EE1 and ISE1 Communications I

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Lecture eight

Lecture Aims

- To introduce Energy spectral density (ESD),
- Input and Output Energy spectral densities,
- To introduce Power spectral density (PSD),
- Input and Output Power spectral densities.

Signal Energy, Parseval's Theorem

Consider an energy signal $g(t)$, Parseval's Theorem states that

$$E_g = \int_{-\infty}^{\infty} |g(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |G(\omega)|^2 d\omega$$

Proof:

$$\begin{aligned} E_g &= \textstyle\int_{-\infty}^{\infty} g(t) g^*(t) dt = \int_{-\infty}^{\infty} g(t) \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} G^*(\omega) e^{-j\omega t} d\omega \right] dt \\ &= \textstyle\frac{1}{2\pi} \int_{-\infty}^{\infty} G^*(\omega) \left[\int_{-\infty}^{\infty} g(t) e^{-j\omega t} dt \right] d\omega \\ &= \textstyle\frac{1}{2\pi} \int_{-\infty}^{\infty} G(\omega) G^*(\omega) d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} |G(\omega)|^2 d\omega \end{aligned}$$

Example

Consider the signal $g(t) = e^{-at}u(t)$ ($a > 0$).
Its energy is

$$E_g = \int_{-\infty}^{\infty} g^2(t)dt = \int_0^{\infty} e^{-2at}dt = \frac{1}{2a}$$

We now determine E_g using the signal spectrum $G(\omega)$ given by

$$G(\omega) = \frac{1}{j\omega + a}$$

It follows

$$E_g = \frac{1}{2\pi}\int_{-\infty}^{\infty} |G(\omega)|^2 d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{1}{\omega^2 + a^2} d\omega = \frac{1}{2\pi a}[tan^{-1}\frac{\omega}{a}]_{-\infty}^{\infty} = \frac{1}{2a}$$

which verifies Parseval's theorem.

Energy Spectral Density

- Parseval's theorem can be interpreted to mean that the energy of a signal $g(t)$ is the result of energies contributed by all spectral components of a signal $g(t)$.

- The contribution of a spectral component of frequency ω is proportional to $|G(\omega)|^2$.

- Therefore, we can interpret $|G(\omega)|^2$ as the energy per unit bandwidth of the spectral components of $g(t)$ centered at frequency ω.

- In other words, $|G(\omega)|^2$ is the energy spectral density of $g(t)$

Energy Spectral Density (continued)

The **energy spectral density (ESD)** $\psi(t)$ is thus defined as

$$\Psi(\omega) = |G(\omega)|^2$$

and

$$E_g = \frac{1}{2\pi}\int_{-\infty}^{\infty} \Psi(\omega) d\omega$$

Thus, the ESD of the signal $g(t) = e^{-at}u(t)$ of the previous example is

$$\Psi(\omega) = |G(\omega)|^2 = \frac{1}{\omega^2 + a^2}$$

Energy of modulated signals (important)

Let $g(t)$ be a baseband energy signal with energy E_g.

The energy of the modulated signal $\varphi(t) = g(t)\cos\omega_0 t$ is half the energy of $g(t)$. That is,

$$E_\varphi = \frac{1}{2}E_g.$$

Proof: See attached notes.

The same applies to power signals. That is, if $g(t)$ is a power signal then

$$P_\varphi = \frac{1}{2}P_g.$$

(You will use this result when computing the efficiency of a Full AM system).

Time Autocorrelation Function and ESD

For a real signal the autocorrelation function $\psi_g(t)$ is defined as

$$\psi_g(\tau) = \int_{\infty}^{\infty} g(t)g(t+\tau)dt$$

Do you remember the correlation of two signals (lecture three)? The autocorrelation function measure the correlation between $g(t)$ and all its translated versions.
Notice

$$\psi_g(\tau) = \psi_g(-\tau).$$

and

$$g(\tau) * g(-\tau) = \psi_g(\tau).$$

But, most important...

Time Autocorrelation Function and ESD

...the Fourier transform of the autocorrelation function is the Energy Spectral Density! That is

$$\psi_g(\tau) \Longleftrightarrow \Psi(\omega) = |G(\omega)|^2$$

Proof:

$$\begin{aligned}\mathcal{F}[\psi_g(\tau)] &= \int_{-\infty}^{\infty} e^{-j\omega\tau}[\int_{-\infty}^{\infty} g(t)g(t+\tau)dt]d\tau \\ &= \int_{-\infty}^{\infty} g(t)[\int_{-\infty}^{\infty} g(\tau+t)e^{-j\omega\tau}d\tau]dt\end{aligned}$$

The Fourier transform of $g(\tau+t)$ is $G(\omega)e^{j\omega t}$. Therefore,

$$\mathcal{F}[\psi_g(\tau)] = G(\omega)\int_{-\infty}^{\infty} g(t)e^{j\omega t}dt = G(\omega)G(-\omega) = |G(\omega)|^2$$

ESD of the Input and the Output

If $g(t)$ and $y(t)$ are the input and the corresponding output of a LTI system, then

$$Y(\omega) = H(\omega)G(\omega).$$

Therefore,

$$|Y(\omega)|^2 = |H(\omega)|^2|G(\omega)|^2.$$

This shows that

$$\Psi_y(\omega) = |H(\omega)|^2\Psi_g(\omega).$$

Thus, the output signal ESD is $|H(\omega)|^2$ the input signal ESD.

Signal Power and Power Spectral Density

The power P_g of a real signal $g(t)$ is given by

$$P_g = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} g^2(t) dt.$$

All the results for energy signals can be extended to power signals.
Call $S_g(\omega)$ the Power Spectral Density (PSD) of $g(t)$. Thus,

$$P_g = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_g(\omega) d\omega.$$

$S_g(\omega)$ can be found using the autocorrelation function.

Time autocorrelation Function of Power Signals

The (time) autocorrelation function $\mathcal{R}_g(\tau)$ of a real deterministic power signal $g(t)$ is defined as

$$\mathcal{R}_g(\tau) = \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} g(t)g(t+\tau)dt$$

We have that

$$\mathcal{R}_g(\tau) \Longleftrightarrow S_g(\omega)$$

If $g(t)$ and $y(t)$ are the input and the corresponding output of a LTI system, then

$$S_y(\omega) = |H(\omega)|^2 S_g(\omega).$$

Thus, the output signal PSD is $|H(\omega)|^2$ the input signal PSD.

Conclusions

We learned about

- Energy and Power Spectral Densities,
- Time autocorrelation functions,
- Input and output energies and powers.