E2.5 Signals & Linear Systems

Tutorial Sheet 1 – Introduction to Signals & Systems

(Lectures 1 & 2)

1. Sketch each of the following continuous-time signals, specify if the signal is periodic/non-periodic, odd/even. If the signal is periodic specify its period.

   (i) $x(t) = 2\sin(2\pi t)$

   (ii) $x(t) = \begin{cases} 3e^{-2t}, & t \geq 0 \\ 0, & t < 0 \end{cases}$

   (iii) $x(t) = 1/|t|$

   (iv) $x(t) = \sin(\frac{2\pi}{5}t) + \sin(\frac{2\pi}{3}t)$

   (v) $x(t) = \sin(2\pi t) + \sin(\sqrt{2}\pi t)$

2. Sketch the signal

$$x(t) = \begin{cases} 1-t, & 0 \leq t \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

   Now sketch each of the following and describe briefly in words how each of the signals can be derived from the original signal $x(t)$.

   (i) $x(t+3)$

   (ii) $x(t/3)$

   (iii) $x(t/3+1)$

   (iv) $x(-t+2)$

   (v) $x(-2t+1)$

3. Sketch each of the following signals. For each case, specify if the signal is periodic/non-periodic, odd/even. If the signal is periodic specify its period.

   (i) $x[n] = \cos(n\pi)$

   (ii) $x[n] = \begin{cases} 0.5^{-n}, & n \leq 0 \\ 0, & n > 0 \end{cases}$

   (iii) What is the maximum possible frequency of $e^{j\omega_0 n}$ (compare this result with the case $e^{j\omega_0 t}$)?

4. Consider the rectangular function

$$\Pi(t) = \begin{cases} 1, & |t| < 1/2 \\ 1/2, & |t| = 1/2 \\ 0, & \text{otherwise} \end{cases}$$

(i) Sketch $x(t) = \sum_{k=0}^{1} \Pi(t-k)$

(ii) Sketch $x(t) = \sum_{k=-\infty}^{+\infty} \Pi(t-k)$. (Hint: there is a simple way to express this signal.)

5. Consider a discrete-time signal $x[n]$, fed as input into a system. The system produces the discrete-time output $y[n]$ such that

$$y[n] = \begin{cases} x[n], & n \text{ even} \\ 0, & n \text{ odd} \end{cases}$$

(i) Is the system described above memoryless? Explain.
(ii) Is the system described above causal? Explain.
(iii) Are causal systems in general memoryless? Explain.
(iv) Is the system described above linear and time-invariant? Explain.

6. State with a brief explanation if the following systems are linear/non-linear, causal/non-causal, time-invariant/time-varying.

(i) $y[n] = x[n] - x[n-1]$

(ii) $y[n] = \text{sgn}(x[n])$

(iii) $y[n] = n^2 x[n+2]$

7. State with a brief explanation if the following systems are linear/non-linear, causal/non-causal, time-invariant/time-varying.

(i) $y(t) = x(t)\cos(2\pi f_o t + \phi)$

(ii) $y(t) = A\cos(2\pi f_o t + x(t))$

(iii) $y(t) = \int_{-\infty}^{t} x(\delta)d\delta$

(iv) $y(t) = x(2t)$

(v) $y(t) = x(-t)$

8. To understand better the periodicity properties of a discrete-time sine function, write an M-file in matlab that generates the function $y = \cos(2\pi f_0 n)$ where n=0,1,…,50 and f0 is arbitrary, then plot the function using either ‘plot’ or ‘stem’ commands.

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Tutorial Sheet 2 – System Responses

1. A Linear Time Invariant (LTI) system is specified by system equation

$$(D^2 + 4D + 4)\,y(t) = Df\,(t)$$

a) Find the characteristic polynomial, characteristic equation, characteristic roots and characteristic modes of this system.

b) Find $y_0(t)$, the zero-input component of the response y(t) for t ≥ 0, if the initial conditions are $y_0(0) = 3$, and $\dot{y}_0(0) = -4$.

2. Repeat question one with

$$D(D+1)\,y(t) = (D+2)\,f(t)$$

And initial conditions of $y_0(0) = 1$, and $\dot{y}_0(0) = 1$.

3. Repeat question one with

$$(D^2 + 9)\,y(t) = (3D+2)\,f(t)$$

And initial conditions of $y_0(0) = 0$, and $\dot{y}_0(0) = 6$.

4. Evaluate the following integrals:

a) $\int_{-\infty}^{\infty} f(\tau)\delta(t-\tau)d\tau$

b) $\int_{-\infty}^{\infty} \delta(\tau)f(t-\tau)d\tau$

c) $\int_{-\infty}^{\infty} \delta(t)e^{-j\omega t}dt$

d) $\int_{-\infty}^{\infty} \delta(t-2)\sin \pi t\, dt$

5. Find the unit impulse response of the LTI system specified by the equation

$$\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 3y(t) = \frac{dx}{dt} + 5x(t)\,.$$

6. Find the unit impulse response of the LTI system specified by the equation

$$(D+1)(D^2+5D+6)\,y(t) = (5D+9)\,f(t)\,.$$

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Tutorial Sheet 3 – Zero-state Responses & Convolution

1.* Using direct integration, find the expression for:

a) $y(t) = u(t) * u(t)$

b) $y(t) = e^{-at}u(t) * e^{-bt}u(t)$

c) $y(t) = tu(t) * u(t)$.

2.* Using direct integration, find:

a) $y(t) = \sin t\, u(t) * u(t)$

b) $y(t) = \cos t\, u(t) * u(t)$.

3.* The unit impulse response of an LTI system is $h(t) = e^{-t}u(t)$. Use the convolution table to find this system's zero-state response $y(t)$ if the input $f(t)$ is:

a) $u(t)$

b) $e^{-2t}u(t)$

c) $\sin 3t\, u(t)$

4.** By applying the shift property of convolution, find the system's response (i.e. zero-state response) given that $h(t) = e^{-t}u(t)$ and that the input $f(t)$ is as shown in Fig 4.1.

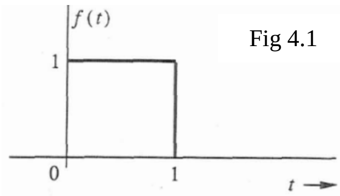


Fig 4.1

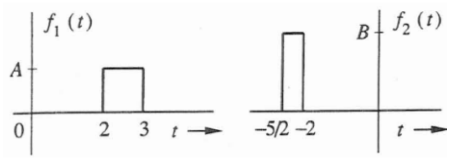
5.** A first-order allpass filter impulse response is given by

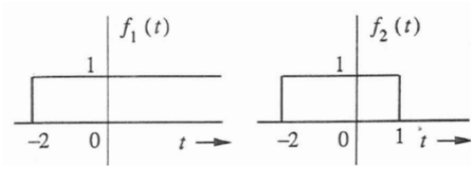
$$h(t) = -\delta(t) + 2e^{-t}u(t).$$

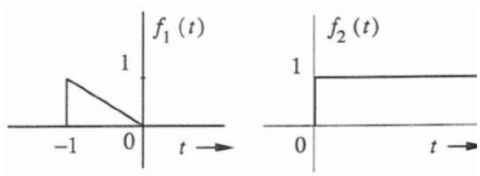
a) Find the zero-state response of this filter for the input $e^{t}u(-t)$.

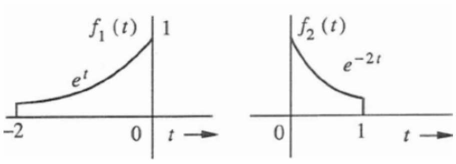
b) Sketch the input and the corresponding zero-state response.

6.** Find and sketch $c(t) = f_1(t) * f_2(t)$ for the pairs of functions shown as follow:

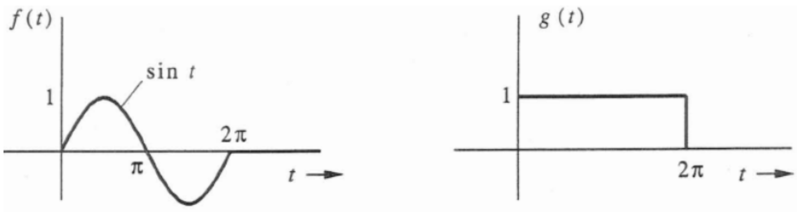
a) 

b) 

c) 

d) 

7.*** Find and sketch $c(t) = f(t) * g(t)$ for the pairs of functions shown below.



8. Matlab exercise. Write a routine in matlab that, given two functions x(t) and h(t), computes y(t)=h(t)*x(t). You may use this routine to verify the convolutions you have computed in 6 and 7.

Also remember that to implement an integral in matlab you have to replace it using trapezoidal approximation of integral. If you have two discrete-time sequences x[n],h[n] the discrete-time convolution is given by

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k]$$

This is slightly different from the formula that the trapezoidal rule would give you. Can you appreciate the difference? If you are lost you can always use and modify the m-files ‘graphical_conv.m’ which has been provided to you.

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Tutorial Sheet 4 – Laplace Transform

(Support Lecture 6)

1.* By direct integration, find the one-sided Laplace transforms of the following functions (please note that if not otherwise stated, we will always consider the unilateral Laplace transform):

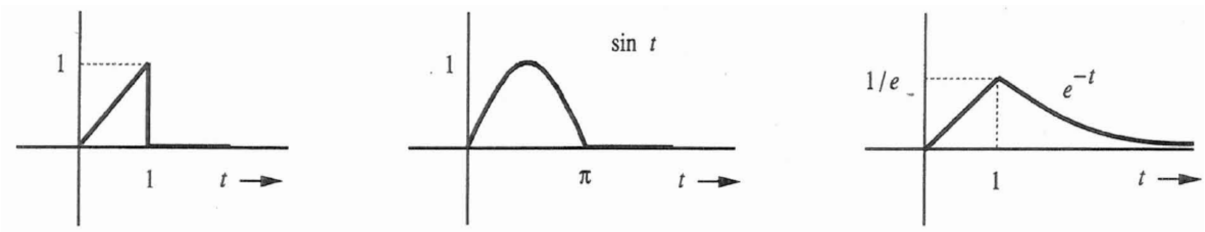
a) $u(t) - u(t-1)$

b) $te^{-t}u(t)$

c) $t\cos\omega_0 t\, u(t)$

d) $e^{-2t}\cos(5t+\theta)\, u(t)$

2.* By direct integration, find the Laplace transforms of the following signals:



(a) (b) (c)

3.* Find the inverse (one-sided) Laplace transforms of the following functions (from now on, if not stated otherwise, we always look for a causal solution):

a) $\dfrac{2s+5}{s^2+5s+6}$

b) $\dfrac{3s+5}{s^2+4s+13}$

c) $\dfrac{(s+1)^2}{s^2-s-6}$

d) $\dfrac{2s+1}{(s+1)(s^2+2s+2)}$

e) $\dfrac{s+3}{(s+2)(s+1)^2}$

4.** Find the Laplace transforms of the following function using the Laplace Transform Table and the time-shifting property where appropriate.

a) $u(t) - u(t-1)$

b) $e^{-(t-\tau)}u(t)$

c) $e^{-t}u(t-\tau)$

d) $\sin[\omega_0(t-\tau)]\,u(t-\tau)$

e) $\sin[\omega_0(t-\tau)]\,u(t)$

5.** Find the inverse Laplace transform of the function:

$$\frac{2s+5}{s^2+5s+6}e^{-2s}.$$

6.*** The Laplace transform of a causal periodic signal can be found from the knowledge of the Laplace transform of its first cycle alone.

a) If the Laplace transform of *f(t)* shown in Fig. 6 a) is *F(s)*, show that G(s), the Laplace transform of *g(t)* shown in Fig. 6 b) is given by:

$$G(s)=\frac{F(s)}{1-e^{-sT_0}} \qquad \text{Re}\, s>0$$

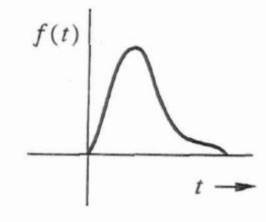


Fig 6 a)

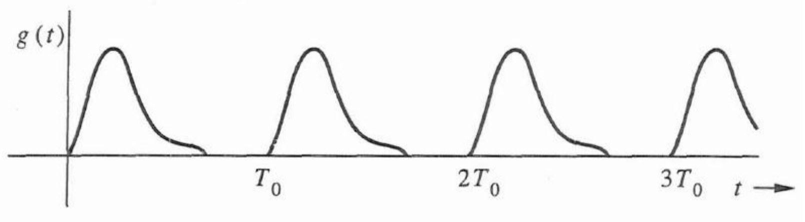
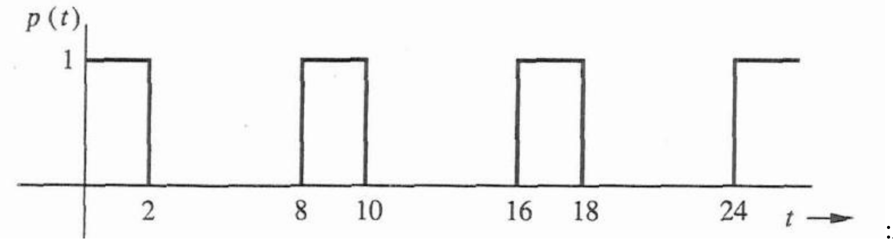


Fig 6 b)

b) Using the results in a), find the Laplace transform of the signal p(t) shown in Fig. 6 c).



:*Fig 6 c)*

(Hint: Remember that $1+x+x^2+x^3+\ldots=\frac{1}{1-x}$ for $|x|<1$.)

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Tutorial Sheet 5 – Laplace Transform & Frequency Response

(Lectures 7 - 9)

1.* Using Laplace transform, solve the following differential equations:

a) $(D^2+3D+2)y(t) = Df(t)$ if $y(0^-) = \dot{y}(0^-) = 0$ and $f(t) = u(t)$

b) $(D^2+4D+4)y(t) = (D+1)f(t)$ if $y(0^-) = 2, \dot{y}(0^-) = 1$ and $f(t) = e^{-t}u(t)$

c) $(D^2+6D+25)y(t) = (D+2)f(t)$ if $y(0^-) = \dot{y}(0^-) = 1$ and $f(t) = 25u(t)$.

2.* For each of the system described by the following differential equations, find the system transfer function.

a) $\dfrac{d^2y}{dt^2} + 11\dfrac{dy}{dt} + 24y(t) = 5\dfrac{df}{dt} + 3f(t)$

b) $\dfrac{d^3y}{dt^3} + 6\dfrac{d^2y}{dt^2} - 11\dfrac{dy}{dt} + 6y(t) = 3\dfrac{d^2f}{dt^2} + 7\dfrac{df}{dt} + 5f(t)$

c) $\dfrac{d^4y}{dt^4} + 4\dfrac{dy}{dt} = 3\dfrac{df}{dt} + 2f(t)$.

3.** For a system with transfer function

$$H(s) = \frac{s+5}{s^2+5s+6}$$

a) Find the zero-state response if the input *f(t)* is

(i) $e^{-4t}u(t)$ (ii) $e^{-3t}u(t)$ (iii) $e^{-4(t-5)}u(t-5)$

b) For this system write the differential equation relating the output *y(t)* to the input *f(t)*.

4.** For the circuit shown in Figure Q4, the switch is in open position for a long time before t = 0, when it is closed instantaneously.

a) Write loop equations in time domain for t ≥ 0.

b) Solve for $y_1(t)$ and $y_2(t)$ by taking the Laplace transform of loop equations found in part a).

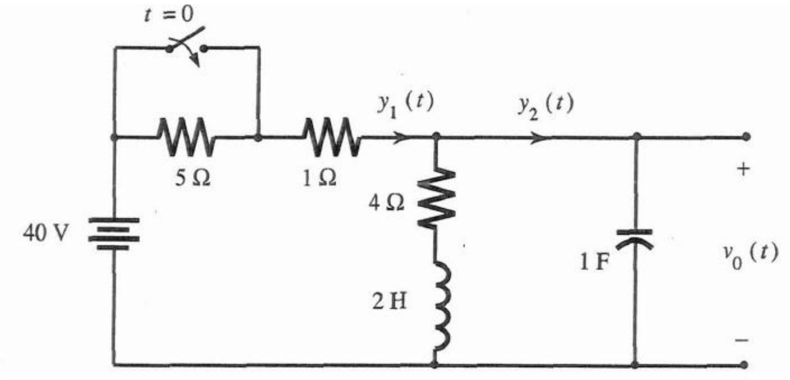

Fig. Q4

5.* Using the initial and final value theorems, find the initial and final values of the zero-state response of a system with the transfer function

$$H(s) = \frac{6s^2 + 3s + 10}{2s^2 + 6s + 5}$$

and the input is

a) $u(t)$

b) $e^{-t}u(t)$.

6.** For a LTI system described by the transfer function

$$H(s) = \frac{s+3}{(s+2)^2}$$

Find the system response to the following inputs:

a) $\cos(2t + 60^\circ)$

b) $\sin(3t - 45^\circ)$

c) e^{j3t}

7.** Using graphical method, draw a rough sketch of the amplitude and phase response of LTI systems whose pole-zero plots are shown in Fig. Q7(a) & (b).

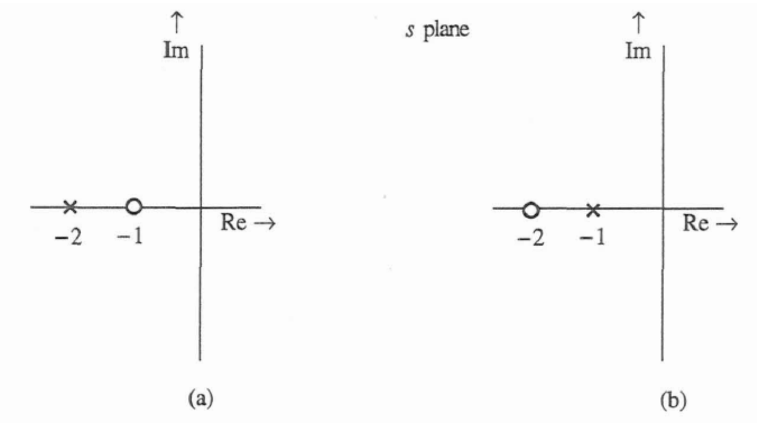

Fig. Q7

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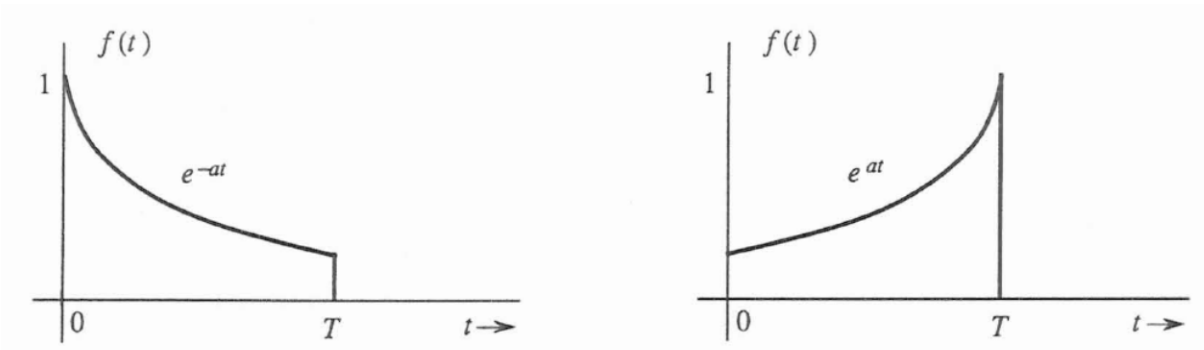
Tutorial Sheet 6 – Fourier Transform

1.* Derive the Fourier transform of the signals *f(t)* shown in Fig. Q1 (a) and (b).



(a) (b)

Figure Q1

2.* Sketch the following functions:

a) $rect(\frac{t}{2})$ b) $rect(\frac{t-10}{8})$

c) $\text{sinc}(\frac{\pi\omega}{5})$ d) $\text{sinc}\,(\frac{\omega-10\pi}{5})$.

3.** Apply the duality property to the appropriate function in the Fourier Transform table and show that:

a) $\frac{1}{2}[\delta(t)+\frac{j}{\pi t}] \Leftrightarrow u(\omega)$

b) $\frac{1}{t} \Leftrightarrow -j\pi\,\text{sng}(\omega)$

c) $\delta(t+T)-\delta(t-T) \Leftrightarrow 2j\sin(T\omega)$

4.** The Fourier transform of the triangular pulse *f(t)* shown in Fig. Q5(a) is given to be:

$$F(\omega)=\frac{1}{\omega^2}(e^{j\omega}-j\omega e^{j\omega}-1)$$

Use this information and the time-shifting and time-scaling properties, find the Fourier transforms of the signals $f_1(t)$ to $f_5(t)$ shown in Fig. Q5 (b)-(f).

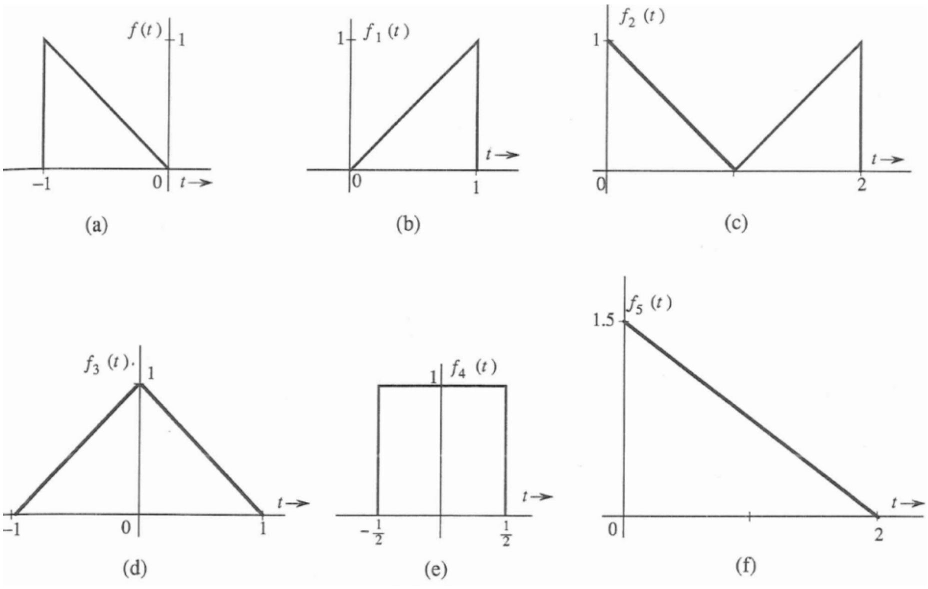


Fig. Q5

5.** The signals in Fig. Q6 (a)-(c) are modulated signals with carrier cos 10t. Find the Fourier transforms of these signals using appropriate properties of the Fourier transform and the FT table given in Lecture 10, slides 13-15. Sketch the amplitude and phase spectra for (a) and (b).

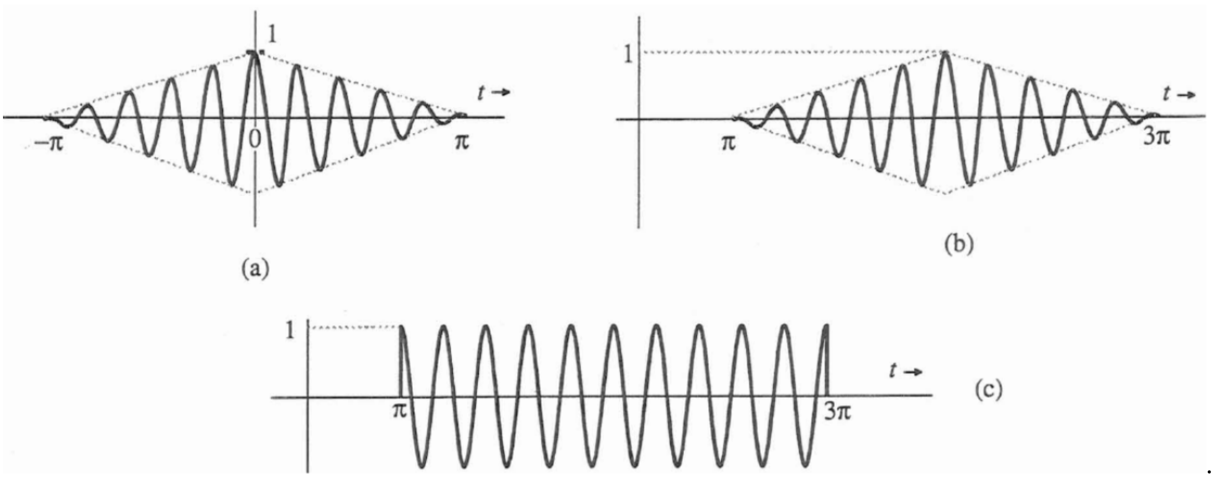


Fig. Q6

6.*** The process of recovering a signal *f(t)* from the modulated signal $f(t)\cos\omega_0 t$ is called **demodulation**. Show that the signal $f(t)\cos\omega_0 t$ can be demodulated by multiplying it with $2\cos\omega_0 t$ and passing the product through a lowpass filter of bandwidth W radians/sec. Assume that $W < \omega$.

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Tutorial Sheet 7 – Sampling

(Lectures 12 - 13)

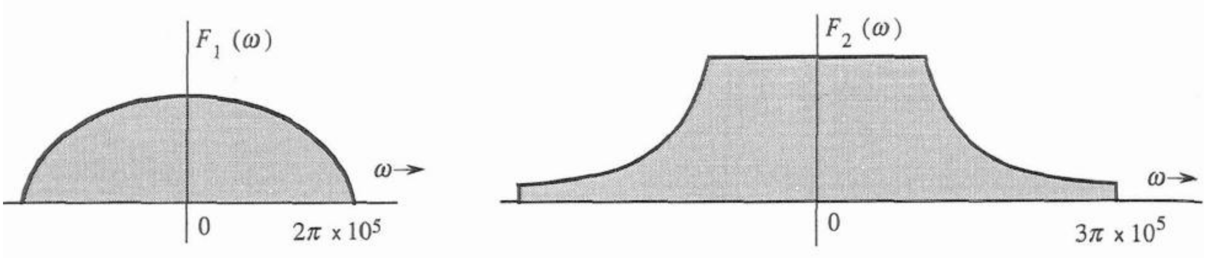
1.* By applying the Parseval's theorem, show that

$$\int_{-\infty}^{\infty} \text{sinc}^2(kx)dx = \frac{\pi}{k}.$$

2.* Fig. Q2 (a) and (b) shows Fourier spectra of signals $f_1(t)$ and $f_2(t)$. Determine the Nyquist sampling rates for the following signals. (Hint: Use the frequency convolution and the width property of the convolution.)

a) $f_1(t)$ b) $f_2(t)$ c) $f_1^2(t)$

d) $f_2^3(t)$ e) $f_1(t)f_2(t)$



(a) (b)

Figure Q2

3.* Signals $f_1(t) = 10^4 rect(10^4 t)$ and $f_2(t) = \delta(t)$ are applied at the inputs of ideal lowpass filters $H_1(\omega) = rect(\frac{\omega}{40,000\pi})$ and $H_2(\omega) = rect(\frac{\omega}{20,000\pi})$. The outputs $y_1(t)$ and $y_2(t)$ of these filters are multiplied to obtain the signal $y(t) = y_1(t)y_2(t)$ as shown in Figure Q3.

a) Sketch $F_1(\omega)$ and $F_2(\omega)$.

b) Sketch $H_1(\omega)$ and $H_2(\omega)$.

c) Sketch $Y_1(\omega)$ and $Y_2(\omega)$.

d) Find the Nyquist sampling rate of $y_1(t)$, $y_2(t)$ and $y(t)$.

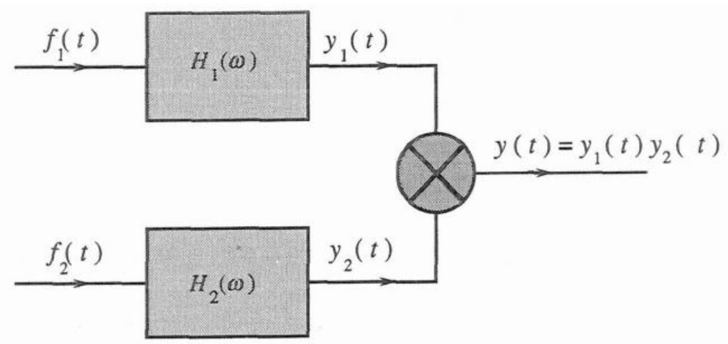


Figure Q3

4.** For the signal $e^{-at}u(t)$, determine the bandwidth of an anti-aliasing filter if the essential bandwidth of the signal contains 99% of the signal energy.

5.** A zero-order hold circuit shown in Fig. Q5 is often used to reconstruct a signal *f(t)* from its samples.

a) Find the unit impulse response of this circuit.

b) Find the transfer function $H(\omega)$, and sketch $|H(\omega)|$.

c) Sketch the output of this circuit for an input *f(t)* which is ¼ cycle of a sinewave.

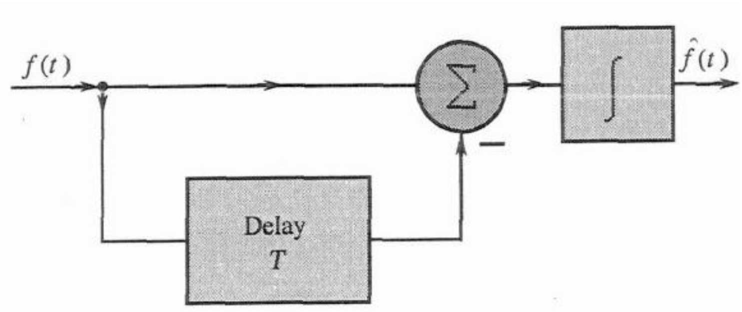


Figure Q5

6. MATLAB exercise: Write a matlab function of the form:

function [t, sinewave] = sinegen(fsig,fsamp,T)

that is able to generate a sampled sinewave of frequency fsig sampled at fsamp on the interval [0,T). Using this function reproduce the figures in slides 14 of lecture 13 which shows aliasing when sampling sinewaves of frequency 1Hz and 6Hz using the sampling rate fs=5Hz. Plot the sampled signals using the command ‘stem’.

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Tutorial Sheet 8 – DFT and z-transform

(Lectures 14 - 15)

1.* For a signal *f(t)* that is time-limited to 10 ms and has an essential bandwidth of 10 kHz, determine N_0, the number of signal samples necessary to compute a power of 2 DFT with a frequency resolution f_0 of at least 50 Hz. Explain if any zero padding is necessary.

2.* Choose appropriate values for N_0 and T and compute the DFT of the signal $e^{-t}\,u(t)$. (Note that the choice of N_0 and T is not unique; it will depend on your assumptions. What is important here is the reasoning that you use to arrive at your answer.)

3. Using the definition of z-transform, show that

(a)* $\gamma^{k-1}u[k-1] \Leftrightarrow \dfrac{1}{z-\gamma}$

(b)** $u[k-m] \Leftrightarrow \dfrac{z}{z^m(z-1)}$

(c)** $\dfrac{\gamma^k}{k!}u[k] \Leftrightarrow e^{\frac{\gamma}{z}}$

4. Using z-transform table given in the lecture notes, show that

(a)* $2^{k+1}u[k-1] + e^{k-1}u[k] \Leftrightarrow \dfrac{4}{z-2} + \dfrac{z}{e(z-e)}$

(b)** $k\gamma^k u[k-1] \Leftrightarrow \dfrac{\gamma z}{(z-\gamma)^2}$ (Hint: $u[k-1] = u[k] - \delta[k]$, $f[k]\delta[k] = f[0]\delta[k]$.)

(c)** $[2^{-k}\cos\frac{\pi}{3}k]u[k-1] \Leftrightarrow \dfrac{0.25(z-1)}{z^2-0.5z+0.25}$

5. Find the causal inverse z-transform of

(a)* $X[z] = \frac{z(z-4)}{z^2 - 5z + 6}$

(b)** $X[z] = \frac{z(e^{-2} - 2)}{(e^{-2} - z)(z-2)}$

(c)** $X[z] = \frac{z(-5z + 22)}{(z+1)(z-2)^2}$

6. MATLAB exercise. Using the m-file sinegen of the previous class problem. Compute and plot the amplitude of the DFT of a sine wave of freq=11Hz sampled with sampf=31Hz with a window size of T=0.25 sec. Use zero padding to increase of 4 the number of frequencies you can see. Finally compute the DFT of the same signal but using a window of size T=2sec. Use the command stem to plor the 3 different DFT, do you understand the differences? Also use the help on line to learn about ‘fft’ and ‘fftshift’.