EE1 and ISE1 Communications I

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Lecture fourteen

Lecture Aims

- To verify bandwidth calculations for FM using single tone modulating signals

Verification of FM bandwidth

- To verify Carson's rule

$$B_{FM} = 2(\Delta f + B) = 2\left(\frac{k_f m_p}{2\pi} + B\right)$$

- Consider a single tone modulating sinusoid

$$m(t) = \alpha \cos \omega_m t \qquad a(t) = \int_{-\infty}^{t} m(\tau) d\tau = \frac{\alpha}{\omega_m} \sin \omega_m t$$

- We can express the FM signal as

$$\hat{\varphi}_{FM}(t) = Ae^{j\left(\omega_c t + \frac{k_f \alpha}{\omega_m} \sin \omega_m t\right)}$$

Verification of FM bandwidth

- The angular frequency deviation is $\Delta\omega = k_f m_p = \alpha k_f$
- Since the bandwidth of $m(t)$ is $B = f_m$ Hz, the frequency deviation ratio (or modulation index) is

$$\beta = \frac{\Delta f}{f_m} = \frac{\Delta\omega}{\omega_m} = \frac{\alpha k_f}{\omega_m}$$

- Hence the FM signal becomes

$$\hat{\varphi}_{FM}(t) = Ae^{(j\omega_c t + j\beta \sin \omega_m t)} = Ae^{j\omega_c t}(e^{j\beta \sin \omega_m t})$$

Verification of FM bandwidth

The exponential term $e^{j\beta \sin \omega_m t}$ is a periodic signal with period $2\pi/\omega_m$ and can be expanded by the exponential Fourier series:

$$e^{j\beta \sin \omega_m t} = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_m t}$$

where

$$C_n = \frac{\omega_m}{2\pi} \int_{-\pi/\omega_m}^{\pi/\omega_m} e^{j\beta \sin \omega_m t} e^{-jn\omega_m t} dt$$

Bessel functions

By changing variables $\omega_m t = x$, we get

$$C_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{(j\beta \sin x - nx)} dx$$

This integral is denoted as the Bessel function $J_n(\beta)$ of the first kind and order n. It cannot be evaluated in closed form but it has been tabulated.

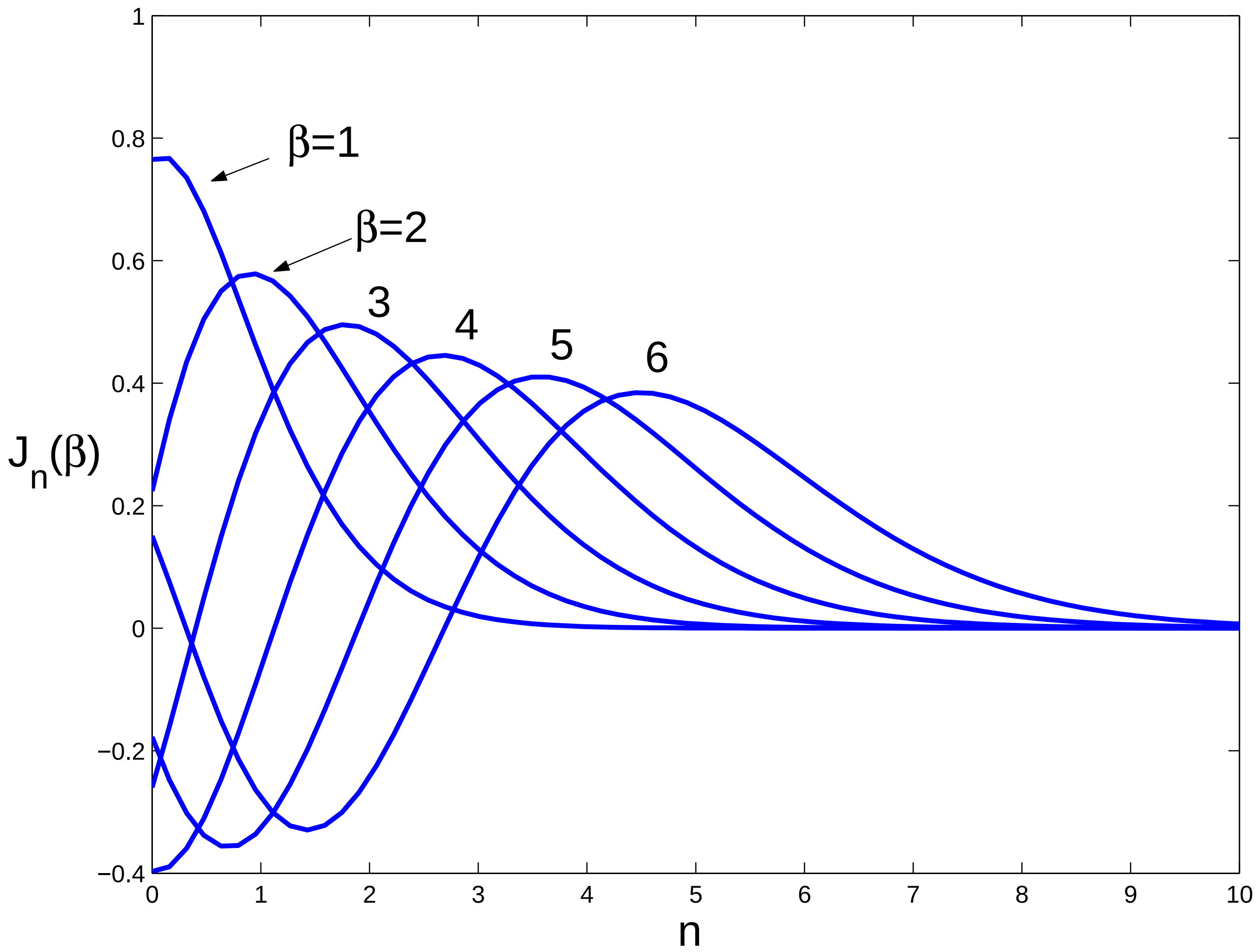
Hence the FM waveform can be expressed as

$$\hat{\varphi}_{FM}(t) = A \sum_{n=-\infty}^{\infty} J_n(\beta) e^{(j\omega_c t + jn\omega_m t)}$$

and

$$\varphi_{FM}(t) = A \sum_{n=-\infty}^{\infty} J_n(\beta) \cos(\omega_c + n\omega_m)t$$

Bessel functions of the first kind



Bandwidth calculation for FM

The FM signal for single tone modulation is

$$\varphi_{FM}(t) = A \sum_{n=-\infty}^{\infty} J_n(\beta) \cos(\omega_c + n\omega_m)t.$$

The modulated signal has 'theoretically' an infinite bandwidth made of one carrier at frequency ω_c and an infinite number of sidebands at frequencies $\omega_c \pm \omega_m$, $\omega_c \pm 2\omega_m$, ..., $\omega_c \pm n\omega_m$, ... However

- for a fixed β, the amplitude of the Bessel function $J_n(\beta)$ decreases as n increases. This means that for any fixed β there is only a finite number of significant sidebands.
- As $n > \beta + 1$ the amplitude of the Bessel function becomes negligible. Hence, the number of significant sidebands is $\beta + 1$.

This means that with good approximation the bandwidth of the FM signal is

$$B_{FM} = 2nf_m = 2(\beta + 1)f_m = 2(\Delta f + B).$$

Example

Estimate the bandwidth of the FM signal when the modulating signal is the one shown in Fig. 1 with period $T = 2 \times 10^{-4}$sec, the carrier frequency is $f_c = 100$MHz and $k_f = 2\pi \times 10^5$.

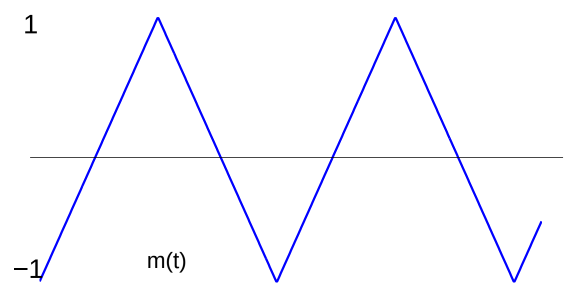


Figure 1: The modulating signal $m(t)$

Repeat the problem when the amplitude of $m(t)$ is doubled.

Example

- Peak amplitude of $m(t)$ is $m_p = 1$.
- Signal period is $T = 2 \times 10^{-4}$, hence fundamental frequency is $f_0 = 5$kHz.
- We assume that the essential bandwidth of $m(t)$ is the third harmonic. Hence the modulating signal bandwidth is $B = 15$kHz.
- The frequency deviation is:

$$\Delta f = \frac{1}{2\pi} k_f m_p = \frac{1}{2\pi}(2\pi \times 10^5)(1) = 100\text{kHz}.$$

- Bandwidth of the FM signal:

$$B_{FM} = 2(\Delta f + B) = 230\text{kHz}.$$

Example

- Doubling amplitude means that $m_p = 2$.
- The modulating signal bandwidth remains the same, i.e., $B = 15$kHz.
- The new frequency deviation is:

$$\Delta f = \frac{1}{2\pi} k_f m_p = \frac{1}{2\pi}(2\pi \times 10^5)(2) = 200\text{kHz}.$$

- The new bandwidth of the FM signal is:

$$B_{FM} = 2(\Delta f + B) = 430\text{kHz}.$$

Example

Now estimate the bandwidth of the FM signal if the modulating signal is time expanded by a factor 2.

- The time expansion by a factor 2 reduces the signal bandwidth by a factor 2. Hence the fundamental frequency is now $f_0 = 2.5$kHz and $B = 7.5$kHz.
- The peak value stays the same, i.e., $m_p = 1$ and

$$\Delta f = \frac{1}{2\pi} k_f m_p = \frac{1}{2\pi}(2\pi \times 10^5)(1) = 100\text{kHz}.$$

- The new bandwidth of the FM signal is:

$$B_{FM} = 2(\Delta f + B) = 2(100 + 7.5) = 215\text{kHz}.$$

Second Example

An angle modulated signal with carrier frequency $\omega_c = 2\pi \times 10^5$rad/s is given by:

$$\varphi_{FM}(t) = 10\cos(\omega_c t + 5\sin 3000t + 10\sin 2000\pi t).$$

- Find the power of the modulated signal
- Find the frequency deviation Δf
- Find the deviation ration $\beta = \frac{\Delta f}{B}$
- Estimate the bandwidth of the FM signal.

Second Example

- The carrier amplitude is 10 therefore the power is $P = 10^2/2 = 50$.
- The signal bandwidth is $B = 2000\pi/2\pi = 1000$Hz.
- To find the frequency deviation we find the instantaneous frequency:

$$\omega_i = \frac{d}{dt}\theta(t) = \omega_c + 15,000\cos 3000t + 20,000\pi\cos 2000\pi t.$$

The angle deviation is the maximum of $15,000\cos 3000t + 20,000\pi\cos 2000\pi t$. The maximum is: $\Delta\omega = 15,000 + 20,000\pi$rad/s. Hence, the frequency deviation is

$$\Delta f = \frac{\Delta\omega}{2\pi} = 12,387.32\text{Hz}.$$

- The modulation index is

$$\beta = \frac{\Delta f}{B} = 12.387.$$

- The bandwidth of the FM signal is: $B_{FM} = 2(\Delta f + B) = 26,774.65$Hz.

Conclusions

- Verified bandwidth calculation for FM using single tone modulating signal.
- Examined Bessel functions and their properties.
- Examined two examples and calculated FM bandwidths.