ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE

Imperial College London

Sampling Moments and Reconstructing Signals with Finite Rate of Innovation: Shannon meets Strang-Fix

Pier Luigi Dragotti, Martin Vetterli, Pancham Shukla, and Thierry Blu

Pancham Shukla (PhD Student)

Communications and Signal Processing Group

Electrical and Electronic Engineering Department

Imperial College London

Email: p.shukla@imperial.ac.uk

Outline

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  - Annihilating filter method (Prony’s method)
  - Sampling of streams of Diracs and piecewise polynomial signals.

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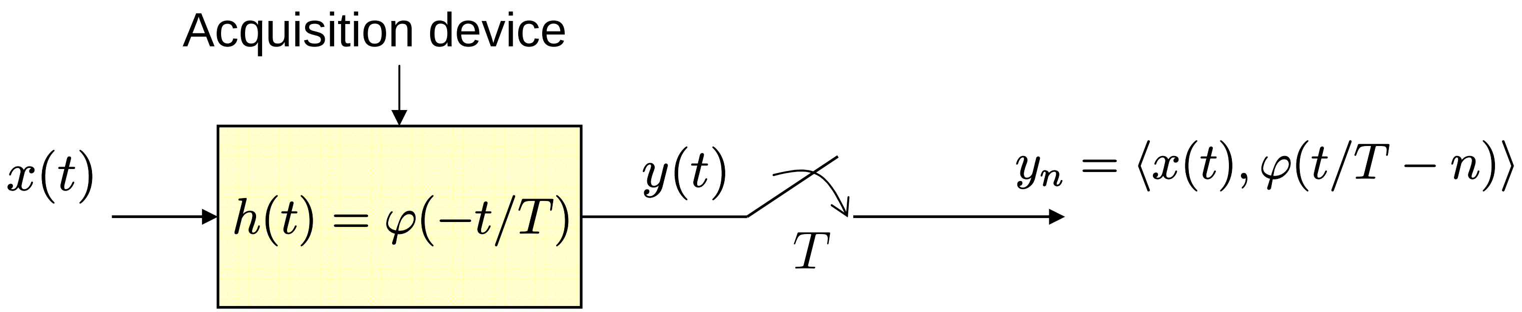
  - Polynomial reproduction in 2-D, and moments from samples.
  - Sampling sets of Diracs, bilevel polygons, polygonal lines, and quadrature domains (e.g. circles, ellipses, cardioids).

- Application (image super-resolution)

- Conclusion

Introduction: Problem statement and motivation

We consider uniform sampling!



Given the samples $y_n = \langle x(t), \varphi(t/T - n) \rangle$, we want to reconstruct $x(t)$.

Natural questions:

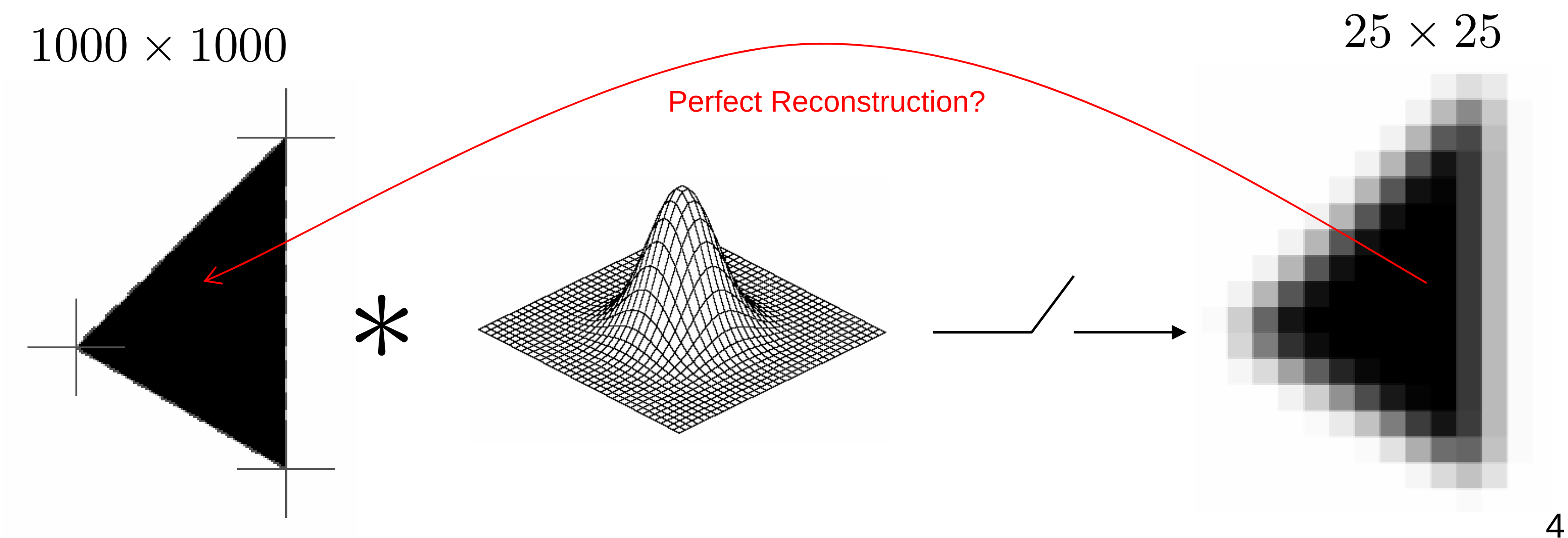
- What signals $x(t)$ can be sampled?
- What kernels $\varphi(t)$ can be used?
- What reconstruction algorithm?

Is there any life beyond 'bandlimited-sinc' space?

Introduction: Sampling for sparsity

Why sampling?

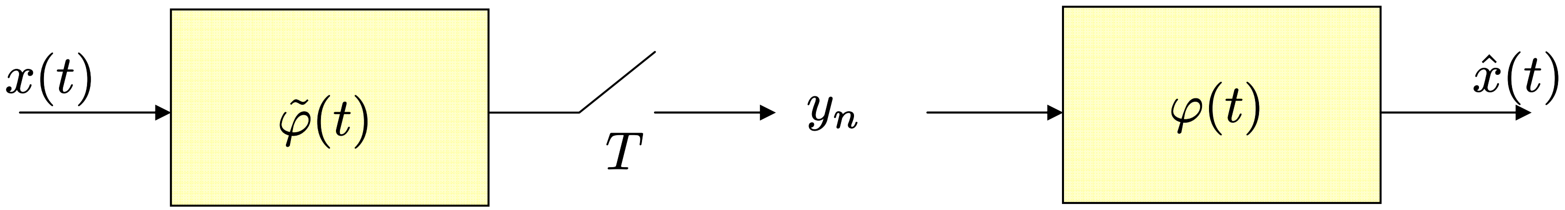
- Many natural phenomena are continuous and required to be observed and processed by sampling.
- Important for hybrid analog/digital processing.
- Related to the notion of sparsity of signals; important in data transmission and storage.
- Useful in image resolution enhancement and super-resolution.



Introduction: Classical to FRI

Classical sampling formulation:

- Sampling of $x(t)$ is equivalent to projecting $x(t)$ onto the shift-invariant subspace $V = \mathrm{span}\{\varphi(t/T - n)\}_{n\in\mathbb{Z}}$.
- If $x(t) \in V$, perfect reconstruction is possible.
- Reconstruction process is linear: $\hat{x}(t) = \sum_n y_n \, \varphi(t/T - n)$.
- For bandlimited signals $\varphi(t) = \mathrm{sinc}(t)$.



What is special about $x(t)$? – bandlimited!

The signal $\hat{x}(t) = \sum_n y_n \, \varphi(t/T - n)$ has a finite number $\rho = 1/T$ of degrees of freedom per unit time.

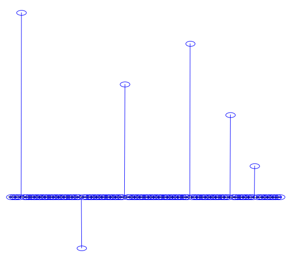
Intuition: If the number of samples y_n per unit of time is greater than or equal to the degrees of freedom ρ then we can reconstruct $x(t)$ from its samples y_n

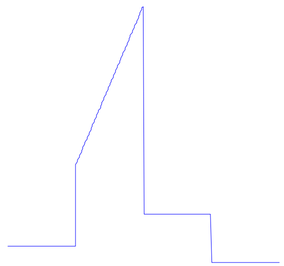
Introduction: Signals with Finite Rate of Innovation (FRI)

Definition [VetterliMB02]: The number ρ of degrees of freedom per unit time is called rate of innovation. A signal with a finite ρ is called signal with finite rate of innovation.

Notice: Many signals that do not belong to shift-invariant subspace have finite rate of innovation. That means non-bandlimited but parametric signals!

Examples: Streams of Diracs and piecewise polynomials.
(e.g. a stream of K Diracs has 2K degrees of freedom: amplitudes and positions.)





These signals can be sampled using infinite support sinc and Gaussian kernels [VetterliMB02].

Introduction: Sampling kernels

Possible classes of kernels
(Ideally as general as possible and of compact support)

Class 1. Any kernel $\varphi(t)$ that can reproduce polynomials (satisfy Strang-Fix conditions):

$$\sum_n c_{m,n}\, \varphi(t-n) = t^m, \quad m = 0, 1, \ldots, N$$

E.g. any scaling function (wavelet theory), B-splines

Class 2. Any kernel $\varphi(t)$ that can reproduce exponentials

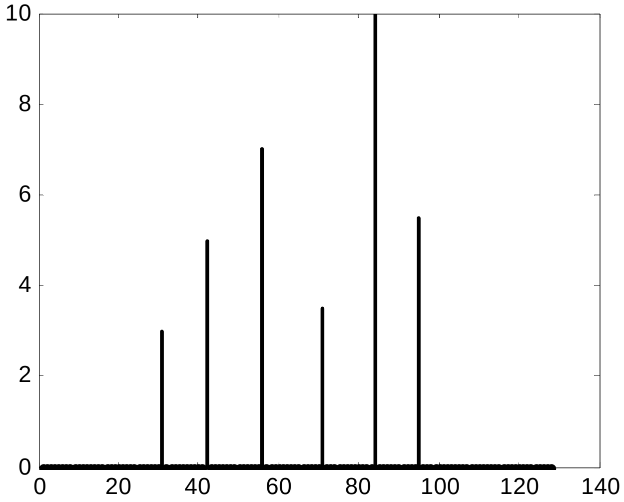
E.g. E-splines [UnserB05].
Useful in sampling piecewise sinusoidal signals. [BerentD-ICASSP06]

Class 3. Any kernel with rational Fourier transform

Linear differential acquisition devices: most electrical, mechanical, and acoustic systems. E.g. sampling the step response of an R-C circuit.

We focus on the Class 1 kernels that can reproduce polynomials. The polynomial reproduction property of the kernel allows us to reproduce the moments.

1-D case: Sampling Diracs with kernels that reproduce polynomials



Assume that $x(t)$ is a stream of K Diracs:
$x(t) = \sum_{k=0}^{K-1} a_k\, \delta(t - t_k)$ and let $T = 1$.

Q: Given the samples $y_n = \langle x(t), \varphi(t-n) \rangle$, how can we find the locations t_k and amplitudes a_k of the Diracs?

Assume that the kernel $\varphi(t)$ can reproduce polynomials up to degree $N \geq 2K - 1$:

$$\sum_n c_{m,n}\, \varphi(t-n) = t^m, \quad m = 0, 1, \ldots, N.$$

1-D case: Sampling of Diracs

Computing $\tau_m = \sum_n c_{m,n}\, y_n, \quad m = 0, 1, \ldots, N$, we have that

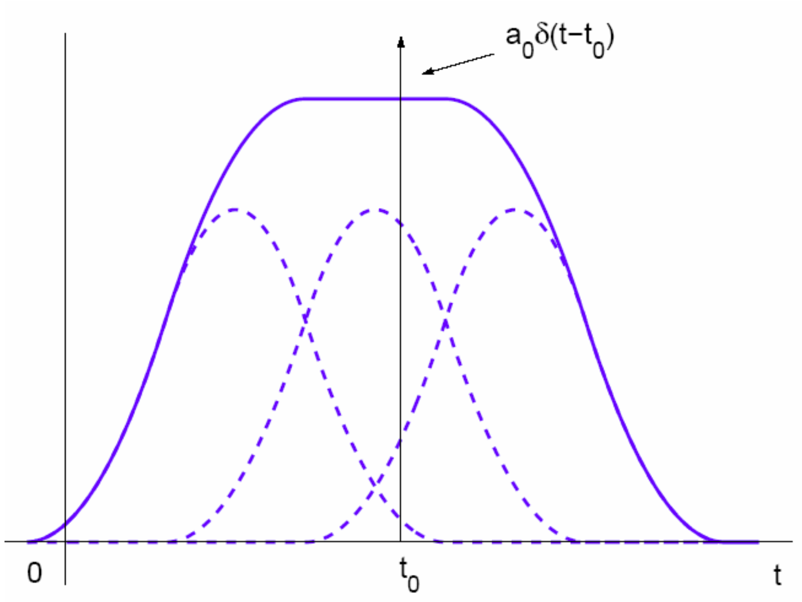
$$\begin{aligned}
\tau_m &= \sum_n c_{m,n}\, y_n \\
&= \left\langle x(t), \sum_n c_{m,n} \varphi(t-n) \right\rangle \\
&= \int_{-\infty}^{\infty} x(t)\, t^m dt, \quad \text{(moments of } x(t)) \\
&= \sum_{k=0}^{K-1} a_k t_k^m, \quad m = 0, 1, \ldots, N
\end{aligned}$$

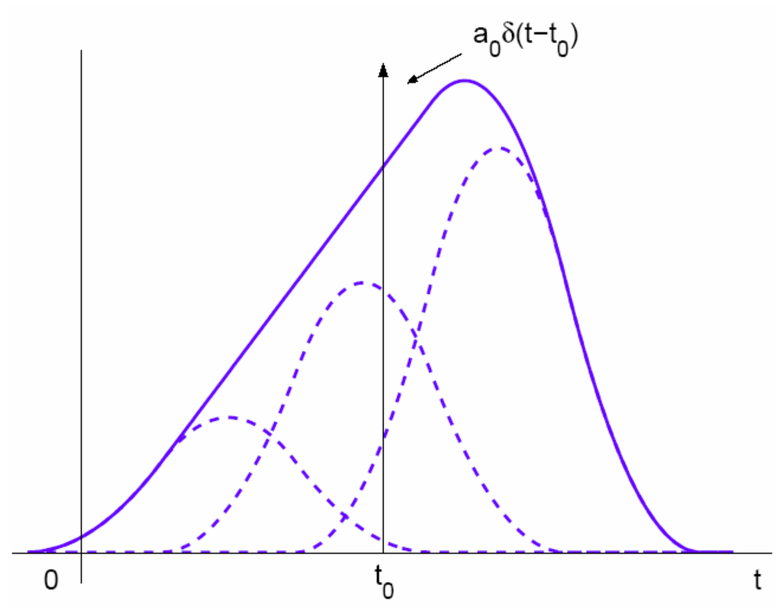
We thus obtain the moments τ_m of $x(t)$ from the linear combinations of samples y_n and coefficients $c_{m,n}$.

It is possible to retrieve the locations t_k and amplitudes a_k of K Diracs from the moments $\tau_m = \sum_{k=0}^{K-1} a_k\, t_k^m, \quad m = 0, 1, \ldots, N$ using *annihilating filter method*.

1-D case: Sampling of Diracs

However, for $K = 1$ Dirac, we only need two moments, and thus, a kernel $\varphi(t)$ that can reproduce polynomials at least up to degree $N = 2K - 1 = 1$.

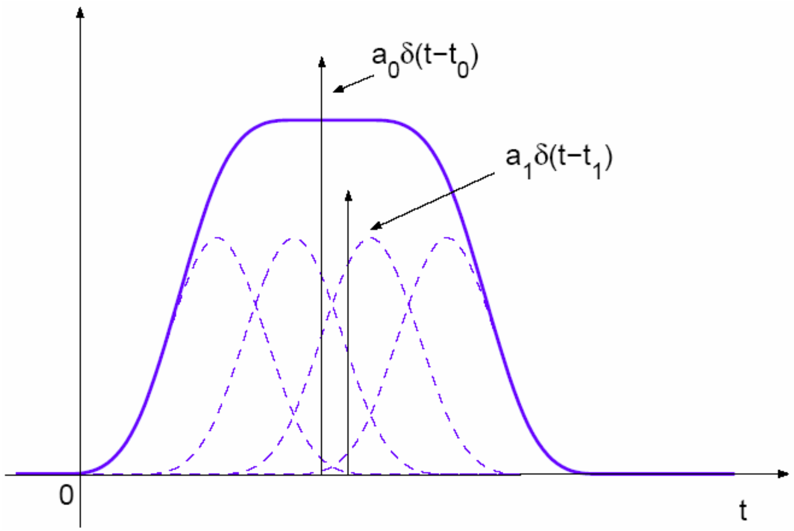




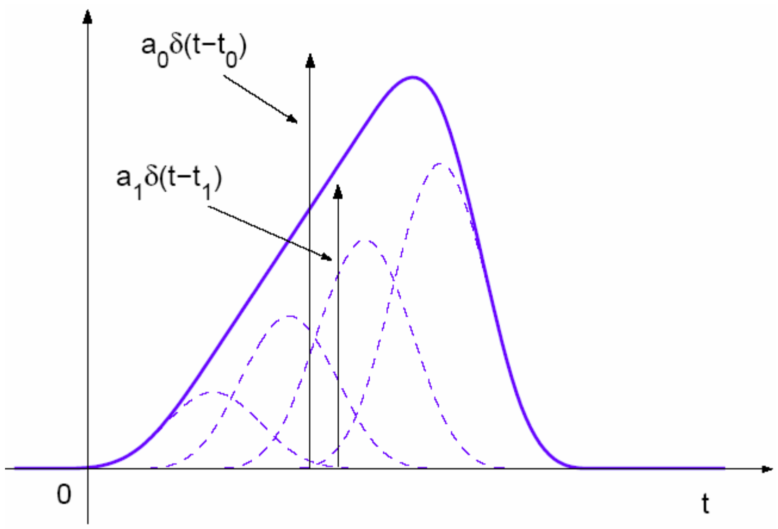
$$\sum_n y_n =< a_0\delta(t-t_0), \sum_n \varphi(t-n) >= \int_{-\infty}^{\infty} a_0\delta(t-t_0)\sum_n \varphi(t-n)\,dt = a_0 \sum_n \varphi(t_0-n) = a_0$$

$$\sum_n c_{m,n}\, y_n =< a_0\delta(t-t_0), c_{1,n}\sum_n \varphi(t-n) >= a_0 \sum_n c_{1,n}\,\varphi(t_0-n) = a_0 t_0$$

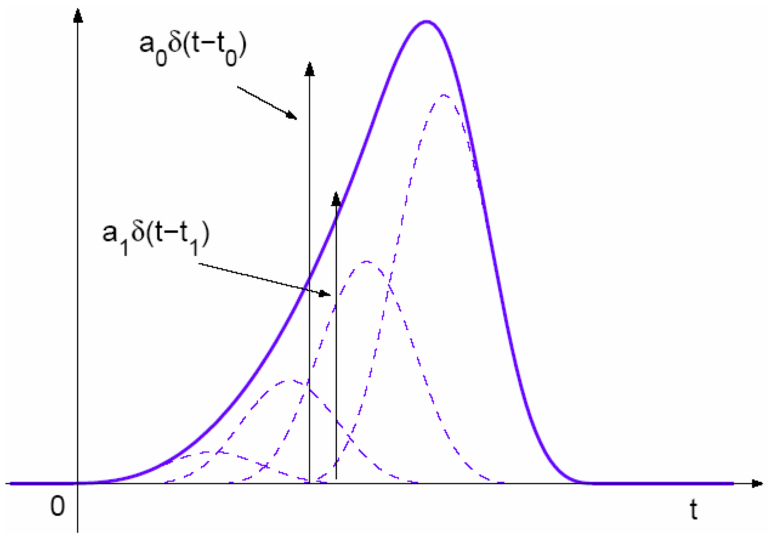
1-D case: Sampling of Diracs



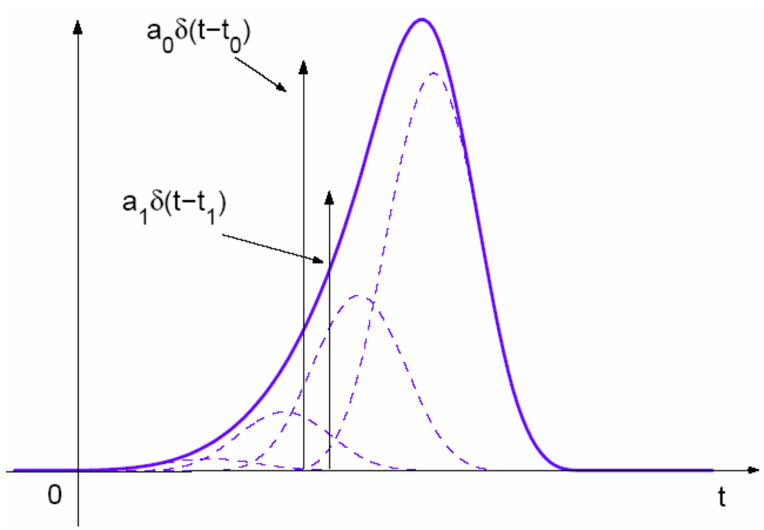
$$\tau_0 = \sum_n y_n = a_0 + a_1$$



$$\tau_1 = \sum_n c_{1,n} y_n = a_0 t_0 + a_1 t_1$$



$$\tau_2 = \sum_n c_{2,n} y_n = a_0 t_0^2 + a_1 t_1^2$$



$$\tau_3 = \sum_n c_{3,n} y_n = a_0 t_0^3 + a_1 t_1^3$$

1-D case: Annihilating filter method

1. Design a filter h_m such that the convolution $h_m * \tau_m = \sum_{i=0}^{m} h_i \tau_{m-i} = 0$.

   The z-transform of the filter h_m is $H(z) = \prod_{k=0}^{K-1} \left(1 - t_k z^{-1}\right)$.

$$\begin{bmatrix} \tau_{K-1} & \tau_{K-2} & \cdots & \tau_0 \\ \tau_K & \tau_{K-1} & \cdots & \tau_1 \\ \vdots & \vdots & \ddots & \vdots \\ \tau_{N-1} & \tau_{N-2} & \cdots & \tau_{N-K} \end{bmatrix} \begin{bmatrix} h_1 \\ h_2 \\ \vdots \\ h_K \end{bmatrix} = - \begin{bmatrix} \tau_K \\ \tau_{K+1} \\ \vdots \\ \tau_N \end{bmatrix}.$$

   This is a classic Yule-Walker system with a unique solution for distinct Diracs.

2. From h_m, find the roots of $H(z)$. This gives the Dirac locations t_k.

3. Solve the first K equations in $\tau_m = \sum_{k=0}^{K-1} a_k t_k^m$.
   This gives us the amplitudes a_k.

$$\begin{bmatrix} 1 & 1 & \cdots & 1 \\ t_0 & t_1 & \cdots & t_{K-1} \\ \vdots & \vdots & \ddots & \vdots \\ t_0^{K-1} & t_1^{K-1} & \cdots & t_{K-1}^{N-1} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_{K-1} \end{bmatrix} = \begin{bmatrix} \tau_0 \\ \tau_1 \\ \vdots \\ \tau_{K-1} \end{bmatrix}.$$

   This is a classic Vandermonde system with unique solution for distinct t_k.

1-D case: Sampling streams of Diracs

Proposition 1

Assume a sampling kernel $\varphi(t)$ *that can reproduce polynomials up to degree* $N \geq 2K-1$ *and of compact support* L. *A stream of* K *Diracs* $x(t) = \sum_{k=0}^{K-1} a_k \, \delta(t-t_k)$ *is uniquely determined from the samples defined by* $y_n = <\varphi(t/T-n), x(t)>$ *if there are at most* K *Diracs in an interval of size* KLT.

- Since the kernel is of compact support, samples of properly isolated groups of (at most K) Diracs do not influence each other.
- Therefore, Proposition 1 can be extended for an infinite stream of Diracs using a sequential local algorithm by relaxing the interval size of K Diracs from KLT to 2KLT. This helps to isolate the groups of at most K Diracs.
- There is a trade-off between local rate of innovation and complexity in the reconstruction process.

This also applies to a stream of differentiated Diracs:

$$x^{(R)}(t) = \sum_{k=0}^{K-1} \sum_{r=0}^{R-1} a_{k,r} \, \delta^{(r)}(t-t_k)$$

K Diracs with $\hat{K} = KR$ weights can be sampled using a kernel that can reproduce polynomials up to degree $N \geq 2\hat{K}-1$ or $N \geq 2KR-1$.

1-D case: Sampling piecewise constant signals

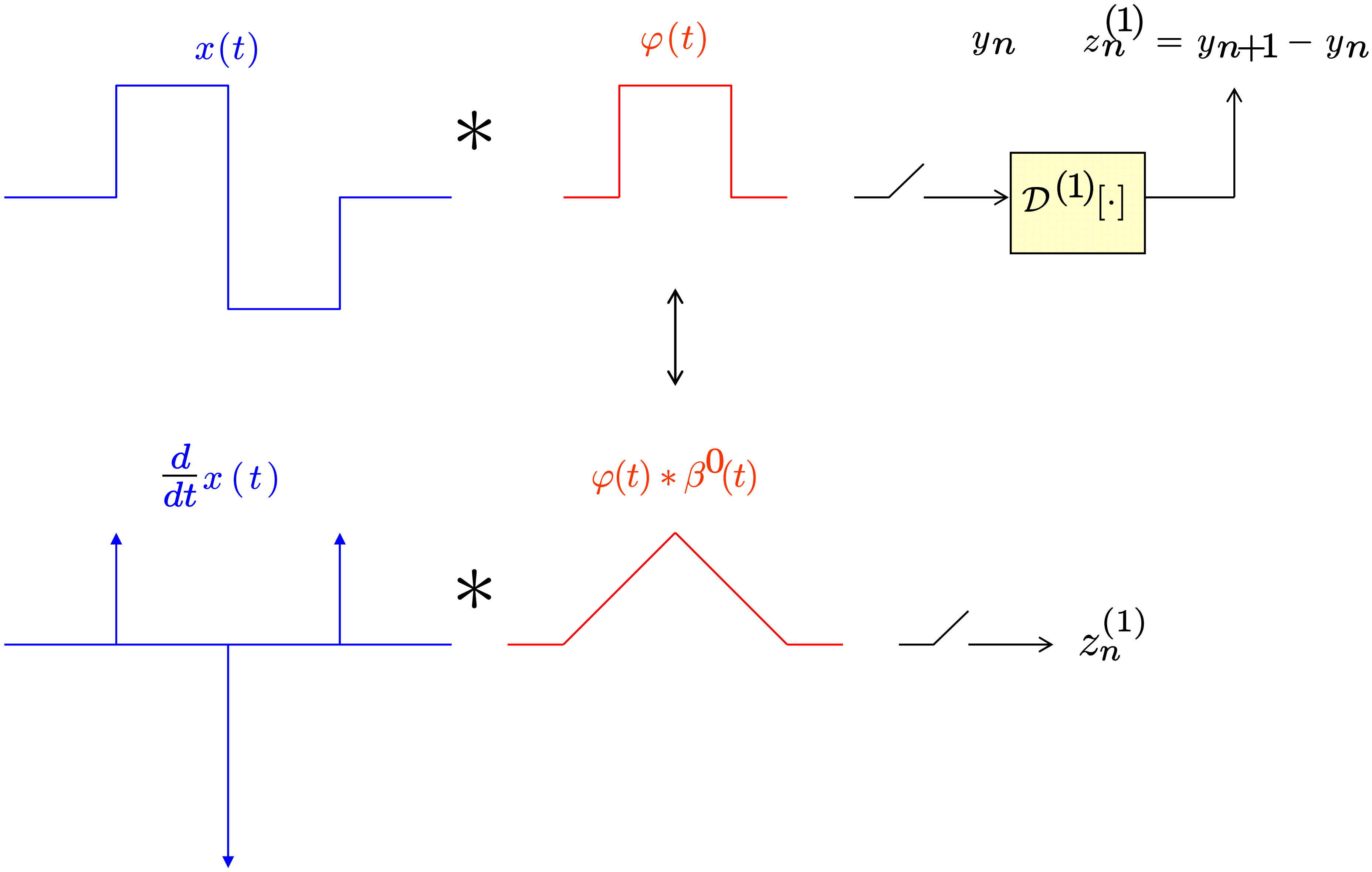
Insight: The derivative of a piecewise constant is a stream of Diracs. Thus by computing the derivative of piecewise constant signal, we can sample it.

Given the samples y_n compute the finite difference $z_n^{(1)} = y_{n+1} - y_n$, we have

$$\begin{aligned} z_n^{(1)} = y_{n+1} - y_n &= \langle x(t), \varphi(t-n-1) - \varphi(t-n) \rangle \\ &= \frac{1}{2\pi} \langle X(w), \hat{\varphi}(w) e^{-jwn} (e^{-jw} - 1) \rangle \qquad \text{Parseval} \\ &= \frac{1}{2\pi} \left\langle X(w), -jw\, \hat{\varphi}(w)\, e^{-jwn} \left(\frac{1 - e^{-jw}}{jw} \right) \right\rangle \\ &= - \left\langle x(t), \frac{d}{dt} \left[\varphi(t-n) * \beta^0(t-n) \right] \right\rangle \\ &= \left\langle \frac{d}{dt} x(t), \varphi(t-n) * \beta^0(t-n) \right\rangle \end{aligned}$$

Thus the samples $z_n^{(1)}$ are related to the derivative of $x(t)$.

1-D case: Sampling piecewise constant signals



1-D case: Sampling piecewise polynomial signals

Similarly,

Piecewise polynomial $\xrightarrow{\text{Higher order derivative}}$ Differentiated Diracs

$x(t)$ $\qquad\qquad$ $x^{(R)}(t) = \sum_{k=0}^{K-1} \sum_{r=0}^{R-1} a_{k,r}\, \delta^{(r)}(t - t_k)$

$$x(t) \xrightarrow{\varphi(t)} y_n \xrightarrow{\mathcal{D}^{(R)}[\cdot]} z_n^{(R)} \xrightarrow{\tau_m} \begin{bmatrix} t_k \\ a_{k,r} \end{bmatrix} \rightarrow x^{(R)}(t) \rightarrow x(t)$$

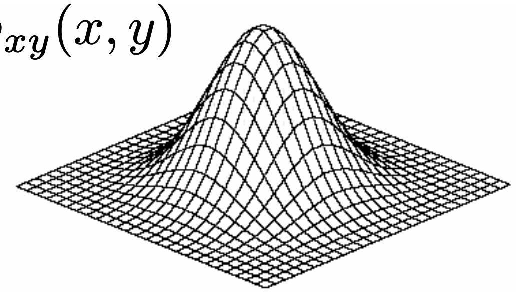
2-D case: Polynomial reproduction in 2-D

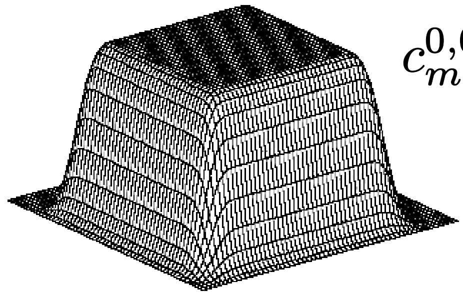
The 2-D sampling kernel is a separable kernel given by the tensor product of two 1-D functions that can reproduce polynomials: $\varphi_{xy}(x,y)=\varphi(x)\varphi(y)$. Therefore, it follows that,

$$\sum_m \sum_n c_{m,n}^{\alpha,\beta}\,\varphi_{xy}(x-m,y-n)=x^\alpha y^\beta,$$

where $\gamma=\alpha+\beta,\ \gamma=0,1,\ldots,N.$

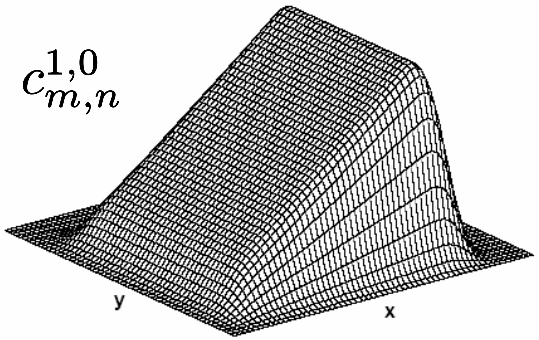
$\varphi_{xy}(x,y)$

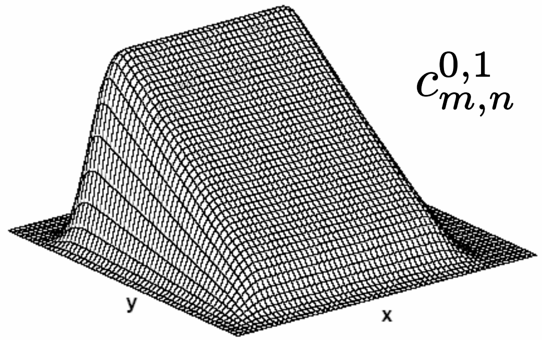




$c_{m,n}^{0,0}$

$c_{m,n}^{1,0}$





$c_{m,n}^{0,1}$

2-D case: Moments from samples

In 2-D, we observe the samples of a signal $g(x,y)$ as given by

$$y_{m,n} = \langle g(x,y), \varphi_{xy}(x/T_x - m, y/T_y - n)\rangle.$$

The polynomial reproduction property of $\varphi_{xy}(x,y)$ allows us to retrieve the (geometric and complex) moments of the signal $g(x,y)$ from its samples $y_{m,n}$:

Geometric moments:

$$\begin{aligned}
\mathcal{M}_{\alpha,\beta} &= \int\int_{\Omega} g(x,y)\, x^{\alpha} y^{\beta}\, dx\, dy \\
&= \int\int_{\Omega} g(x,y) \sum_m \sum_n c_{m,n}^{\alpha,\beta}\, \varphi_{xy}(x-m, y-n)\, dx\, dy \\
&= \sum_m \sum_n c_{m,n}^{\alpha,\beta}\, \langle g(x,y), \varphi_{xy}(x-m, y-n)\rangle \\
&= \sum_m \sum_n c_{m,n}^{\alpha,\beta}\, y_{m,n}
\end{aligned}$$

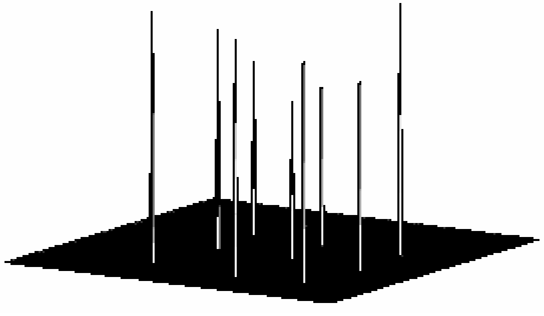
Complex moments:

$$\tau_{\gamma} = \int\int_{\Omega} g(x,y)\,(x+iy)^{\gamma}\, dx\, dy = \sum_{\beta=0}^{\gamma} \binom{\gamma}{\beta}\, i^{\beta}\, \mathcal{M}_{\alpha,\beta},$$

$$\text{where } \gamma = \alpha + \beta,\ i = \sqrt{-1}$$

2-D case: Sets of Diracs

We want to reconstruct a set of K Diracs
$g(x,y) = \sum_{k=0}^{K-1} a_k\, \delta_{xy}(x - x_k, y - y_k)$
from the observed samples
$y_{m,n} = \langle g(x,y), \varphi_{xy}(x - m, y - n)\rangle.$



- In 1-D we use the ability of $\varphi(t)$ to reproduce polynomials for retrieving moments

$$\tau_m = \int_{-\infty}^{\infty} x(t)\, t^m = \sum_{k=0}^{K-1} a_k\, t_k^m, \quad m = 0, 1, \ldots N$$

of the signal $x(t)$. Then used the annihilating fitter method.

- In 2-D we simply need to obtain the complex-moments

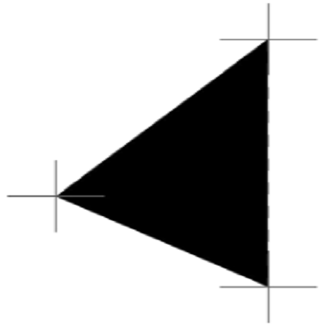
$$\tau_\gamma = \int\int g(x,y) z^\gamma\, dxdy = \int\int g(x,y)(x+iy)^\gamma\, dxdy = \sum_{k=0}^{K-1} a_k\, z_k^\gamma$$

$$\text{where } z = x + iy, \text{ and } \gamma = 0, 1, \ldots, N.$$

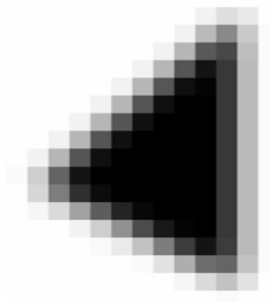
Then using the annihilating filter method, we retrieve the locations $z_k = (x_k + iy_k)$ and amplitudes a_k. For K Diracs, we need $2K$ moments, i.e. $N \geq 2K - 1$.

Bilevel polygonal images

The same applies to polygonal images. However, in this case we need to obtain weighted complex-moments [Davis64] [MilanfarVKW95]. For a given polygon $g(x, y)$ with K corner points, it follows that



$$\begin{aligned}
\hat{\tau}_\gamma &= \gamma(\gamma-1)\int\int_\Omega g(x,y)(z)^{\gamma-2}\,dxdy \\
&= \gamma(\gamma-1)\tau_{\gamma-2} \\
&= \gamma(\gamma-1)\sum_{\beta=0}^{\gamma-2} i^\beta \binom{\gamma-2}{\beta} \sum_m \sum_n c_{m,n}^{\alpha,\beta} y_{m,n} \\
&= \sum_{k=0}^{K} \rho_k\, z_k^\gamma,
\end{aligned}$$

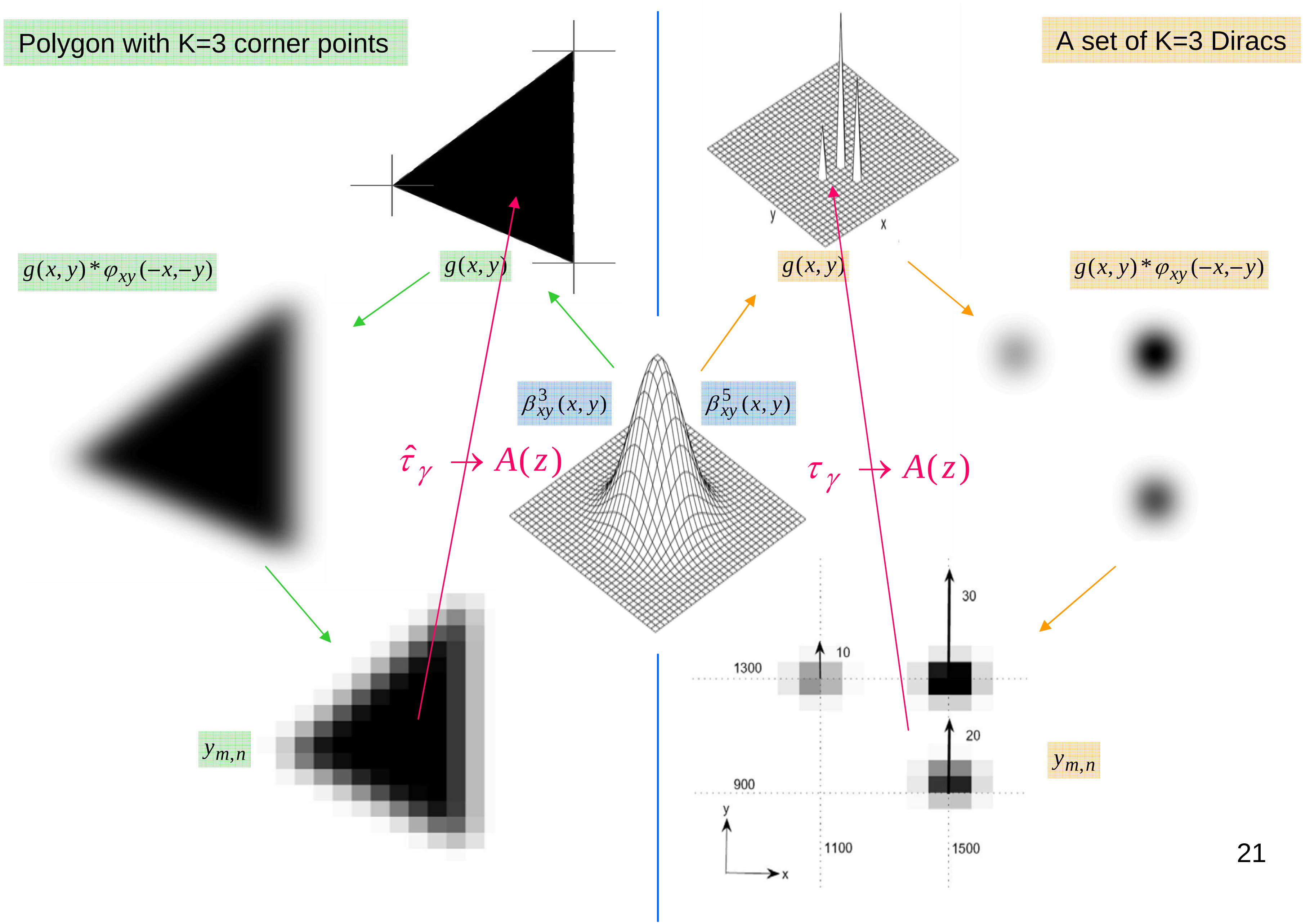


where $\gamma - 2 \in \{0, 1, \ldots, N\}$, $\gamma - 2 = \alpha + \beta$, and $\hat{\tau}_0 = \hat{\tau}_1 = 0$.

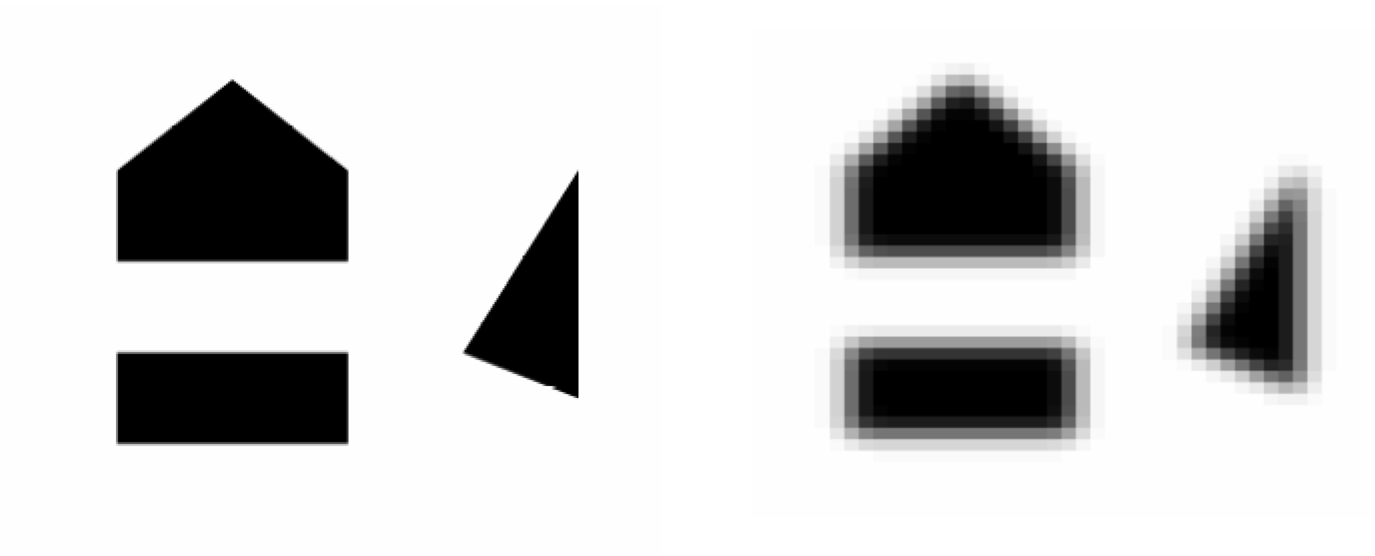
Thus, from the samples $y_{m,n}$ we can estimate complex-moments, and from the complex-moments, using annihilating filter method, the locations $z_k = (x_k + iy_k)$ of the corner points.

To retrieve K corner points, we need $2K$ complex-moments, and therefore, a kernel $\varphi_{xy}(x, y)$ that can reproduce polynomials up to degree $N \geq 2K - 3$.

2-D case: Bilevel polygons and Diracs



2-D case: Bilevel polygonal images

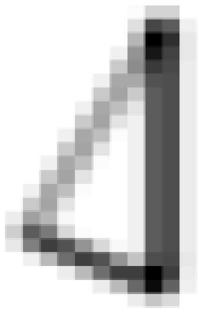


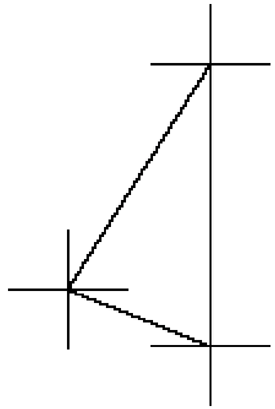
Since the sampling kernel is of compact support, all polygons can be reconstructed independently, given that they are sufficiently apart.

Notice: The polygons must be convex for unique reconstruction.

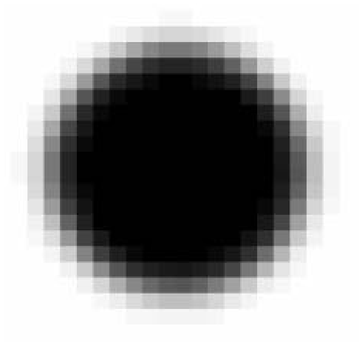
2-D case: Polygonal lines and quadrature domains

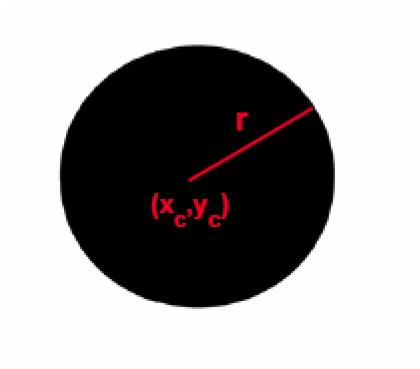
Similarly, by using complex-moments and annihilating filter method we can reconstruct polygonal lines,





and quadrature domains (e.g. circles, ellipses, cardioids) [MilanfarPVGG2000].



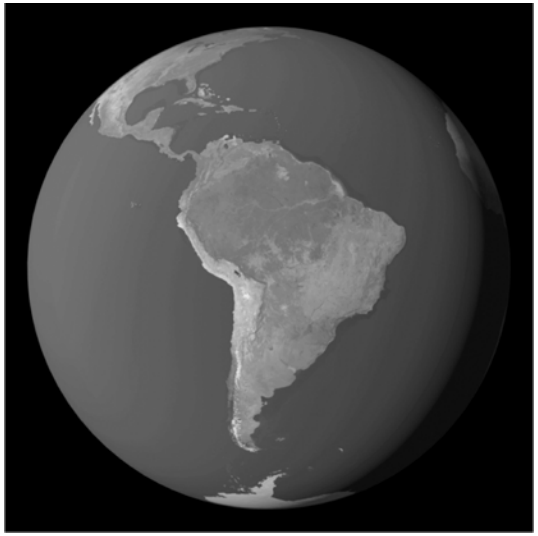


$$\tau_0 = \pi r^2 \Rightarrow r = \sqrt{\tau_0/\pi}$$

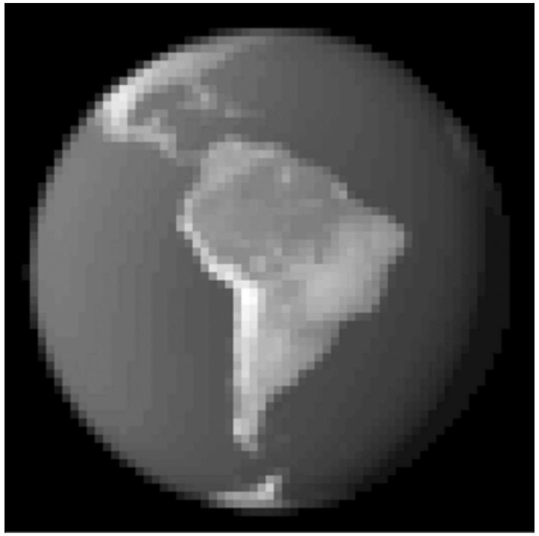
$$z_c = (x_c + i y_c) = \frac{\tau_1}{\tau_0}$$

Application: Image super-resolution

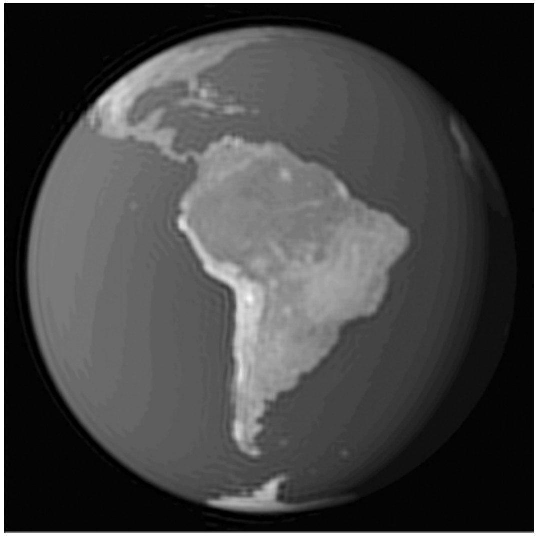
Image registration using continuous-moments from samples [BaboulazD-ICIP06].



Original (2000 × 2000)



Low res. (65 × 65)



Super-res. (2000 × 2000)

- One hundred low resolution and shifted versions of the original image.
- Accurate registration is achieved by retrieving the continuous-moments of the earth from its 100 sets of samples.
- The registered images are then interpolated and restored to achieve super-resolution.

Application: Image super-resolution

Video

Conclusion

- We can sample and perfectly reconstruct a large class of non-bandlimited signals (i.e. Signals with Finite Rate of Innovation) in 1-D, and 2-D.
- We can use a rich class of kernels. In particular, the compactly supported kernels that reproduce polynomials allow us to retrieve the continuous-moments of the signals from their samples.
- The retrieval of continuous-moments from samples is useful in many applications, e.g. super-resolution image registration.

Publications

1. P L Dragotti, M Vetterli and T Blu, "Sampling moments and reconstructing signals of finite rate of innovation: Shannon meets Strang-Fix," IEEE Trans. Sig. Proc. February 2006, submitted.
2. P L Dragotti, M Vetterli and T Blu, "Exact sampling results for signals with finite rate of innovation using Strang-Fix conditions and local kernels," Proc. of IEEE International Conference on Acoustics, Speech and Signal Processing, (ICASSP05), Philadelphia, USA, March 2005.
3. P Shukla and P L Dragotti, "Sampling schemes for 2-D signals with finite rate of innovations using kernels that reproduce polynomials," Proc. of IEEE International Conference on Image Processing (ICIP05), Genova, Italy, September 2005.
4. L Baboulaz and P L Dragotti, "Distributed acquisition and image super-resolution based on continuous moments from samples," IEEE International Conference on Image Processing (ICIP06), Atlanta, USA, October 2006, to appear.

Questions?