ON THE EXPONENTIAL REPRODUCING KERNELS FOR SAMPLING SIGNALS WITH FRI

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Content

- Background on FRI
  - Signals with FRI
  - The sampling & reconstruction process
  - Sampling kernels
- The noisy scenario
  - A subspace approach
  - Prewhitening
  - Modified E-Spline kernels

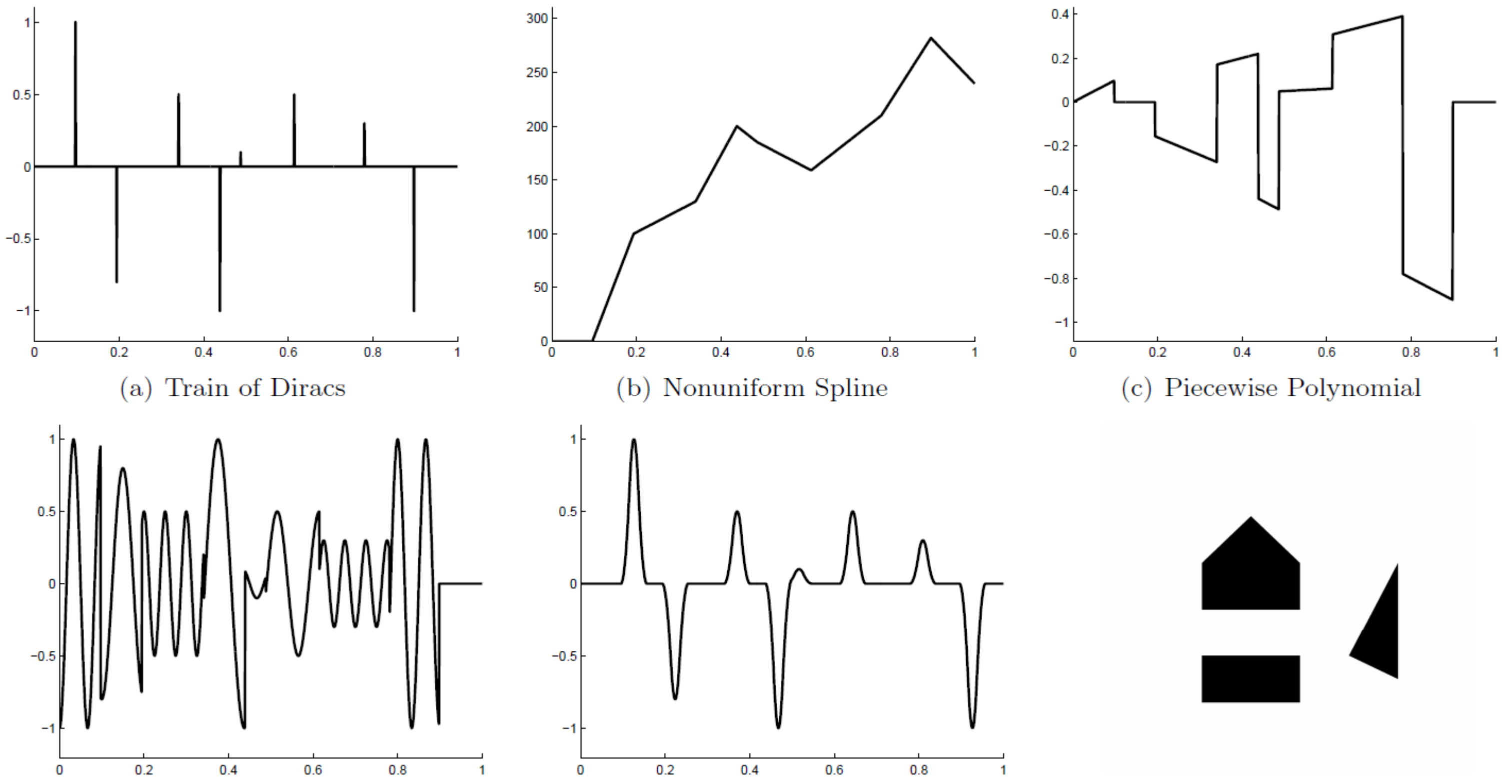
Signals with FRI (i)

- Signals with finite rate of innovation
  - Finite amount of degrees of freedom
  - Parametric representation
    - Known "shape" $g_r(t)$
    - Unknown realisation (location t_k, amplitude $\gamma_{k,r}$, …)
- Mathematically

$$x(t) = \sum_{k\in\mathbb{Z}} \sum_{r=0}^{R-1} \gamma_{k,r} g_r(t - t_k).$$

Signals with FRI (ii)

- Examples of signals with FRI

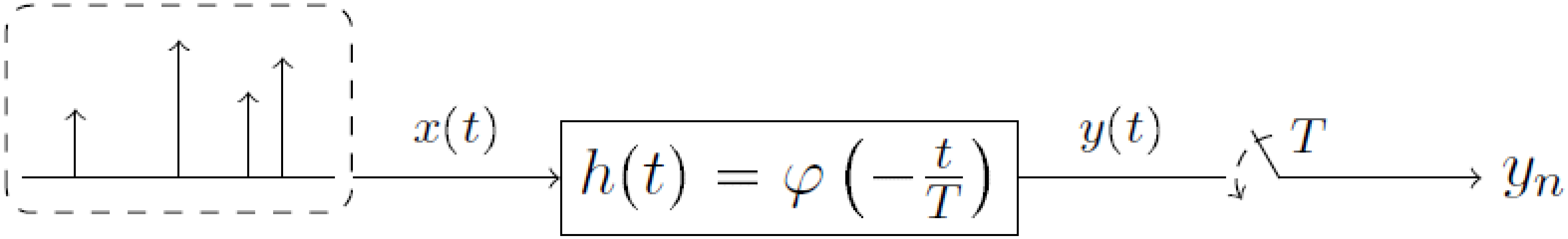


(a) Train of Diracs (b) Nonuniform Spline (c) Piecewise Polynomial

(d) Piecewise Sinusoidal (e) Stream of Pulses (f) 2D set of Bilevel Polygons

Sampling a train of Diracs (i)

- 1. Obtain the input *measurements*
  - Traditional linear scheme



  - Input characterised by (t_k, a_k) $k = 0, \ldots, K-1$
  - Set of N samples $y_n = \langle x(t), h(t - nT) \rangle$

Sampling a train of Diracs (ii)

- **2.** *Modify* the samples
  - Obtain new measurements
    - *Linear transform* $\mathbf{s} = \mathbf{C}\mathbf{y}$
    - *Set of values* s_m for $m = 0, \ldots, P$
  - *Power series equivalence*

$$s_m = f\{y_n\} = \sum_{k=0}^{K-1} \hat{a}_k u_k^m$$

    - Related to the locations and amplitudes
    - Classical spectral estimation problem: *harmonic retrieval*

Sampling a train of Diracs (iii)

- **3.** *Retrieve* the input parameters (t_k, a_k)
  - Prony's method --- Annihilating filter method

$$h_m * s_m = 0 \qquad \hat{h}(z) = \prod_{k=0}^{K-1} (1 - u_k z^{-1})$$

$$\mathbf{S}\mathbf{h} = \mathbf{0}$$

    - Toeplitz matrix **S** is rank deficient
    - Obtain t_k from **h** (null-space of **S**)
  - Find a_k using equation $s_m = f\{y_n\} = \sum_{k=0}^{K-1} \hat{a}_k u_k^m$

Sampling kernels (i)

- **Finite** support
  - Exponential reproducing kernels (*Dragotti et al*)

$$\sum_{n\in\mathbb{Z}} \boxed{c_{m,n}}\varphi(t-n) = \boxed{e^{\alpha_m t}}$$

$$c_{m,n} = e^{\alpha_m n} c_{m,0}$$

$$s_m = \sum_n \boxed{c_{m,n}} y_n$$

$$\mathbf{s} = \mathbf{C}\mathbf{y}$$

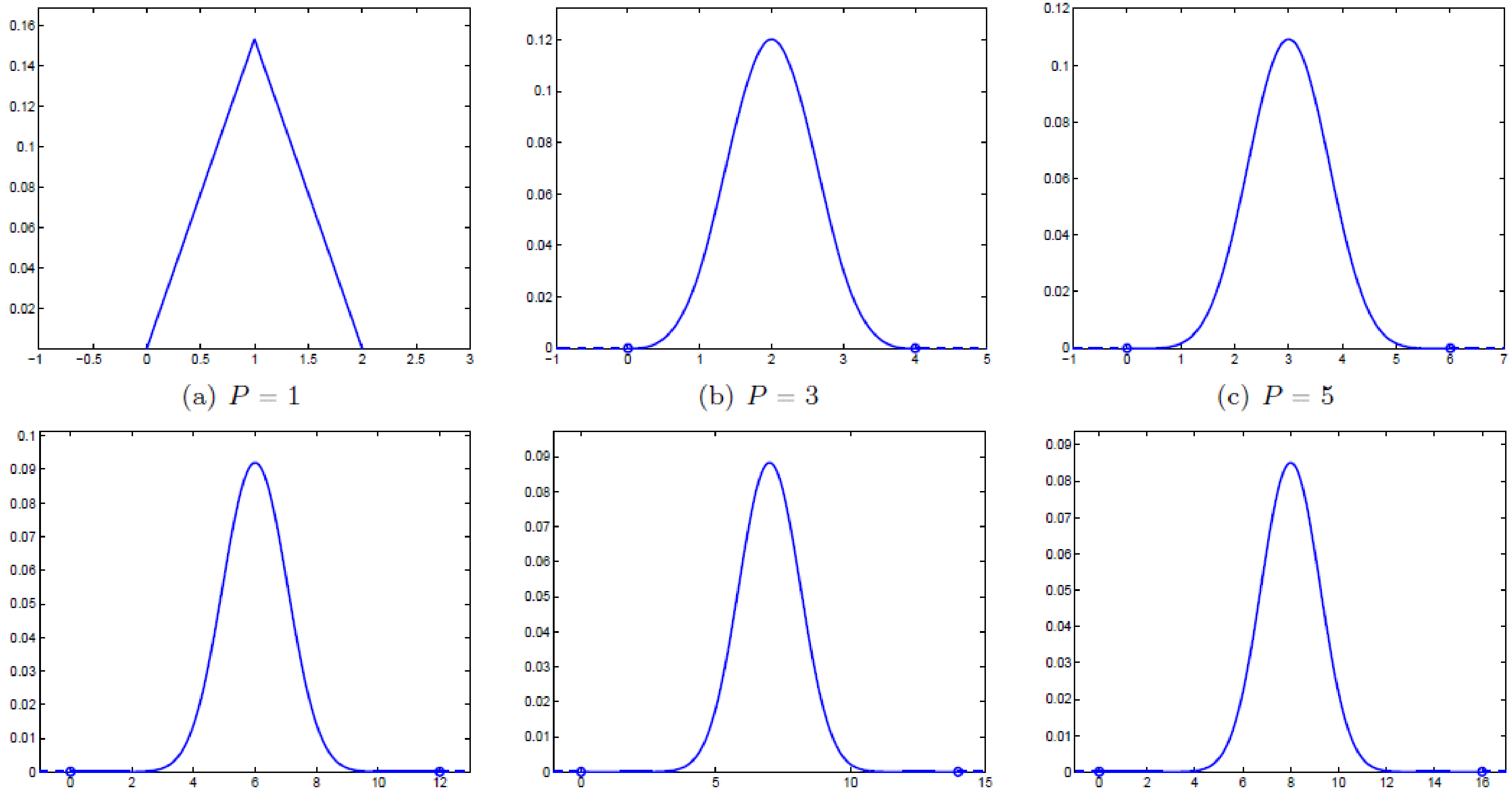
E-Splines

$$\hat{\beta}_{\vec{\alpha}_P}(\omega) = \prod_{m=0}^{P} \left(\frac{1 - e^{\alpha_m - j\omega}}{j\omega - \alpha_m} \right)$$

$$\varphi(t) = \gamma(t) * \beta_{\vec{\alpha}_P}(t)$$

Sampling kernels (ii)

- Kernel examples (E-Splines)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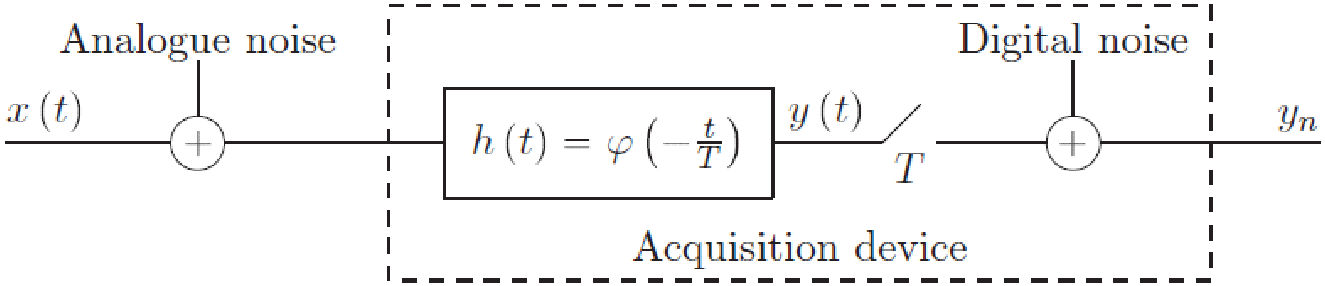
The noisy scenario

- Sampling scheme



- Consider only digital noise: AWGN(0, σ)

$$\tilde{y}_n = \langle x(t), h(t - nT) \rangle + \epsilon_n$$

- Degrades performance reconstruction algorithms

A Subspace Approach (i)

- Measurements change

$$\hat{s}_m = f\{y_n + \epsilon_n\} = f\{y_n\} + f\{\epsilon_n\} \qquad \mathbf{s} = \mathbf{C}\mathbf{y} + \mathbf{C}\mathbf{e}$$

- Toeplitz matrix **S** changes too

$$\mathbf{S}\mathbf{h} = \mathbf{0}$$

$$\begin{pmatrix} s_L & s_{L-1} & \cdots & s_0 \\ s_{L+1} & s_L & \cdots & s_1 \\ \vdots & \vdots & \ddots & \vdots \\ s_P & s_{P-1} & \cdots & s_{P-L} \end{pmatrix}$$

$$\hat{\mathbf{S}} = \mathbf{S} + \mathbf{B}.$$

A Subspace Approach (ii)

- Assume the term **B** is due to AWGN

$$\mathbf{s} = \mathbf{C}\mathbf{y} + \mathbf{C}\mathbf{e} \qquad \hat{\mathbf{S}} = \mathbf{S} + \mathbf{B}.$$

$$\mathbf{S}\mathbf{h} = \mathbf{0} \qquad \|\hat{\mathbf{S}}\mathbf{h}\|^2 \text{ s.t. } \|\mathbf{h}\|^2 = 1$$

- Covariance matrix $\mathbf{B}^*\mathbf{B} = \sigma^2\mathbf{I}$
- **SVD** is able to separate signal and noise subspaces
  - **h** vector corresponding to the noise subspace in SVD
  - Total Least Squares, Cadzow

A Subspace Approach (iii)

- **B** for exp rep kernels is due to *coloured* noise

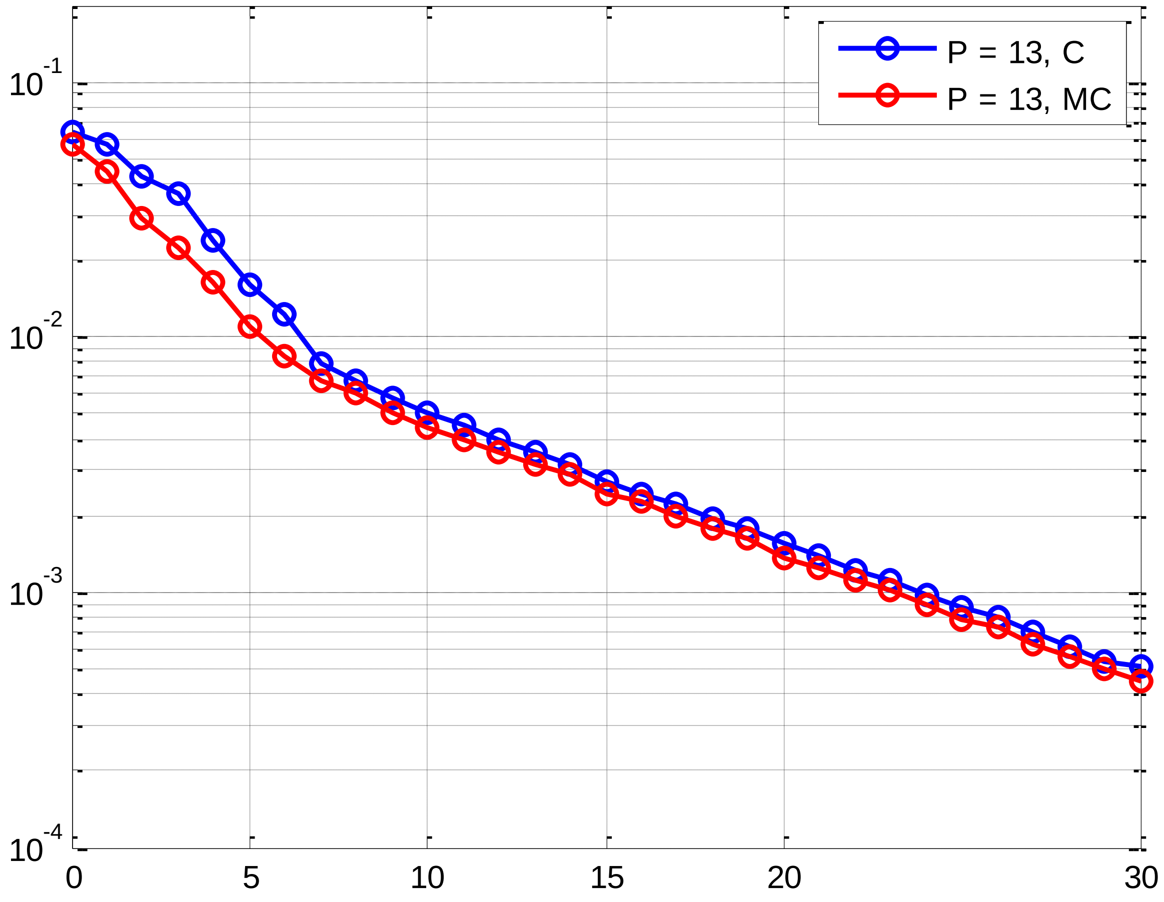
$$\mathbf{s} = \mathbf{C}\mathbf{y} + \boxed{\mathbf{C}\mathbf{e}} \qquad \hat{\mathbf{S}} = \mathbf{S} + \mathbf{B}.$$

$$\mathbf{S}\mathbf{h} = 0 \qquad \|\hat{\mathbf{S}}\mathbf{h}\|^2 \text{ s.t. } \|\mathbf{h}\|^2 = 1$$

- Covariance matrix $\mathbf{R} = \lambda\mathbf{B}^*\mathbf{B}$ $\quad \mathbf{R} = \mathbf{Q}^T\mathbf{Q}$
- *pre-whiten* $\hat{\mathbf{S}}' = \hat{\mathbf{S}}\mathbf{Q}^{-1}$ $\quad (\mathbf{B}\mathbf{Q}^{-1})^*(\mathbf{B}\mathbf{Q}^{-1}) = \sigma^2\lambda^{-1}\mathbf{I}$
- **SVD** is *now* able to separate subspaces
  - Modified TLS or Cadzow

A Subspace Approach (iv)

- Simulations



Modifying E-Splines (i)

- Coloured noise term

$$\mathbf{s} = \mathbf{C}\mathbf{y} + \boxed{\mathbf{C}\mathbf{e}}$$

$$c_{m,n} = \mathrm{e}^{\alpha_m n} c_{m,0} \qquad \varphi(t) = \gamma(t) * \beta_{\vec{\alpha}_P}(t)$$

- *Goal*: **C** to have orthonormal columns
  - Orthogonal $\alpha_m = j\omega_m = j\frac{2\pi m}{N}$
  - Orthonormal $|c_{m,0}| = 1$
    - Then, we have a DFT like transform $s_m = \sum_n c_{m,n} y_n$

Modifying E-Splines (ii)

- Orthonormality
  - Then $|c_{m,0}| = 1$ is equivalent to

$$\left|\hat{\tilde{\varphi}}\left(\frac{2\pi m}{N}\right)\right| = 1, \quad m = 0, 1, \ldots, P.$$

  - Now, the dual is related to the kernel as

$$\hat{\tilde{\varphi}}(\omega) = \frac{\hat{\varphi}(\omega)}{\sum_{k\in\mathbb{Z}} |\hat{\varphi}(\omega + 2\pi k)|^2}$$

  - Considering the transforms

$$\hat{\beta}_{\vec{\alpha}_P}(\omega) = \prod_{m=0}^{P} \mathrm{e}^{-j\frac{\omega-\omega_m}{2}} \operatorname{sinc}\left(\frac{\omega-\omega_m}{2}\right)$$

$$\hat{\varphi}(\omega) = \hat{\gamma}(\omega)\hat{\beta}_{\vec{\alpha}_P}(\omega)$$

$$\alpha_m = j\omega_m = j\frac{2\pi m}{N}$$

$$\hat{\tilde{\varphi}}(\omega_m) = \frac{\hat{\varphi}(\omega_m)}{|\hat{\varphi}(\omega_m)|^2}$$

Modifying E-Splines (iii)

- Then $\left|c_{m,0}\right| = 1$ or

$$\left|\hat{\tilde{\varphi}}\left(\frac{2\pi m}{N}\right)\right| = 1, \quad m = 0, 1, \ldots, P.$$

- Implies that

$$\hat{\tilde{\varphi}}(\omega_m) = \frac{\hat{\varphi}(\omega_m)}{|\hat{\varphi}(\omega_m)|^2} \qquad \alpha_m = j\omega_m = j\frac{2\pi m}{N}$$

$$|\hat{\varphi}(\omega_m)| = |\hat{\gamma}(\omega_m)\hat{\beta}_{\vec{\alpha}_P}(\omega_m)| = 1$$

Modifying E-Splines (iv)

$$|\hat{\varphi}(\omega_m)| = |\hat{\gamma}(\omega_m)\hat{\beta}_{\vec{\alpha}_P}(\omega_m)| = 1 \qquad \varphi(t) = \gamma(t) * \beta_{\vec{\alpha}_P}(t)$$

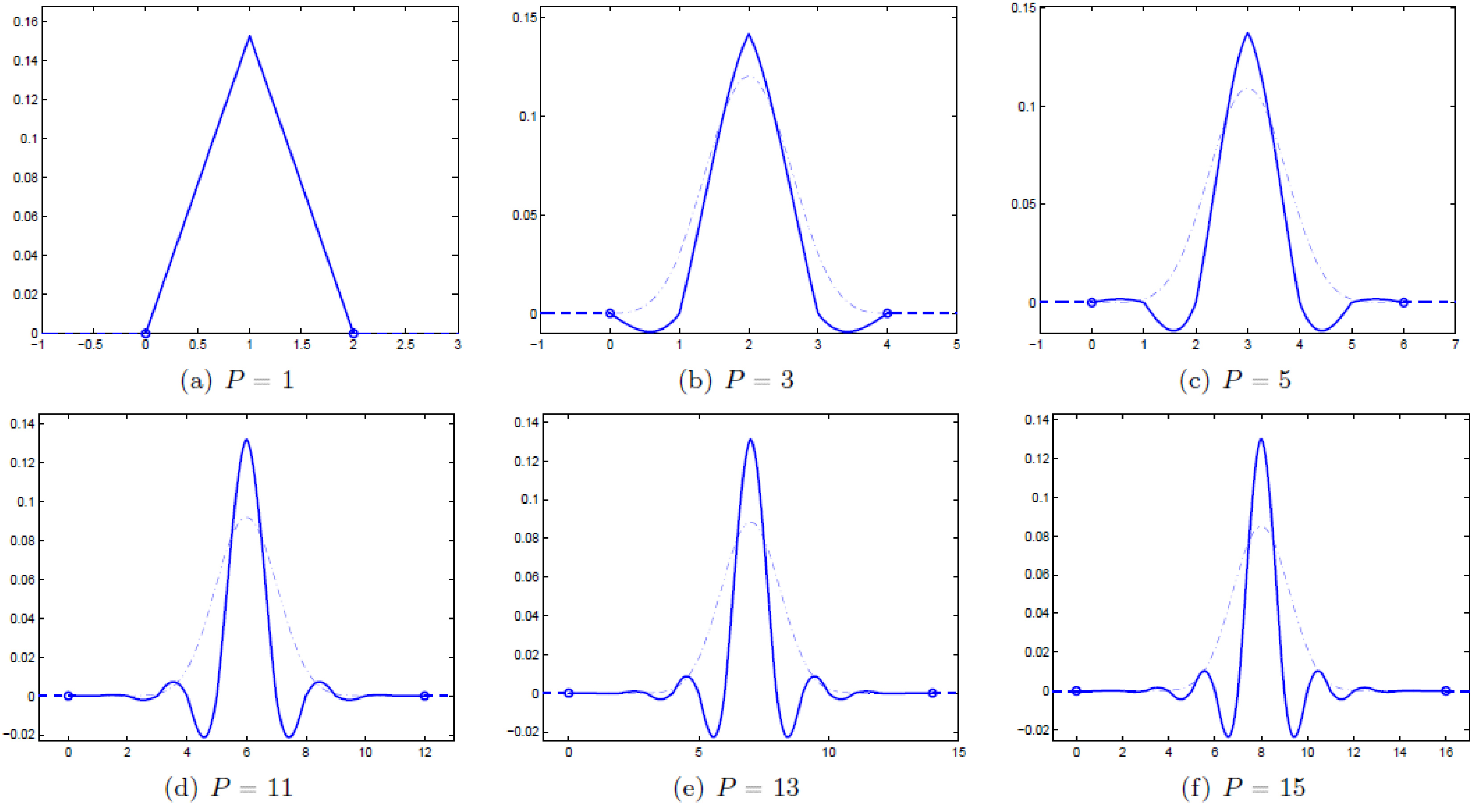
$$\Leftrightarrow \quad |\hat{\gamma}(\omega_m)| = |\hat{\beta}_{\vec{\alpha}_P}(\omega_m)|^{-1}$$

- Polynomial $\sum_i d_i(j\omega)^i$ interpolate $(\omega_m, |\hat{\beta}_{\vec{\alpha}_P}(\omega_m)|^{-1})$
  - Only find coefficients
  - Maximal-order minimal-support kernel

$$\hat{\varphi}(\omega) = \hat{\beta}_{\vec{\alpha}_P}(\omega) \sum_{i=0}^{P-1} d_i(j\omega)^i \qquad \varphi(t) = \sum_{i=0}^{P-1} d_i \beta_{\vec{\alpha}_P}^{(i)}(t)$$

Modifying E-Splines (v)

- Kernel examples



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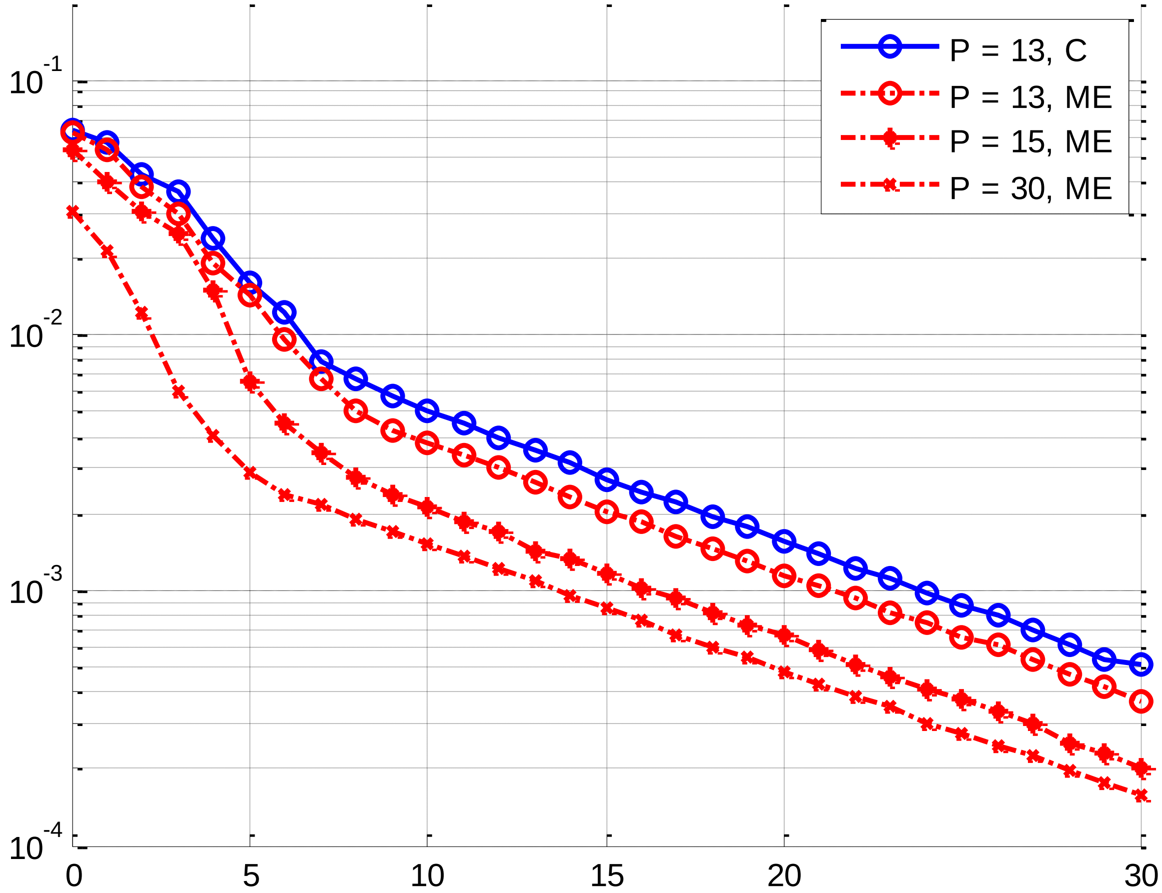
Modifying E-Splines (vi)

- Simulations



Conclusions

- **Noisy FRI scenario**
  - Introduced FRI and explained extension
  - *Modified* TLS / Cadzow for coloured noise
  - *Redesigned* kernels
- Future work
  - Subspace denoising: alternative improvements
  - Other approaches?
  - Adaptive filtering

Questions

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The Sum of Sincs (i)

- Consider a modified E-spline s.t.
  - P even
  - The number of samples $N = P + 1$
  - Kernel centred in zero

$$\varphi'(t) = \varphi\left(t + \frac{P+1}{2}\right) \qquad \varphi(t) = \gamma(t) * \beta_{\vec{\alpha}_P}(t)$$

  - Satisfies

$$|\hat{\varphi}(\omega_m)| = |\hat{\gamma}(\omega_m)\hat{\beta}_{\vec{\alpha}_P}(\omega_m)| = b_m$$

- We use the periodic extension of the kernel

$$b(t) = \sum_{l \in \mathbb{Z}} \varphi'(t + lN)$$

The Sum of Sincs (ii)

- Applying the Poisson summation formula

$$b(t) = \sum_{l\in\mathbb{Z}} \varphi'(t+lN) = \frac{1}{P+1}\sum_{k\in\mathbb{Z}} \hat{\varphi}'\left(\frac{2\pi k}{P+1}\right) \mathrm{e}^{\frac{2\pi k}{P+1}t}$$

  where

$$\hat{\varphi}'(\omega) = \gamma(\omega)\prod_{m=0}^{P} \mathrm{sinc}\left(\frac{\omega-\omega_m}{2}\right)$$

- Consider now all the possible values of *k*, and the subset $\mathcal{K} = \{k : k = \frac{2m-P}{2}, m = 0, \ldots, P\}$

  then

$$\omega_k = \frac{2\pi k}{P+1} \qquad \begin{aligned} \hat{\varphi}'(\omega_k) &= b_k \quad k \in \mathcal{K}, \\ \hat{\varphi}'(\omega_k) &= 0 \quad\; k \notin \mathcal{K} \end{aligned}$$

The Sum of Sincs (iii)

- In total we have that

$$b(t) = \sum_{l \in \mathbb{Z}} \varphi'(t + lN) = \frac{1}{P+1} \sum_{k \in \mathbb{Z}} \hat{\varphi}' \left(\frac{2\pi k}{P+1} \right) \mathrm{e}^{\frac{2\pi k}{P+1} t}$$

becomes

$$b(t) = \frac{1}{P+1} \sum_{k=-\frac{P}{2}}^{\frac{P}{2}} b_k \mathrm{e}^{\frac{2\pi k}{P+1} t}$$

- And, finally, with a change of variable we get the SoS kernel

$$b\left(\frac{x}{T}\right) = g(x) = \mathrm{rect}\left(\frac{x}{\tau}\right) \frac{1}{N} \sum_{k \in \mathcal{K}} b_k \mathrm{e}^{\frac{2\pi k}{\tau} x}$$