SAMPLING SCHEMES FOR 2-D SIGNALS WITH FINITE RATE OF INNOVATION USING KERNELS THAT REPRODUCE POLYNOMIALS

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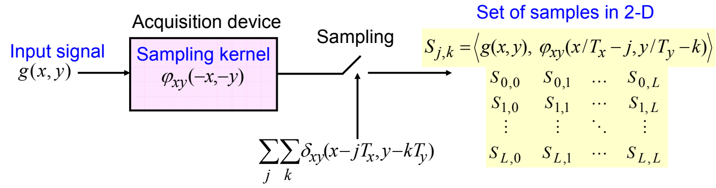
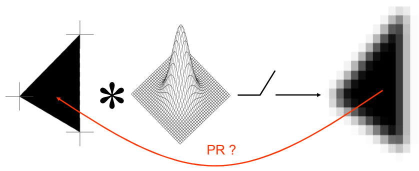
1. INTRODUCTION AND MOTIVATION

A generic sampling setup in 2-D



Motivation

Image resolution enhancement and super-resolution photogrammetry.



What signals? Non-bandlimited signals but with a finite number of degrees of freedom (rate of innovation), and thus known as signals with Finite Rate of Innovation (FRI) [Vetterli et al] [1,2]. E.g., Streams of Diracs, non-uniform splines, and piecewise polynomials. **Present Focus: Sets of 2-D Diracs, bilevel and planar polygons.**

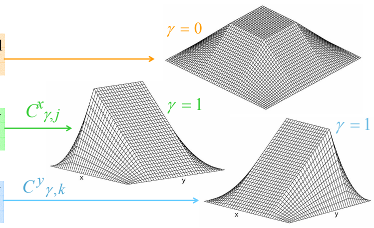
Reconstruction algorithms? Polynomial reproduction, Complex-moments, Annihilating filter, and Directional derivatives.

Sampling kernel and properties? Any kernel φ(x,y) that is of **compact support** and can **reproduce polynomials** of degrees $\gamma = 0,1,2\ldots\Gamma-1$ such that

Partition of unity: $\sum_{j\in Z}\sum_{k\in Z}\varphi_{xy}(x-j,y-k)=1$

Reproduction of polynomials along x-axis: $\sum_{k\in Z}\sum_{j\in Z} C^x_{\gamma,j}\varphi_{xy}(x-j,y-k)=x^\gamma$

Reproduction of polynomials along y-axis: $\sum_{k\in Z}\sum_{j\in Z} C^y_{\gamma,k}\varphi_{xy}(x-j,y-k)=y^\gamma$



e.g., B-splines and Daubechies scaling functions are valid kernels.

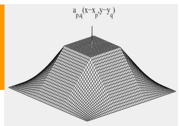
2. SAMPLING OF SIGNALS WITH FINITE RATE OF INNOVATION (FRI) IN 2-D

• SETS OF 2-D DIRACS (LOCAL RECONSTRUCTION)

Consider $g(x,y)=\sum_j\sum_k a_{j,k}\,\delta_{xy}(x-x_j,y-y_k)$ and $\varphi_{xy}(x,y)$ with support $L_x \times L_y$ such that there is at most one Dirac $a_{p,q}\delta_{xy}(x-x_p,y-y_q)$ in an area of size $L_xT_x \times L_yT_y$.

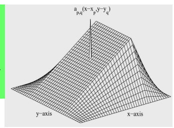
From the partition of unity the **amplitude** is determined as:

$$a_{p,q}=\sum_{j=1}^{L_x}\sum_{k=1}^{L_y} S_{j,k}$$

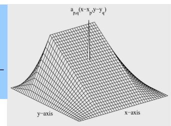


And using polynomial reproduction properties along **x-axis** and **y-axis**, **the coordinate position** (x_p, y_p) is determined as:

$$x_p=\frac{\sum_{j=1}^{L_x}\sum_{k=1}^{L_y} C^x_{1,j} S_{j,k}}{a_{p,q}}$$



$$y_q=\frac{\sum_{j=1}^{L_x}\sum_{k=1}^{L_y} C^y_{1,k} S_{j,k}}{a_{p,q}}$$



• BILEVEL POLYGONS & DIRACS using COMPLEX-MOMENTS (GLOBAL APPROACH)

Moments are used to characterize unspecified objects. [Shohat et al 1943, Elad et al. 2004]. For a convex, bilevel polygon $g(x,y)$ with N corner points, and an analytic function $h(z)=z^n,\ z=x+\sqrt{-1}\,y$ in closure O, the complex-moments of the polygon follow [Milanfar et al.][3]:

$$\sum_{i=1}^{N}\rho_i z_i^n=\iint_O g(x,y)h''(z)\,dxdy$$
$$=\iint_O g(x,y)(z^n)''\,dxdy$$
$$=n(n-1)\iint_O g(x,y)z^{(n-2)}\,dxdy$$
$$=n(n-1)\,\tau^s_{n-2} \text{ (simple complex-moment)}$$
$$=\tau^w_n \text{ (weighted complex-moment)}$$

The z_i can be retrieved from τ^w_n using annihilating filter $A(z)$ (Prony's like method) such that $A[i]*\tau^w_n=0$.

where ρ_i are complex weights and z_i are corner point coordinates of the bilevel polygon $g(x,y)$, $n=2,3,\ldots 2N+1$.

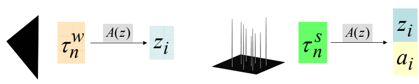
Theorem [Milanfar et al.] [3]: **For a given non-degenerate, simply connected, and convex polygon in the complex Cartesian plane, all its N corner points are uniquely determined by its weighted complex-moments τ^w_n up to order 2N-1.**

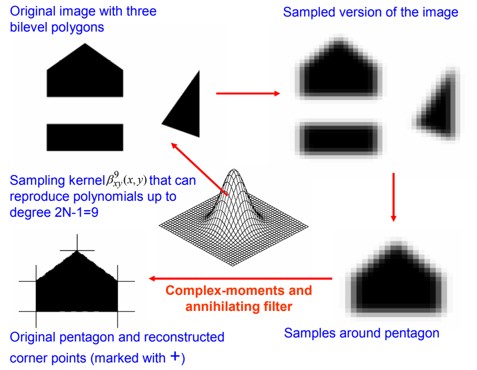
A **sampling perspective** to above theorem follows…

As the kernel $\varphi_{xy}(x,y)$ can reproduce polynomials up to degree 2N-1 (γ = 0,1...2N-1), all 2N moments τ^w_n are determined from weighted sums of the samples $S_{j,k}$. For instance,

$$\tau^w_3 = 3\cdot(3-1)\cdot\sum_j\sum_k\left(C^x_{1,j}+\sqrt{-1}\,C^y_{1,k}\right)^{(3-2)} S_{j,k}=\sum_{i=1}^{N}\rho_i z_i^3$$

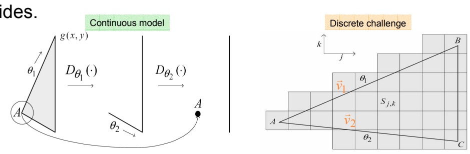
Now for both bilevel polygon and set of Diracs, using **complex-moments** and **annihilating filter method**, it is straightforward to see that





• PLANAR POLYGONS based on DIRECTIONAL DERIVATIVES (LOCAL APPROACH)

Intuitively, for a planar polygon, two successive directional derivatives along two adjacent sides of the polygon results into a 2-D Dirac at the corner point formed by the respective sides.



Lattice theory [4] and [Convey and Sloan]

Link directional derivatives → discrete differences.

Subsampling over integer lattices.

Local directional kernels in the framework of 2-D Dirac sampling (local reconstruction).

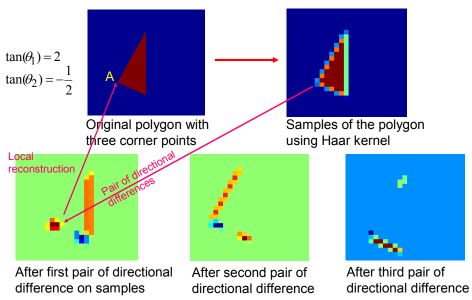
Sampling matrix

$$V_A=\begin{bmatrix}\vec v_1\\ \vec v_2\end{bmatrix}=\begin{bmatrix}v_{1,1} & v_{1,2}\\ v_{2,1} & v_{2,2}\end{bmatrix}$$

$\theta_1=\tan^{-1}\left(\frac{v_{1,2}}{v_{1,1}}\right)$ $\theta_2=\tan^{-1}\left(\frac{v_{2,2}}{v_{2,1}}\right)$

A planar triangle needs three pairs of directional differences to get decomposed in three 2-D Diracs.

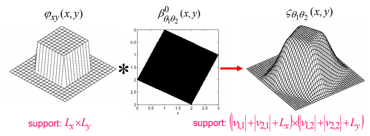
$\tan(\theta_1)=2$, $\tan(\theta_2)=-\frac{1}{2}$



$$\frac{D_{\theta_1}\left[D_{\theta_2}\left[S_{j,k}\right]\right]}{|\det(V_A)|}=\left\langle\frac{\partial}{\partial\theta_2}\left(\frac{\partial}{\partial\theta_1}\left(g(x,y)\right)\right),\ \varsigma_{\theta_1\theta_2}(x,y)\right\rangle \text{ where } \varsigma_{\theta_1\theta_2}(x,y)=\frac{\beta^0_{\theta_1\theta_2}(x,y)*\varphi_{xy}(x,y)}{|\sin(\theta_2-\theta_1)|}$$

(Dirac at A, Modified kernel)

At each corner point → Independent modified (directional) kernel $\varsigma_{\theta_1\theta_2}(x,y)$.



The directional kernel $\varsigma_{\theta_1\theta_2}(x,y)$ can reproduce polynomials of degrees 0 and 1 in both x and y directions.

Assume only one corner point in support of directional kernel. Using local reconstruction scheme, the amplitude and the position of Dirac at any corner point (e.g. at point A) follows:

$$a_{p,q}=\frac{\sum_j\sum_k D_{\theta_2}\left[D_{\theta_1}\left[S_{j,k}\right]\right]}{|\det(V_A)|}\qquad x_p=\frac{\sum_j\sum_k C^x_{1,j}\,D_{\theta_2}\left[D_{\theta_1}\left[S_{j,k}\right]\right]}{a_{p,q}\,|\det(V_A)|}\qquad y_q=\frac{\sum_j\sum_k C^y_{1,k}\,D_{\theta_2}\left[D_{\theta_1}\left[S_{j,k}\right]\right]}{a_{p,q}\,|\det(V_A)|}$$

3. CONCLUSION AND ONGOING WORK

Conclusion: Local and global sampling choices for the classes of 2-D FRI signals with varying degrees of complexity.

Current investigations: Sampling of more general shapes such as circles, ellipses, and polygons containing polygonal holes inside.

Future plans: Extension of our 2-D sampling results in higher dimensions and effect of noise. Integration of sampling results with wavelet footprints for image resolution enhancement and super-resolution photogrammetry.

4. KEY REFERENCES

1. M Vetterli, P Marziliano, and T Blu, "Sampling signals with finite rate of innovation," IEEE Trans. on Signal Processing, 50(6): 1417-1428, June 2002.
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3. P Milanfar, G Verghese, W Karl, and A Willsky, "Reconstructing polygons from moments with connections to array processing," IEEE Trans. on Signal Processing, 43(2): 432-443, February 1995.
4. V Velisavljevic, B Beferull-Lozano, M Vetterli, and P L Dragotti, "Discrete multi-directional wavelet bases," Proc. IEEE ICIP, Barcelona, Spain, September 2003.

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