Exponential Reproducing Kernels for Sparse Sampling

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***Abstract*—The theory of Finite Rate of Innovation (FRI) broadened the traditional sampling paradigm to certain classes of parametric signals. In this paper we review the ideal FRI sampling scheme and some techniques to combat noise. We then present alternative and more effective denoising methods for the case of exponential reproducing kernels.**

I. Introduction

In [1] and [2] it was shown how certain classes of non-bandlimited signals can be sampled and perfectly reconstructed. These signals can be completely characterised by their rate of innovation. In the presence of noise, the ideal approaches become unstable and alternative methods are required [3]. This paper focuses on the optimal use of exponential reproducing kernels introduced in [2] for the noisy scenario.

II. Sampling Signals with FRI

Consider a stream of K Diracs at locations t_k, with amplitudes a_k and of duration τ seconds. If we sample the signal with an exponential reproducing kernel $\varphi\left(-\frac{t}{T}\right)$ we obtain the measurements $y_n = \langle x(t), \varphi\left(\frac{t}{T}-n\right)\rangle$, for $n = 0, 1, \ldots, N-1$. Here N is the number of samples and we use a sampling period $T = \frac{\tau}{N}$.

An exponential reproducing kernel is any function $\varphi(t)$ that satisfies $\sum_{n\in\mathbb{Z}} c_{m,0}\mathrm{e}^{\alpha_m(n-t)}\varphi(t-n) = 1$ with $\alpha_m \in \mathbb{C}$ for appropriate coefficients $c_{m,n} = c_{m,0}\mathrm{e}^{\alpha_m n}$. Equivalently we can write

$$c_{m,0}\int_{-\infty}^{\infty} \mathrm{e}^{-\alpha_m t}\varphi(t)\mathrm{d}t = 1. \tag{1}$$

Furthermore, any composite function of the form $\varphi(t) = \gamma(t) * \beta_{\vec{\alpha}_P}(t)$, where $\beta_{\vec{\alpha}_P}(t)$ is an E-Spline [4], is able to reproduce the set $\mathrm{e}^{\alpha_m t}$, $m = 0, 1, \ldots, P$.

Reconstructing the input is a two step process [2]. First, the samples y_n are linearly combined to get the new measurements $s_m = \sum_{n=0}^{N-1} c_{m,n} y_n$. These are equivalent to a power series involving the locations t_k and amplitudes a_k for $\alpha_m = \alpha_0 + m\lambda$. Second, the unknown parameters can be retrieved using the classical Prony's method. The key ingredient is the annihilating filter, for which the following holds [3]:

$$\mathbf{Sh} = \mathbf{0} \tag{2}$$

i.e. the Toeplitz matrix $\mathbf{S}$ is rank deficient. Note that we require $P \geqslant 2K-1$.

III. Working in the Presence of Noise

When the sampling process is not ideal we obtain a corrupted version of the measurements $\hat{y}_n = y_n + \epsilon_n$. The Toeplitz matrix of (2) then becomes $\hat{\mathbf{S}} = \mathbf{S} + \mathbf{B}$ and is no longer rank deficient. When the noise term $\mathbf{B}$ is additive white Gaussian (AWGN) it is reasonable to look for a solution that minimises $\|\hat{\mathbf{S}}\mathbf{h}\|^2$ s.t. $\|\mathbf{h}\| = 1$ [3]. This is a classical total-least-square (TLS) problem that can be solved using singular value decomposition (SVD). The solution is further improved by denoising $\hat{\mathbf{S}}$ using, for instance, Cadzow algorithm.

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Modified TLS and E-Splines

For exponential reproducing kernels $\mathbf{B}$ is due to coloured noise. In order for SVD to provide a reliable separation of the signal and noise subspaces it becomes necessary to "pre-whiten" the noise. If we know the covariance matrix of the noise $\mathbf{R}$ up to a constant factor λ, we can factor it: $\mathbf{R} = \lambda\mathbf{B}^*\mathbf{B} = \mathbf{Q}^T\mathbf{Q}$ and recover the appropriate subspaces by considering the SVD of $\hat{\mathbf{S}}' = \hat{\mathbf{S}}\mathbf{Q}^{-1}$.

It is also possible to control the term $\mathbf{B}$ by designing an appropriate sampling kernel. Consider the matrix $\mathbf{C}$ of size $(P+1)\times N$ with coefficients $c_{m,n}$ at locations (m,n). If we want the noise to be white we need the matrix $\mathbf{C}$ to have orthonormal rows. This is achieved by making them orthogonal with $\alpha_m = j\omega_m = j\frac{2\pi m}{N}$ and then orthonormal by setting $|c_{m,0}| = 1$, which is achieved using (1):

$$|\hat{\varphi}(\omega_m)| = |\hat{\gamma}(\omega_m)\hat{\beta}_{\vec{\alpha}_P}(\omega_m)| = 1, \tag{3}$$

where $\hat{\varphi}(\cdot)$ is the Fourier transform of $\varphi(t)$. Among the kernels satisfying (3), we are interested in the one with the shortest support. This kernel can be formed as a linear combination of various derivatives of the original E-Spline. It is a variation of the maximal-order minimal-support kernels of [5] and is still able to reproduce exponentials. Now, solving the problem in the Fourier domain we only need to determine a polynomial that interpolates $(\omega_m, |\hat{\beta}_{\vec{\alpha}_P}(\omega_m)|^{-1})$.

IV. Simulation Results

Fig. 1 shows the modified E-Spline kernels ('ME') have the best performance, which improves with increasing order P. The modified Cadzow algorithm ('MC') marginally beats the original ('C').

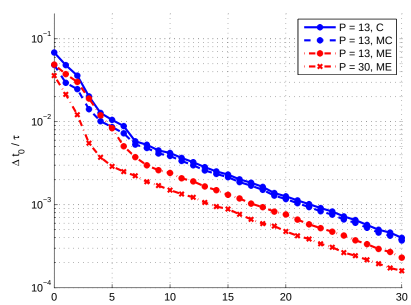

Figure 1. *Retrieval of $K = 2$ Diracs in the presence of noise*. We use $\tau = 1$ seconds, $N = 31$ samples and average over 1000 realisations.

References

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