EE1 and ISE1 Communications I

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Lecture thirteen

Lecture Aims

- To Study the bandwidth of angle modulated waves
  - Narrow-Band Angle Modulation
  - Carson's rule

Bandwidth of Angle Modulated waves

In order to study bandwidth of FM waves, define

$$a(t) = \int_{-\infty}^{t} m(\alpha) d\alpha$$

and

$$\hat{\phi}_{FM}(t) = Ae^{j[\omega_c t + k_f a(t)]} = Ae^{jk_f a(t)} e^{j\omega_c t}$$

The frequency modulated signal is

$$\phi_{FM}(t) = \text{Re}\{\hat{\phi}_{FM}(t)\}$$

Bandwidth of Angle Modulated waves

Expanding the exponential $e^{jk_f a(t)}$ in power series yields

$$\hat{\phi}_{FM}(t) = A\left[1 + jk_f a(t) - \frac{k_f^2}{2!}a^2(t) + \cdots + j^n\frac{k_f^n}{n!}a^n(t) + \cdots\right]e^{j\omega_c t}$$

and

$$\begin{aligned}\phi_{FM}(t) &= \text{Re}\{\hat{\phi}_{FM}(t)\} \\ &= A\left[\cos\omega_c t - k_f a(t)\sin\omega_c t - \tfrac{k_f^2}{2!}a^2(t)\cos\omega_c t + \tfrac{k_f^3}{3!}a^3(t)\sin\omega_c t + \cdots\right]\end{aligned}$$

Narrow-Band Angle Modulation

The signal $a(t)$ is the integral of $m(t)$. It can be shown that if $M(\omega)$ is bandlimited to B, $A(\omega)$ is also bandlimited to B.
If $|k_f a(t)| \ll 1$ then all but the first term are negligible and

$$\phi_{FM}(t) \simeq A[\cos \omega_c t - k_f a(t) \sin \omega_c t]$$

This case is called **narrow-band FM**.

Similarly, the **narrow-band PM** is given by

$$\phi_{PM}(t) \simeq A[\cos \omega_c t - k_p m(t) \sin \omega_c t]$$

Narrow-Band Angle Modulation

Comparison of narrow band FM with Full AM.

Narrow band FM

$$\phi_{FM}(t) \simeq A[\cos\omega_c t - k_f a(t)\sin\omega_c t]$$

Full AM

$$[A + m(t)]\cos\omega_c t = A\cos\omega_c t + m(t)\cos\omega_c t$$

Narrow band FM and full AM require a transmission bandwidth equal to $2B$ Hz. Moreover, the above equations suggests a way to generate narrowband FM or PM signals by using DSB-SC modulators.

Wide-Band FM

- Assume that $|k_f a(t)| \ll 1$ is not satisfied.

- Cannot ignore higher order terms, but power series expansion analysis becomes complicated.

- The precise characterization of the FM bandwidth is mathematically intractable.

- Use an empirical rule (Carson's rule) which applies to most signals of interests.

Bandwidth equation

- Take the angular frequency deviation as $\Delta\omega = k_f m_p$ and frequency deviation as $\Delta f = \frac{k_f m_p}{2\pi}$.

- The transmission bandwidth of an FM signal is, with good approximation, given by

$$B_{FM} = 2(\Delta f + B) = 2\left(\frac{k_f m_p}{2\pi} + B\right)$$

Carson's rule

- The formula
$$B_{FM} = 2(\Delta f + B) = 2\left(\frac{k_f m_p}{2\pi} + B\right)$$
goes under the name of **Carson's rule**.

- If we define frequency deviation ratio as $\beta = \frac{\Delta f}{B}$

- Bandwidth equation becomes

$$B_{FM} = 2B(\beta + 1)$$

Wide-Band PM

- All results derived for FM can be applied to PM.
- Angular frequency deviation $\Delta\omega = k_p \dot{m}_p$ and frequency deviation $\Delta f = \frac{k_p \dot{m}_p}{2\pi}$ where we assume $\dot{m}_p = [\dot{m}(t)]_{max}$ and $|\dot{m}(t)_{min}| = \dot{m}_p$.
- The bandwidth for the PM signal will be

$$B_{PM} = 2\left(\frac{k_p \dot{m}_p}{2\pi} + B\right) = 2(\Delta f + B)$$

Conclusions

Examined

- Narrowband FM
- Wideband FM and PM
- Carson's rule