 3.

3 Inversion of Diffusion Fields

The propagation of a diffusion field follows the diffusion equation:

$$\frac{\partial}{\partial t}u(\mathbf{x},t) = \mu\nabla^2 u(\mathbf{x},t) + f(\mathbf{x},t), \tag{2}$$

where $u(\mathbf{x},t)$ is the field and μ is the diffusivity of the medium through which the field propagates. We assume that the field is monitored by a sensor network over a 2-D region Ω as shown in Fig. 2. Moreover we assume the field is generated by M sources localised in space and time. Therefore the source can be written as follows:

$$f(\mathbf{x},t) = \sum_{m=1}^{M} c_m \delta(\mathbf{x}-\xi_m, t-\tau_m). \tag{3}$$

Given the above assumption, the inversion problem reduces to the problem of retrieving the location and activation time of the sources from the sensor measurements. This is because once $f(\mathbf{x},t)$ has been estimated, the field $u(\mathbf{x},t)$ can be obtained by convolving f with the heat kernel. The problem is therefore sparse, because the whole field is driven by a small number of free parameters.

As in the previous application, we want to estimate some full-field measurements of the diffusion field from the spatio-temporal sensor readings. In the previous case, we used the exponential reproduction formula to have an exact mapping between the samples and the Fourier transform of the original signal at specific frequencies. In this new case, this is not possible and we can only obtain approximate full-field measurements. The aim is to estimate the generalized measurements of the form:

$$\mathcal{Q}(k) = \langle \Psi_k(\mathbf{x})\Gamma(t), f\rangle = \int_\Omega\int_t \Psi_k(\mathbf{x})\Gamma(t) f(\mathbf{x},t)\mathrm{d}t\mathrm{d}V, \tag{4}$$

where $\Psi_k(\mathbf{x}) = e^{-k(x_1+jx_2)}$ and $\Gamma(t)$ is a properly chosen window. By replacing (3) into (4), we obtain:

$$\mathcal{Q}(k) = \sum_{m=1}^{M} c_m \Gamma(\tau_m) e^{-k(\xi_{1,m}+j\xi_{2,m})}.$$

This is a sum of exponentials and the source locations can then be estimated from this sum using Prony's method - a method commonly used in array signal processing [20]. The activation times are estimated in a similar way [21]. It is possible to show [21] that these generalised samples can be obtained from the boundary and interior sensor measurements when Ψ_k is analytic, which is the case here. We refer to [21] for further details.

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