ON EXTENSIONS AND APPLICATIONS OF FRI THEORY

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Outline

- ***Part I*: Dealing with noise effectively in FRI**
  - *Overview*: the FRI sampling scheme
  - Modified TLS and Cadzow
  - Alternative exponential reproducing kernels
- ***Part II*: Sparse Characterization of Neuronal Signals through FRI theory**
  - Neurons & Neuronal activity
    - Action Potentials
    - Calcium Transients
  - Modelling Neuronal Signals

Dealing with noise effectively in FRI

Part I: Content

- Background on FRI
  - Signals with FRI
  - The sampling & reconstruction process
  - Sampling kernels
- The noisy scenario
  - A subspace approach
  - Prewhitening
  - Modified E-Spline kernels

Background on FRI (i)

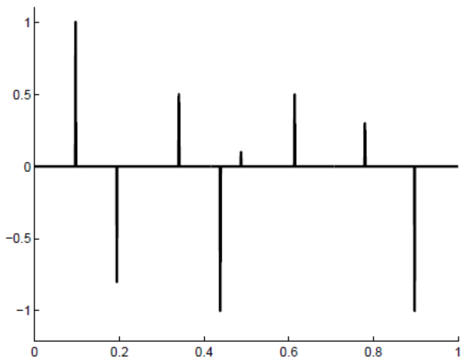
- Signals with finite rate of innovation (FRI)
  - Parametric representation
  - Known "shape" $g_r(t)$
  - Unknown realisation (location t_k, amplitude $\gamma_{k,r}$, ...)
- Mathematically

$$x(t) = \sum_{k \in \mathbb{Z}} \sum_{r=0}^{R-1} \gamma_{k,r} g_r(t - t_k).$$

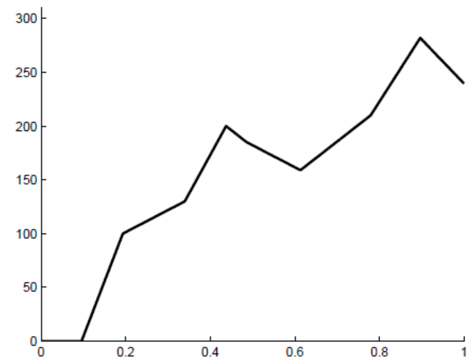
$$\rho = \lim_{\tau \to \infty} \frac{1}{\tau} C_x \left(-\frac{\tau}{2}, \frac{\tau}{2} \right).$$

Background on FRI (ii)

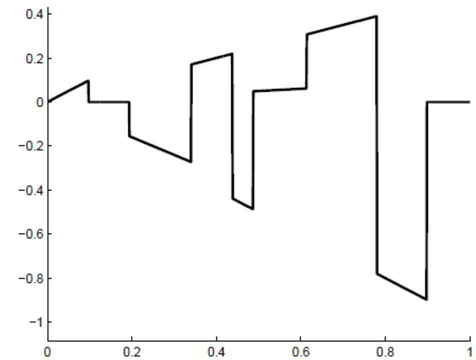
- Examples of signals with FRI



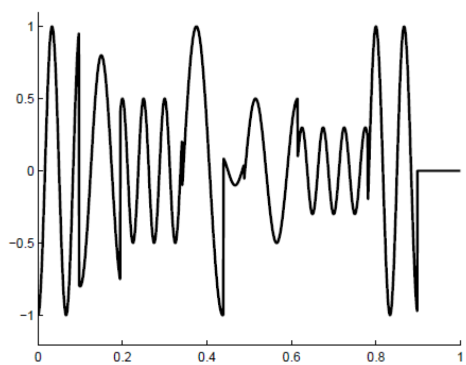
(a) Train of Diracs



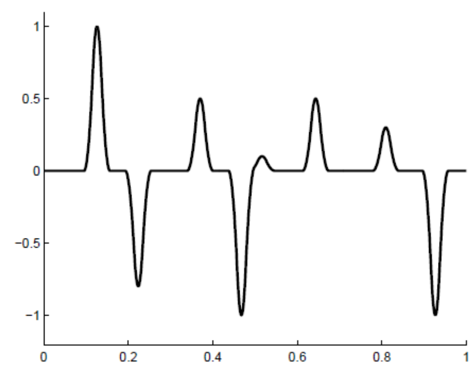
(b) Nonuniform Spline



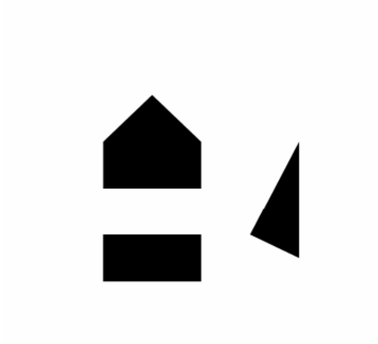
(c) Piecewise Polynomial



(d) Piecewise Sinusoidal



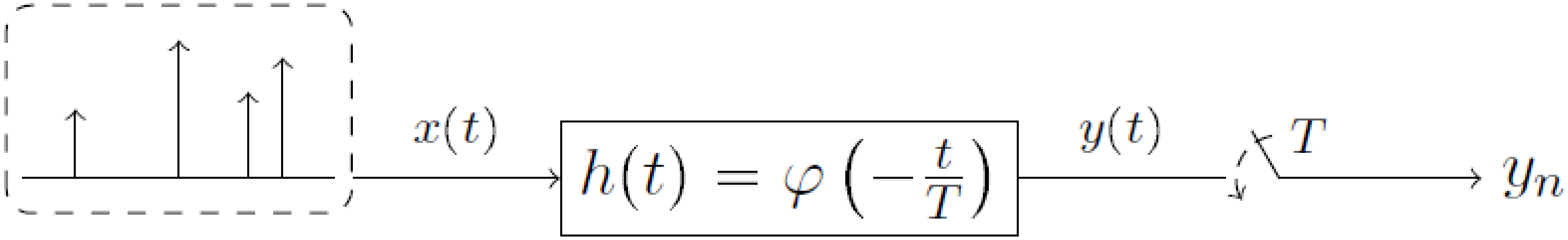
(e) Stream of Pulses



(f) 2D set of Bilevel Polygons

The sampling process (i)

- **1.** Obtain the input *measurements*



$$y_n = \langle x(t), h(t - nT) \rangle$$

The sampling process (ii)

- **2.** *Modify* the samples
  - Sequence of new measurements s_m
  - *Power series*

$$s_m = f\{y_n\} = \sum_{k=0}^{K-1} \hat{a}_k u_k^m \qquad \mathbf{s} = \mathbf{C}\mathbf{y}$$

  - Related to the locations and amplitudes
  - Classical spectral estimation problem: *harmonic retrieval*

The sampling process (iii)

- **3.** *Retrieve* the parameters
  - Prony's method --- Annihilating filter method

$$h_m * s_m = 0 \qquad \hat{h}(z) = \prod_{k=0}^{K-1} (1 - u_k z^{-1})$$

$$\mathbf{Sh} = \mathbf{0}$$

  - Toeplitz matrix **S** is rank deficient
  - Obtain **h** (null-space)
  - Find **a** using equation $s_m = f\{y_n\} = \sum_{k=0}^{K-1} \hat{a}_k u_k^m$

Sampling kernels (i)

- **Infinite** support
  - Classical kernel (*Vetterli et al*)

$$h_B(t) = B\,\mathrm{sinc}(Bt) \qquad s_m = \underbrace{\boxed{\mathrm{DFT}}}_{\mathbf{C}}\{y_n\}$$

- **Finite** support
  - Poly, Exp reproducing (*Dragotti et al*)

$$\sum_{n\in\mathbb{Z}} \underbrace{\boxed{c_{m,n}}}_{\mathbf{C}}\varphi(t-n) = \boxed{e^{\alpha_m t}}$$

$$s_m = \sum_n \boxed{c_{m,n}} y_n$$

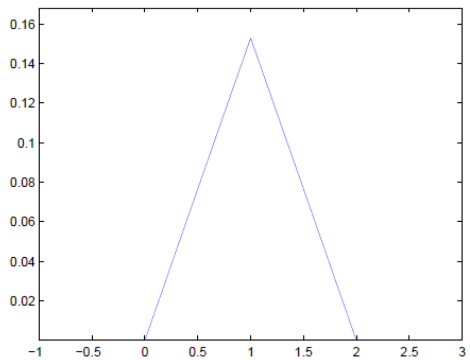
$$c_{m,n} = e^{\alpha_m n} c_{m,0}$$

$$\hat{\beta}_{\vec{\alpha}_P}(\omega) = \prod_{m=0}^{P}\left(\frac{1-e^{\alpha_m - j\omega}}{j\omega - \alpha_m}\right)$$

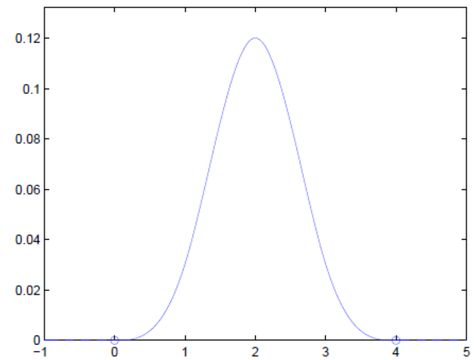
$$\varphi(t) = \gamma(t) * \beta_{\vec{\alpha}_P}(t)$$

Sampling kernels (ii)

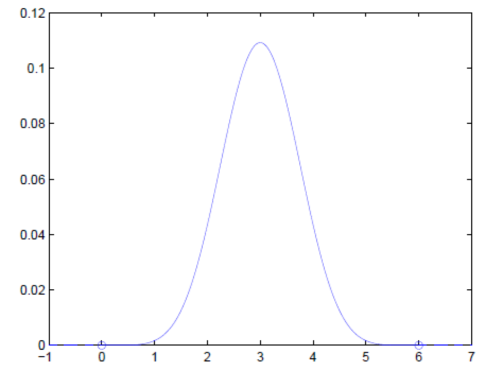
- Kernel examples



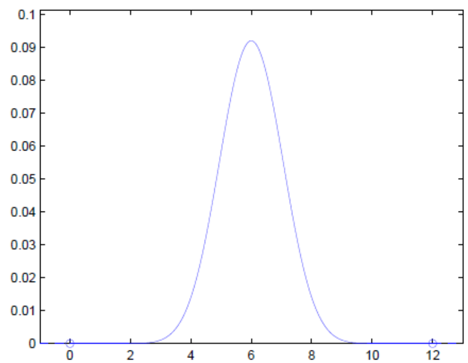
(a) $P = 1$



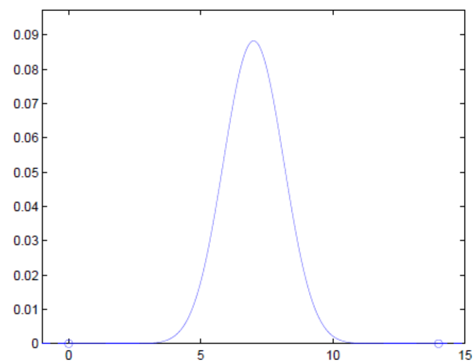
(b) $P = 3$



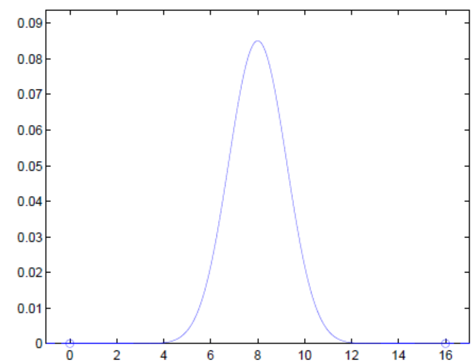
(c) $P = 5$



(d) $P = 11$



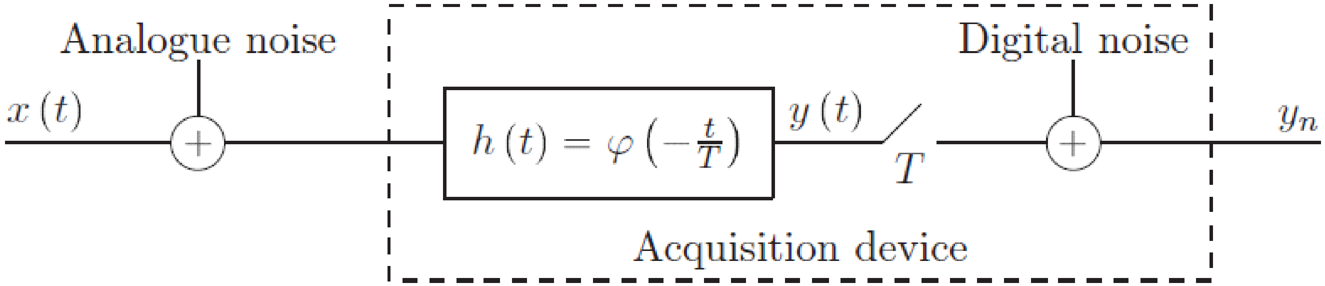
(e) $P = 13$



(f) $P = 15$

The noisy scenario

- Sampling scheme



- Consider only digital noise: AWGN(0, σ)

$$\tilde{y}_n = \langle x(t), h(t - nT) \rangle + \epsilon_n$$

- Degrades performance of basic algorithms

A Subspace Approach (i)

- Measurements change

$$\hat{s}_m = f\{y_n + \epsilon_n\} = f\{y_n\} + f\{\epsilon_n\} \qquad \mathbf{s} = \mathbf{C}\mathbf{y} + \mathbf{C}\mathbf{e}$$

- Toeplitz matrix **S** changes too

$$\mathbf{S}\mathbf{h} = \mathbf{0}$$

$$\begin{pmatrix} s_L & s_{L-1} & \cdots & s_0 \\ s_{L+1} & s_L & \cdots & s_1 \\ \vdots & \vdots & \ddots & \vdots \\ s_P & s_{P-1} & \cdots & s_{P-L} \end{pmatrix}$$

$$\hat{\mathbf{S}} = \mathbf{S} + \mathbf{B}.$$

A Subspace Approach (ii)

- Assume the term **B** is due to AWGN

$$\hat{s}_m = f\{y_n + \epsilon_n\} = f\{y_n\} + f\{\epsilon_n\} \qquad \hat{\mathbf{S}} = \mathbf{S} + \mathbf{B}.$$

$$\mathbf{S}\mathbf{h} = \mathbf{0} \qquad \|\hat{\mathbf{S}}\mathbf{h}\|^2 \text{ s.t. } \|\mathbf{h}\|^2 = 1$$

- Covariance matrix $\mathbf{B}^*\mathbf{B} = \sigma^2\mathbf{I}$ $\qquad \mathbf{R} = \mathbf{Q}^T\mathbf{Q}$

$$\mathrm{E}\{\mathbf{B}^*\mathbf{B}\} = \sigma^2\mathbf{I}$$

- **SVD** is able to separate signal and noise subspaces
  - Total Least Squares, Cadzow
  - **h** vector corresponding to the noise subspace in SVD

A Subspace Approach (iii)

- **B** for exp rep kernels is due to *coloured* noise

$$\hat{s}_m = f\{y_n + \epsilon_n\} = f\{y_n\} + f\{\epsilon_n\} \qquad \mathbf{s} = \mathbf{C}\mathbf{y} + \boxed{\mathbf{C}\mathbf{e}}$$

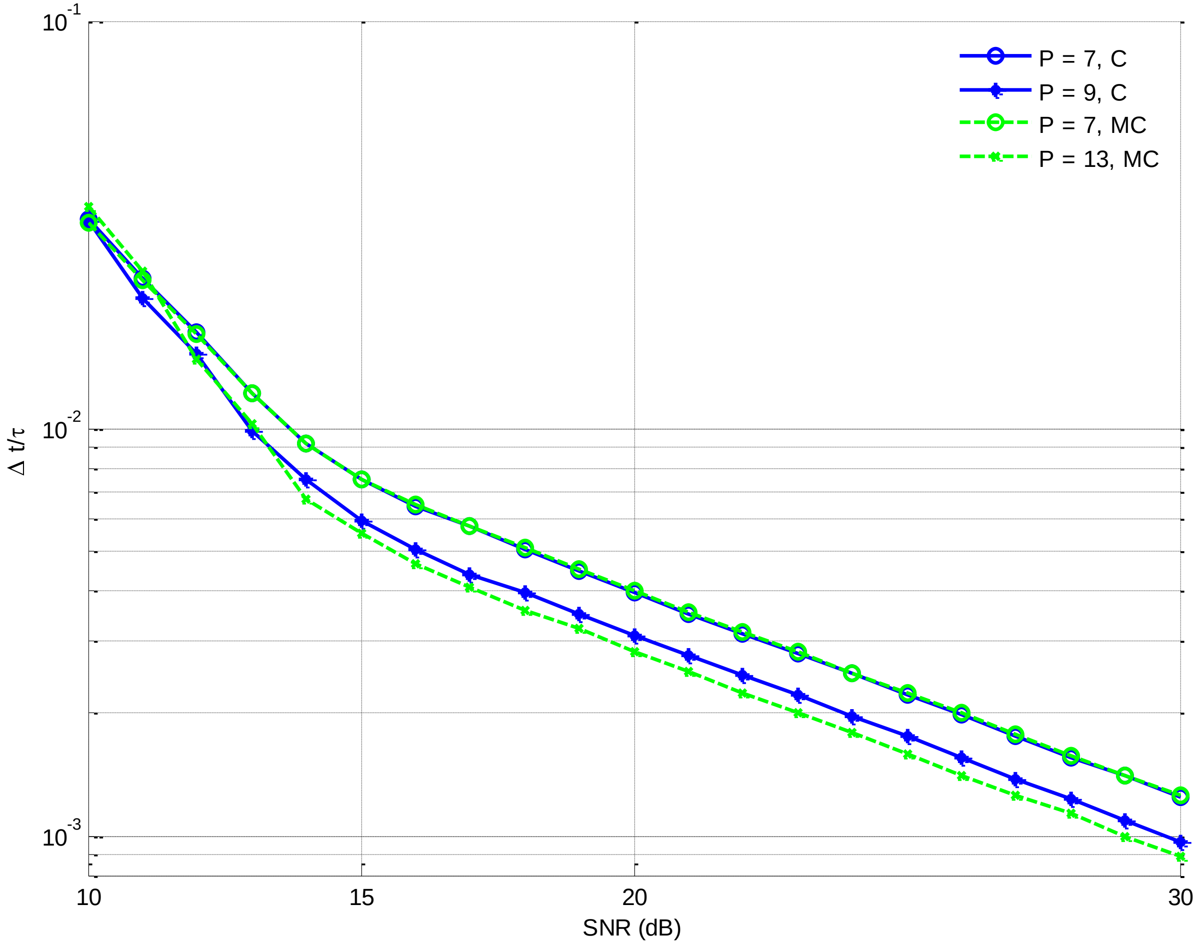
$$\hat{\mathbf{S}} = \mathbf{S} + \mathbf{B}. \qquad \mathbf{S}\mathbf{h} = \mathbf{0} \qquad \|\hat{\mathbf{S}}\mathbf{h}\|^2 \text{ s.t. } \|\mathbf{h}\|^2 = 1$$

$$\mathbf{R} = \lambda \mathrm{E}\{\mathbf{B}^*\mathbf{B}\}$$

- Covariance matrix $\mathbf{R} = \lambda \mathbf{B}^*\mathbf{B}$ $\qquad \mathbf{R} = \mathbf{Q}^T\mathbf{Q}$
- *pre-whiten* $\hat{\mathbf{S}}' = \hat{\mathbf{S}}\mathbf{Q}^{-1} \qquad (\mathbf{B}\mathbf{Q}^{-1})^*(\mathbf{B}\mathbf{Q}^{-1}) = \sigma^2\lambda^{-1}\mathbf{I}$
- **SVD** is *now* able to separate subspaces
  - Modified TLS or Cadzow

A Subspace Approach (iv)

- Simulations



Modifying E-Splines (i)

- Coloured noise term

$$\mathbf{s} = \mathbf{C}\mathbf{y} + \boxed{\mathbf{C}\mathbf{e}}$$

$$c_{m,n} = \mathrm{e}^{\alpha_m n} c_{m,0} \qquad \varphi(t) = \gamma(t) * \beta_{\vec{\alpha}_P}(t)$$

- *Goal*: **C** to have orthonormal columns
  - Orthogonal $\alpha_m = j\omega_m = j\frac{2\pi m}{N}$
  - Orthonormal $|c_{m,0}| = 1$

$$|\hat{\varphi}(\omega_m)| = |\hat{\gamma}(\omega_m)\hat{\beta}_{\vec{\alpha}_P}(\omega_m)| = 1$$

Modifying E-Splines (ii)

$$|\hat{\varphi}(\omega_m)| = |\hat{\gamma}(\omega_m)\hat{\beta}_{\vec{\alpha}_P}(\omega_m)| = 1 \qquad \varphi(t) = \gamma(t) * \beta_{\vec{\alpha}_P}(t)$$

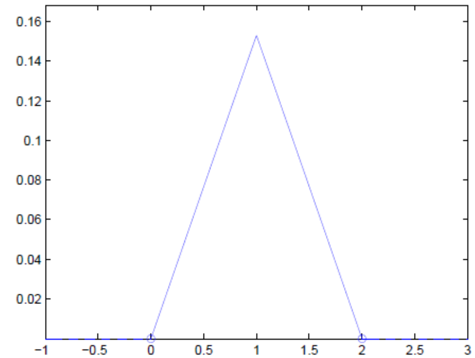
$$\Leftrightarrow \quad |\hat{\gamma}(\omega_m)| = |\hat{\beta}_{\vec{\alpha}_P}(\omega_m)|^{-1}$$

- Polynomial $\sum_i d_i(j\omega)^i$ interpolate $(\omega_m, |\hat{\beta}_{\vec{\alpha}_P}(\omega_m)|^{-1})$
  - Only find coeffs
  - Maximal-order minimal-support kernel

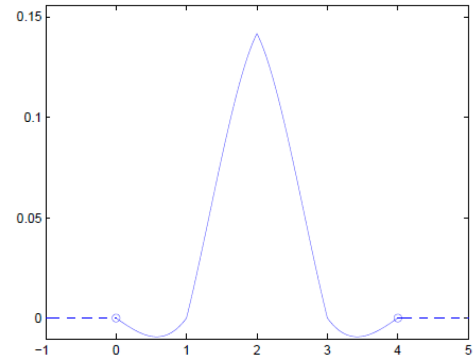
$$\hat{\varphi}(\omega) = \hat{\beta}_{\vec{\alpha}_P}(\omega) \sum_{i=0}^{P-1} d_i(j\omega)^i \qquad \varphi(t) = \sum_{i=0}^{P-1} d_i \beta^{(i)}_{\vec{\alpha}_P}(t)$$

Modifying E-Splines (iii)

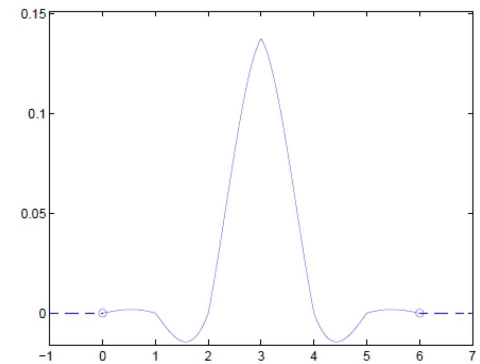
- Kernel examples



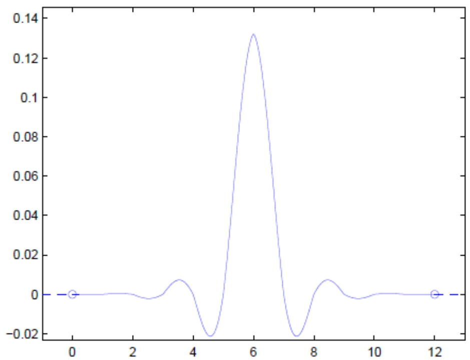
(a) $P = 1$



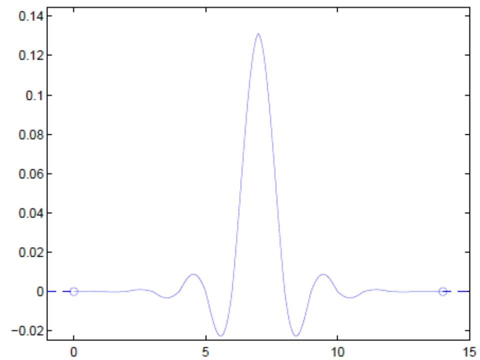
(b) $P = 3$



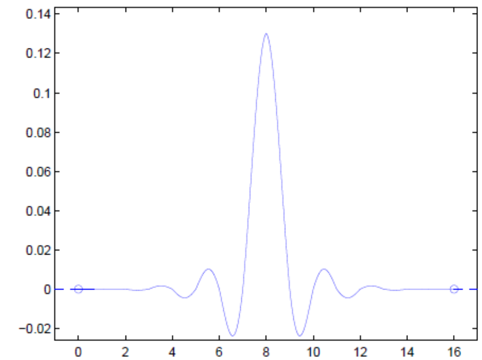
(c) $P = 5$



(d) $P = 11$



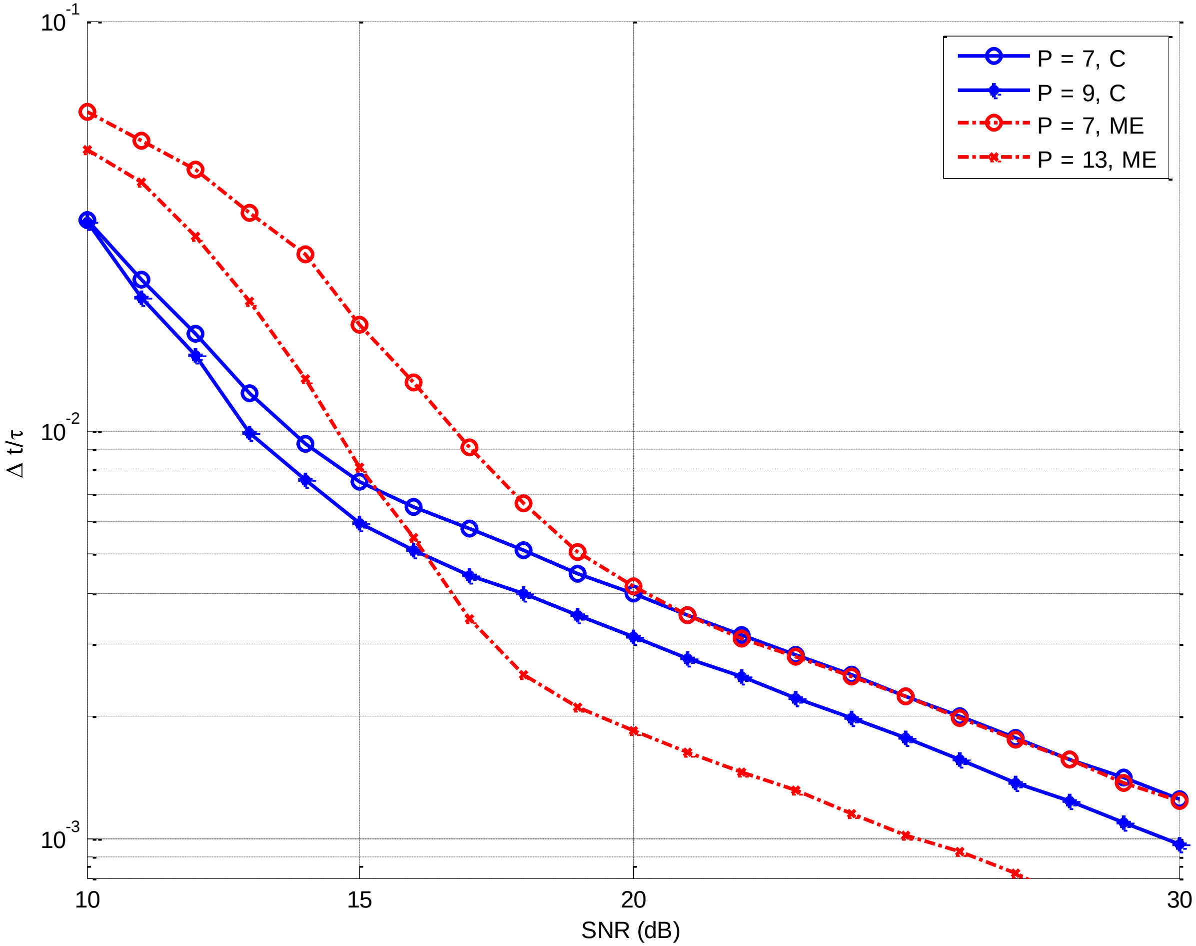
(e) $P = 13$



(f) $P = 15$

Modifying E-Splines (iv)

- Simulations



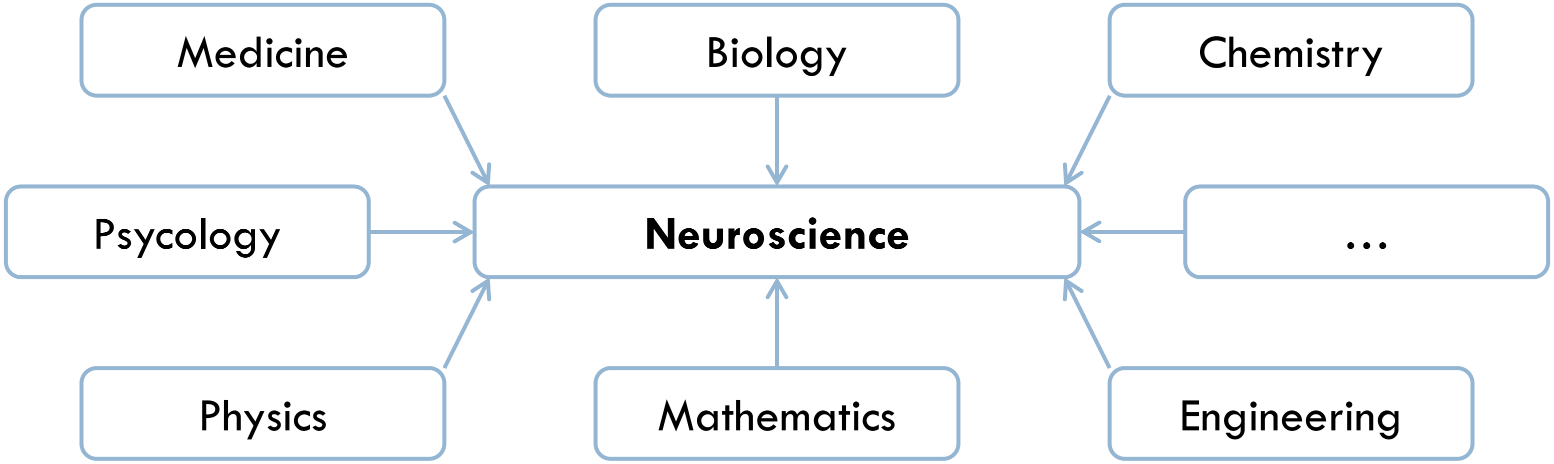
Sparse Characterization of Neuronal Signals through FRI theory

Part II: Content

- Neuroscience
- Brain cells
- Neuronal activity
  - Action Potentials (AP)
  - Calcium Transients
- Modelling Neuronal Signals
  - Sparsity
  - Simulations
- Conclusions

Neuroscience today

- Scientific study of the nervous system
- **Interdisciplinary approach**: best way to improve understanding of the brain



Levels of analysis

- Study of the *nervous system*. In ascending order of complexity:

Molecular Neuroscience

Cellular Neuroscience

Systems Neuroscience

Behavioral Neuroscience

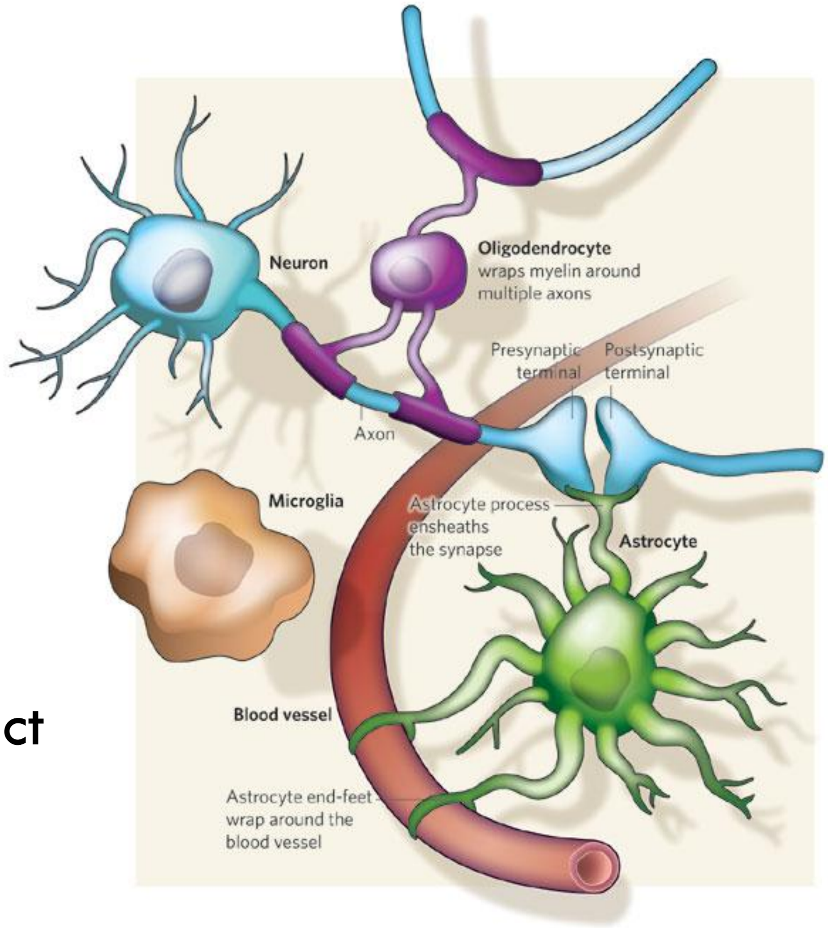
Cognitive Neuroscience

Computational Neuroscience

- Single neuron
- Physiological background
  - Characterise structure to reproduce behaviour
- Modelling
  - Maths / Physics: Hodgkin and Huxley's
  - Electrics: cable theory, Spike Response Model

Brain cells

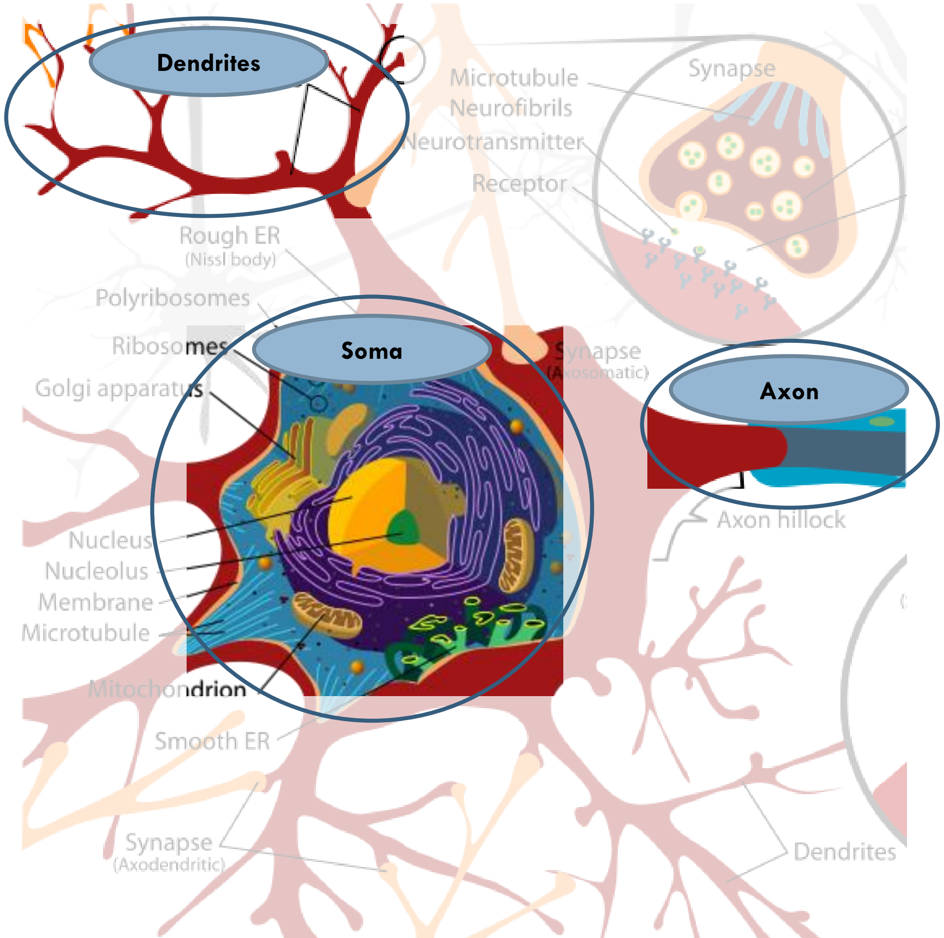
- **Neurons**
  - Sense
  - Communicate
  - React
- Glia (10:1)
  - Insulate
  - Support
  - Nourish
- Different types of glia interact with neurons and the surrounding blood vessels



http://www.nature.com/nature/journal/v457/n7230/fig_tab/457675a_F1.html

Neurons

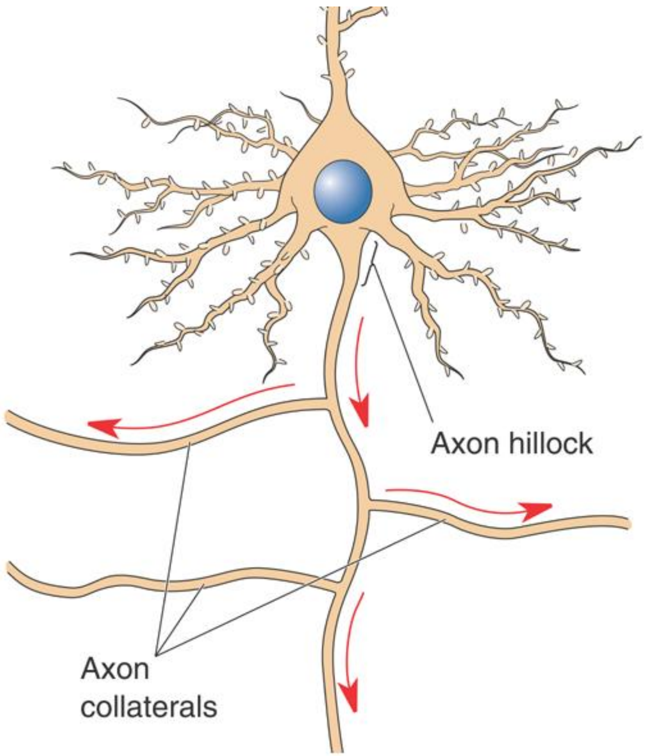
- Nerve cell
- Main parts
  - Soma
  - Axon
  - Dendrites
- Inner / Outer separation
  - Neuronal membrane



Wikipedia: Neuron

The Axon

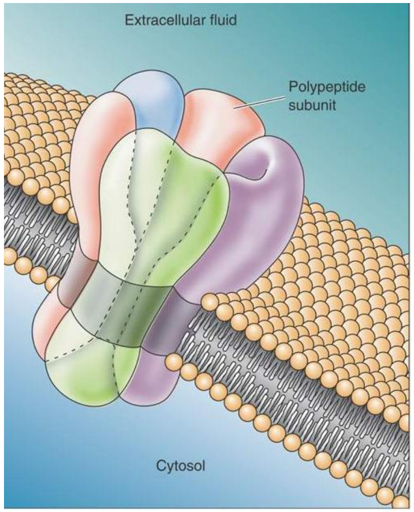
- Unique to neurons
- Transfer information
- Parts
  - Hillock
  - Collaterals
  - Terminal
    - Contact with other neurons (synapse)
    - Axon terminal with dendrites or soma

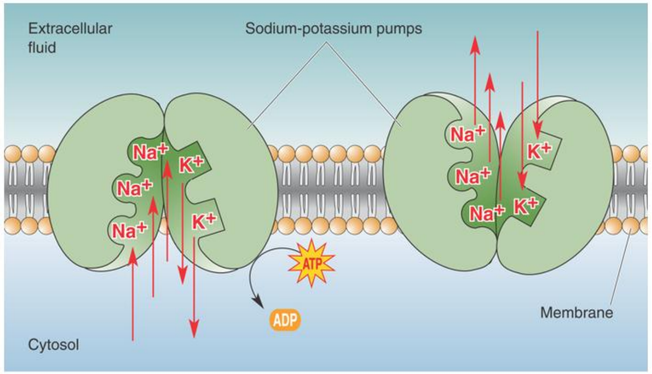


Neuroscience: Exploring the Brain, 3rd Ed. Bear, Connors and Paradiso.

Neurons at rest (i)

- Cytosolic & Extracellular fluids (Na^+, K^+, Ca^{2+}, Cl^-)
- Phospholipid bilayer (membrane)
- Proteins
  - Ion Pumps
  - Ion Channels
- Regulate membrane potential at rest





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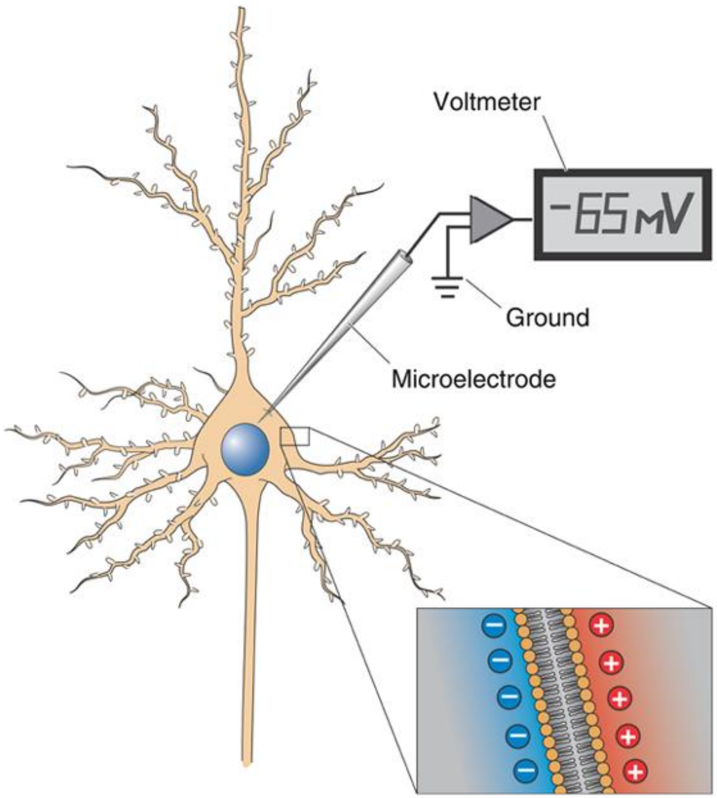
Neurons at rest (ii)

- Ion pumps
  - $[K^+]$ ($[Na^+]$) higher inside (outside)
- Ion channels
  - Initially more permeable to K^+
- Diffusion vs Electrical potential
  - *Balance*: **equilibrium potential**



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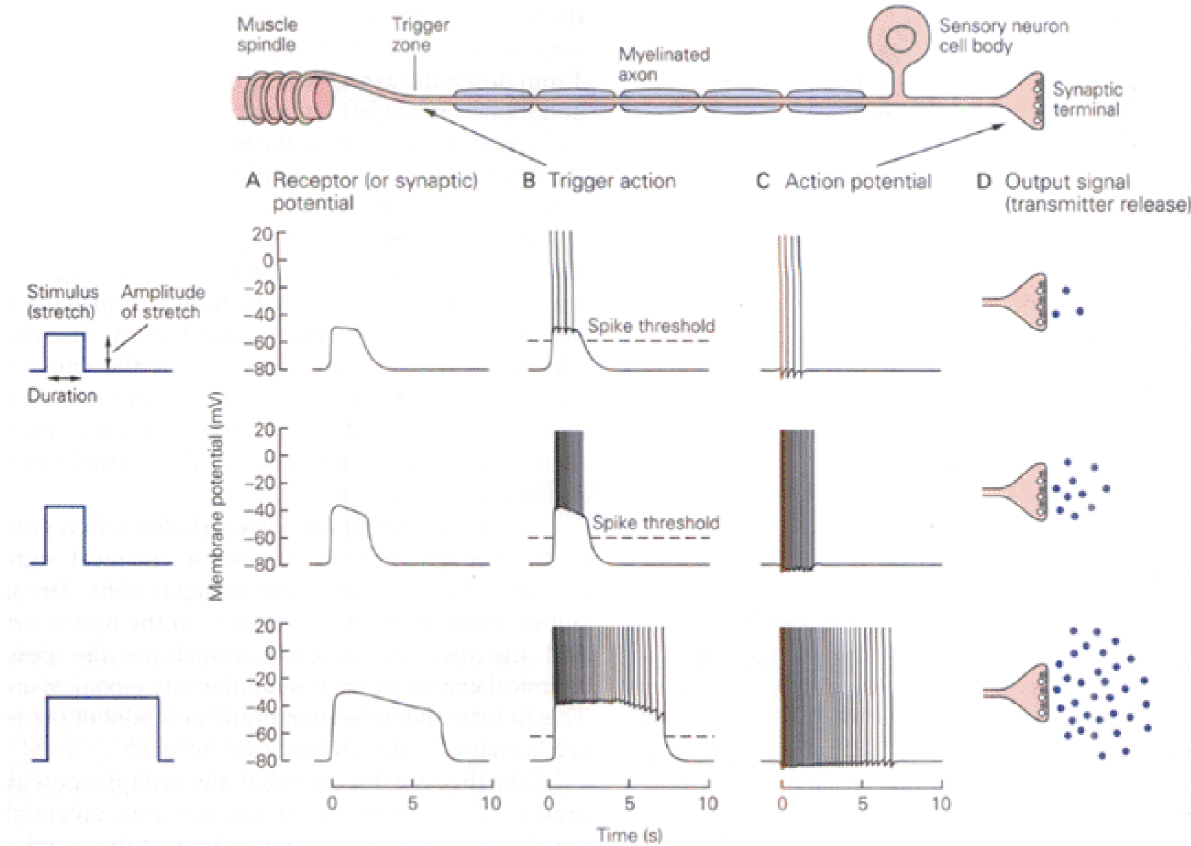
Neuronal activity (i)

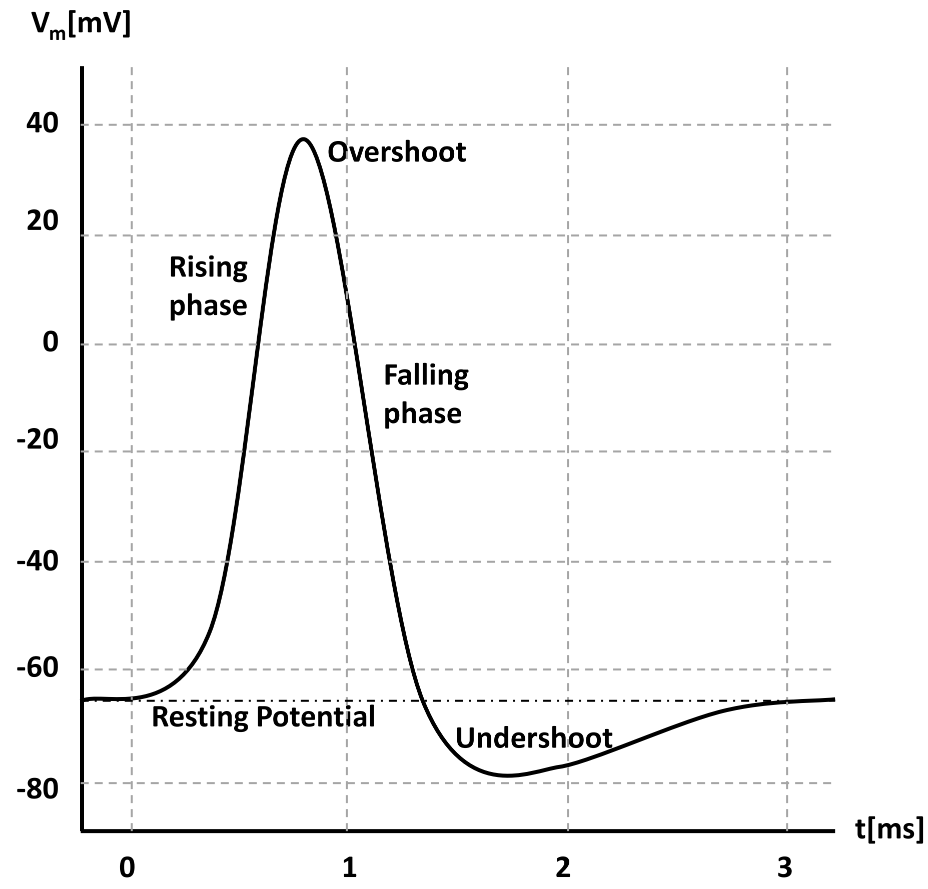
- **Rest**
- **Stimulus**
- **Depolarization**
- **Na ch open**
- **Threshold?**
- **AP**
- **Propagation**
- **Depolarization**
- **Ca ch open**
- **Threshold?**
- **$\Delta[Ca^{2+}]$**



Principles of Neural Science, 4th Ed. Kandel, Schwartz, and Jessel. Copyright © 2000 McGraw Hill

Neuronal activity (ii)

- Axon hillock (voltage-gated sodium channels)
- Absolute/relative refractory period
- Voltage Clamp (Hodgkin and Huxley)



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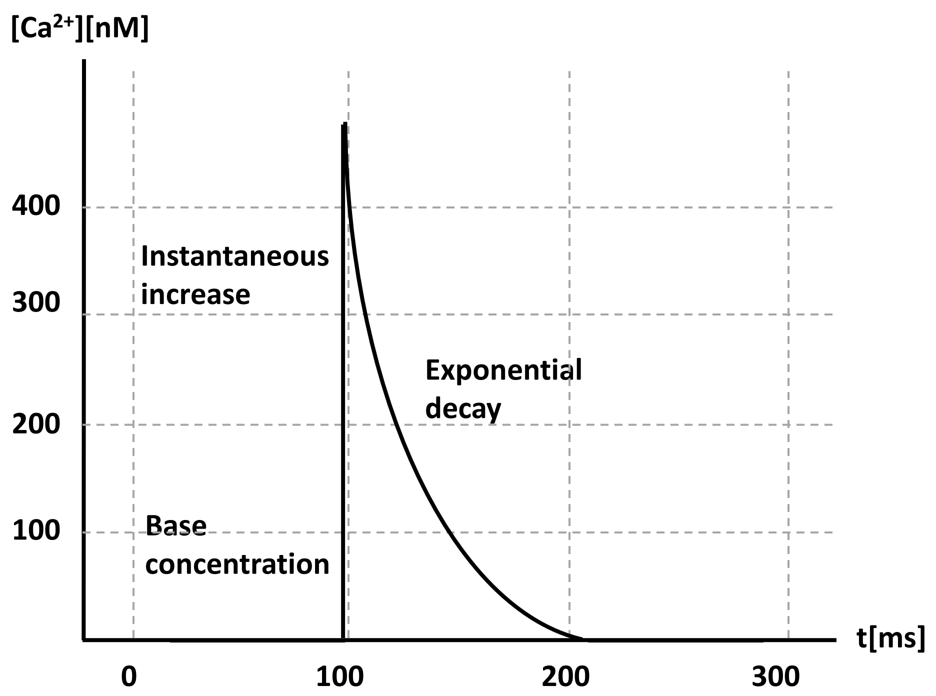
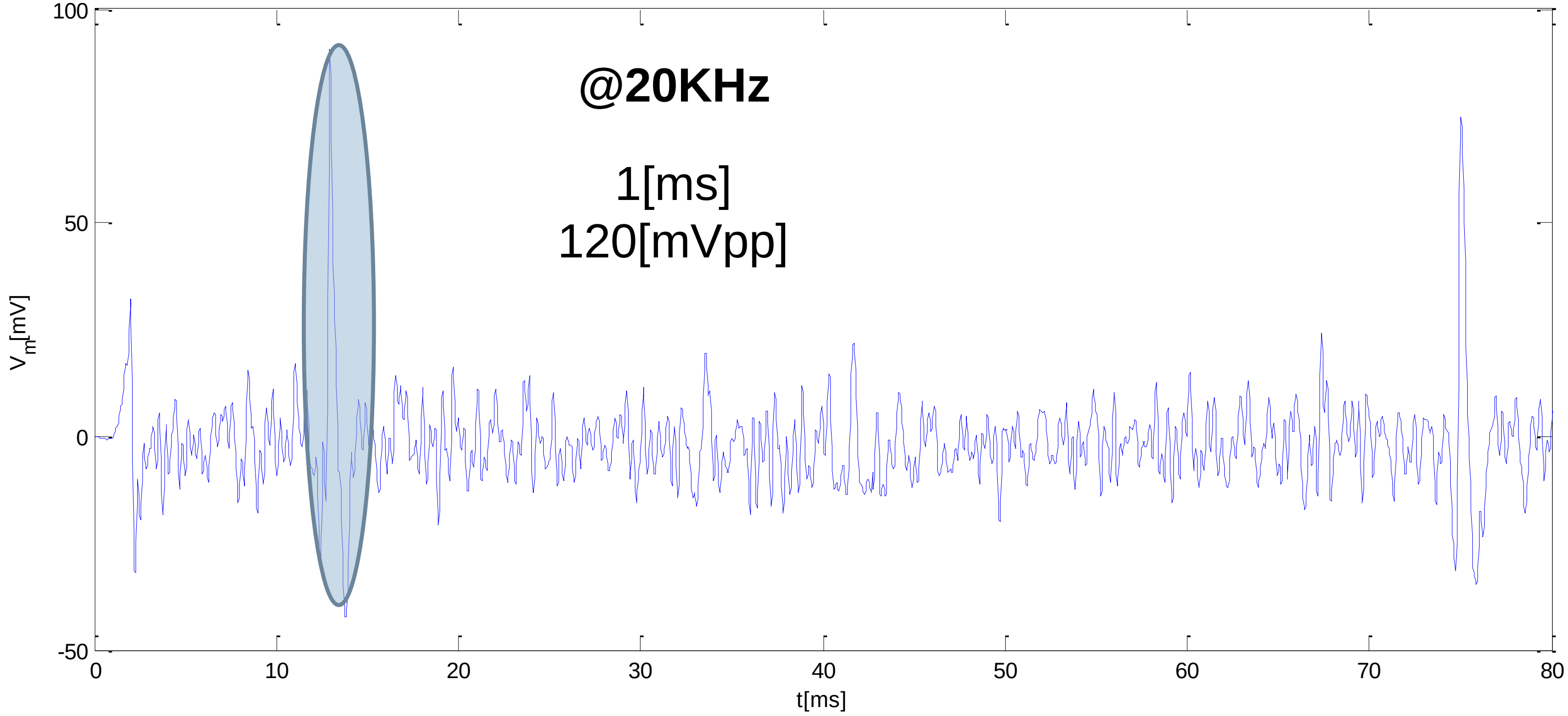
Neuronal activity (iii)

- Dendrite spine heads
- Related to APs (Ca channels)
- (or to other synaptic stimuli, EPSP)
- Calcium imaging

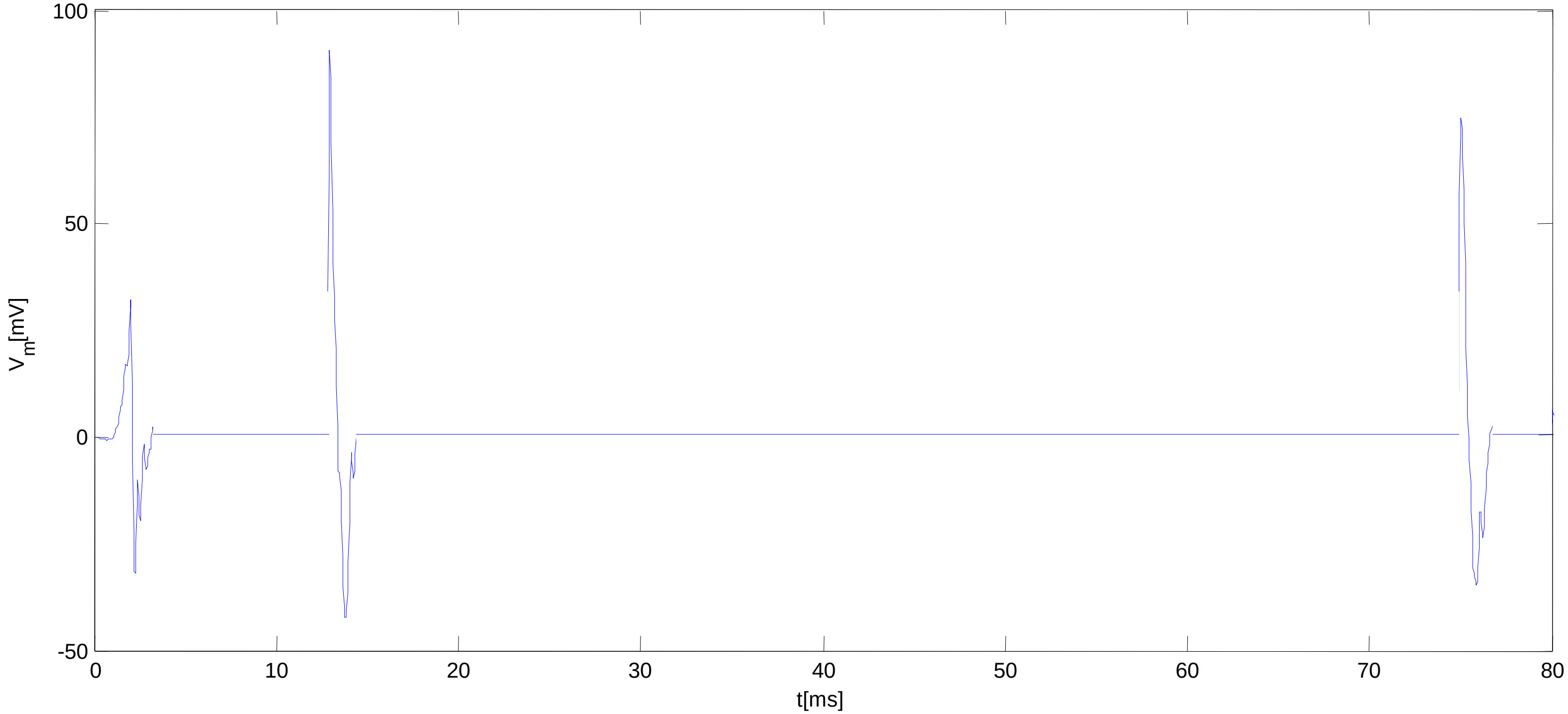


Modelling AP signals



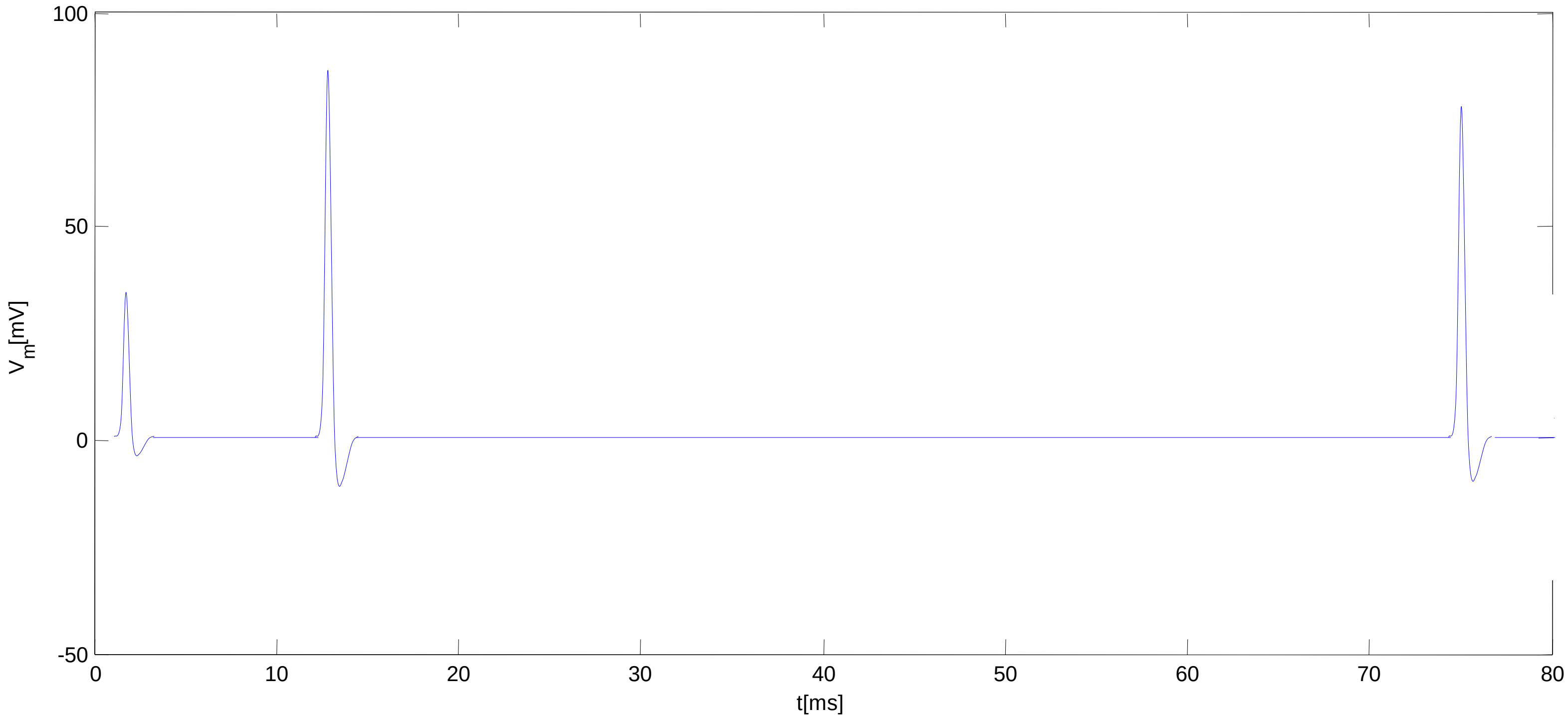
- Real voltage signal
  - Very noisy
  - Sparse? Can we apply FRI?

Inherent sparsity (i)



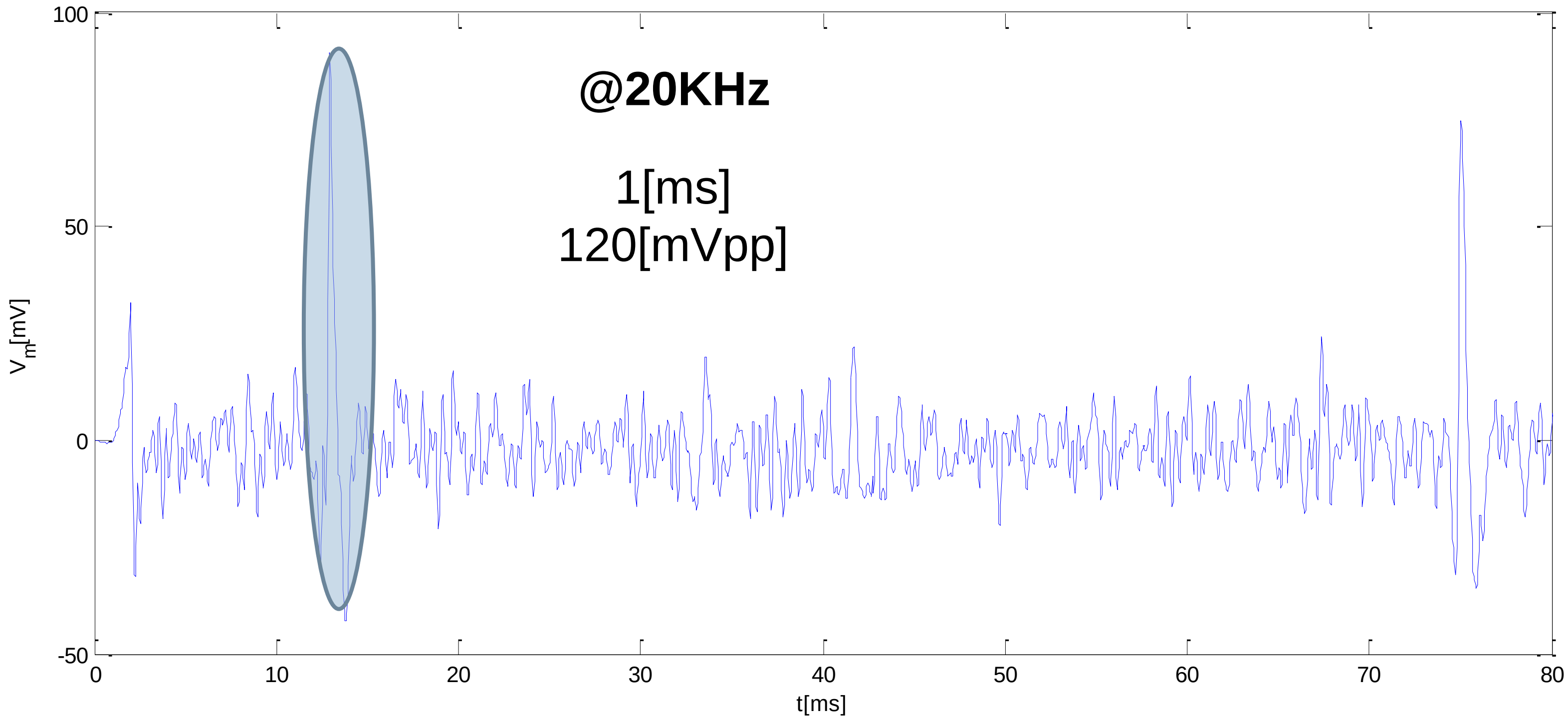
- Try to make it sparse
  - Isolate AP (remove noise)
  - The signal is sparse

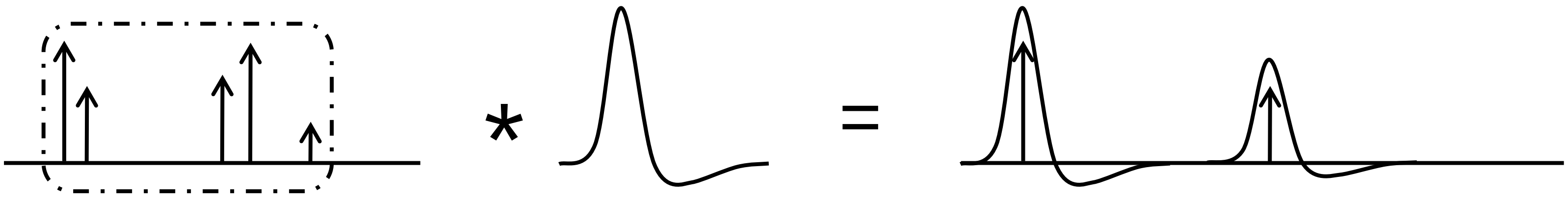
Inherent sparsity (ii)



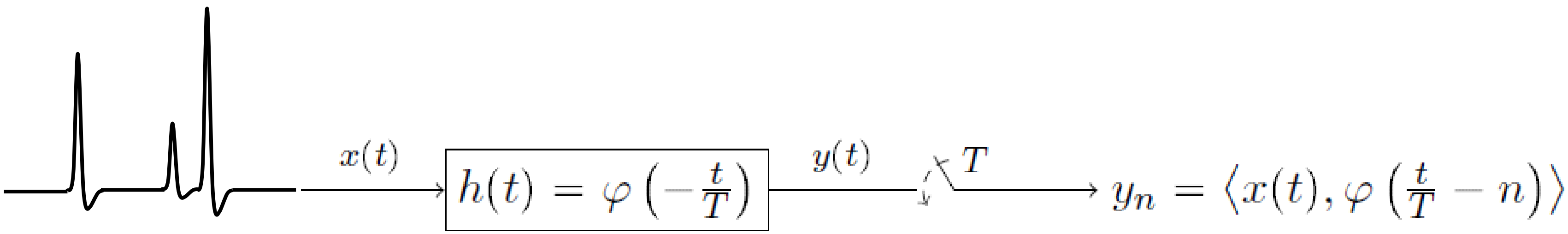
- Simplify further
  - One AP shape only, @(t_k, a_k), $k = 0 \ldots K\text{-}1$
  - If we detect the ideal spikes, we know (t_k, a_k)

Modelling AP signals





Sampling the AP signals

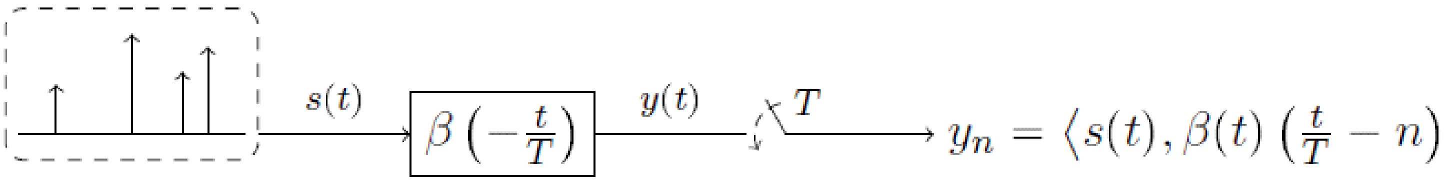


- Find equivalent sampling scheme
  - Rewrite the input
  - Equivalent expression for the samples

$$
\begin{aligned}
x(t) &= \sum_{k\in\mathbb{Z}} a_k \eta(t - t_k) \\
&= \sum_{k\in\mathbb{Z}} a_k \delta(t - t_k) * \eta(t) \\
&= s(t) * \eta(t)
\end{aligned}
$$

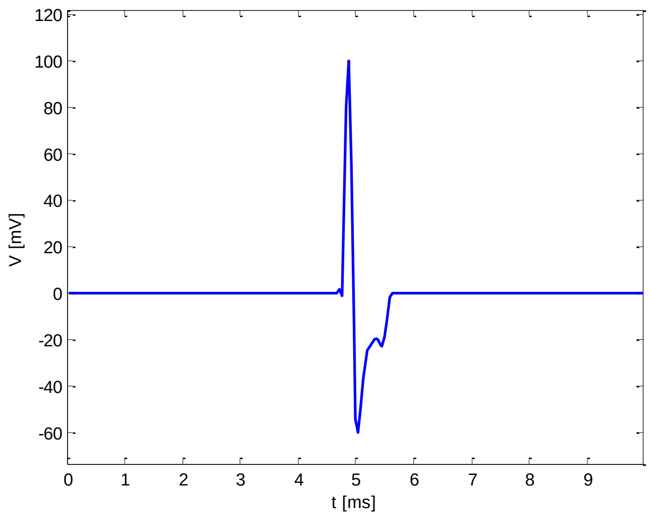
$$
y_n = \left\langle s(t), \beta\left(\frac{t}{T} - n\right)\right\rangle
$$

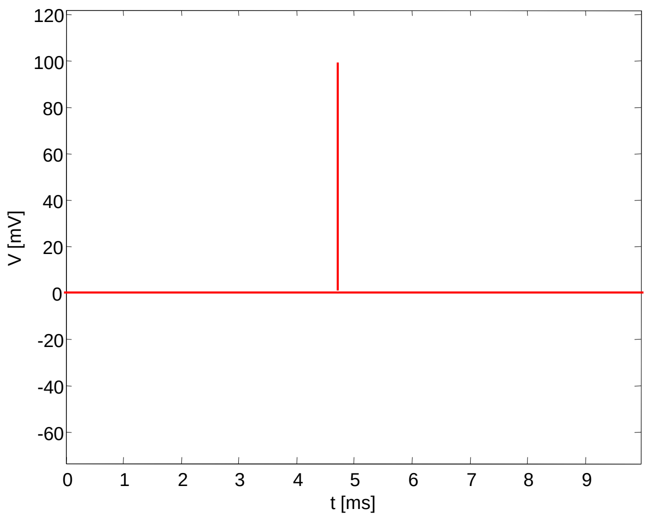
Equivalent sampling scheme



- "Basic" train of deltas scenario
- **Annihilating Filter Method** can be used
  - Sample the original signal
  - Calculate coefficients provided by equivalent scheme
  - Find locations and amplitudes of Diracs

Simulations (i)

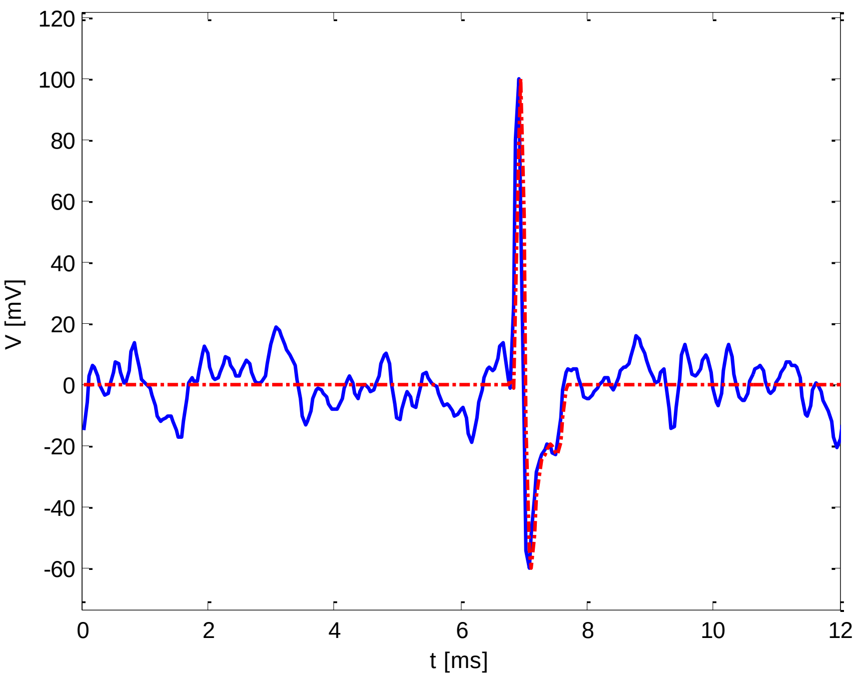




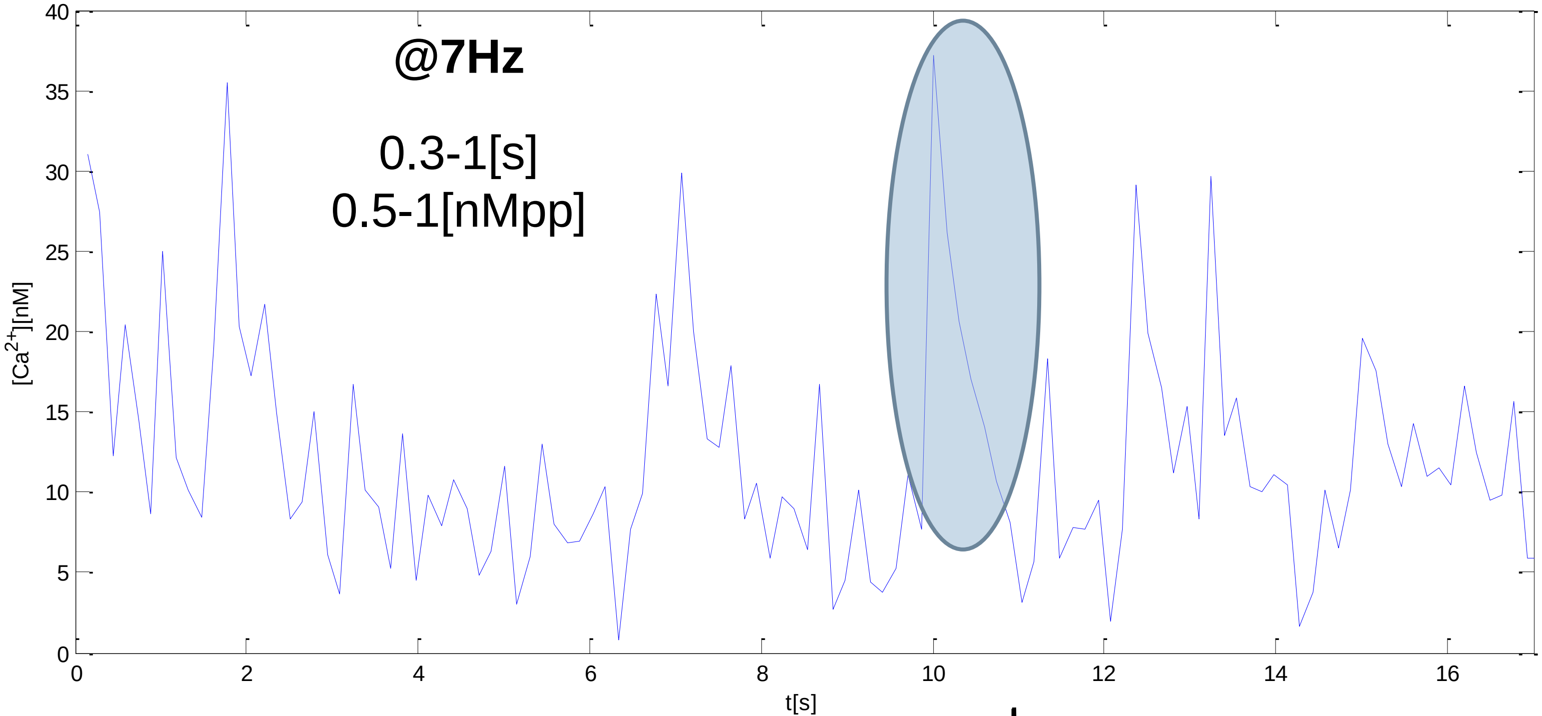
- Ideal scenario
  - Place AP shape at locations t_k with amplitudes a_k
  - Apply Annihilating Filter Method with equivalent scheme coefficients
  - Sampling and Perfect Reconstruction is possible

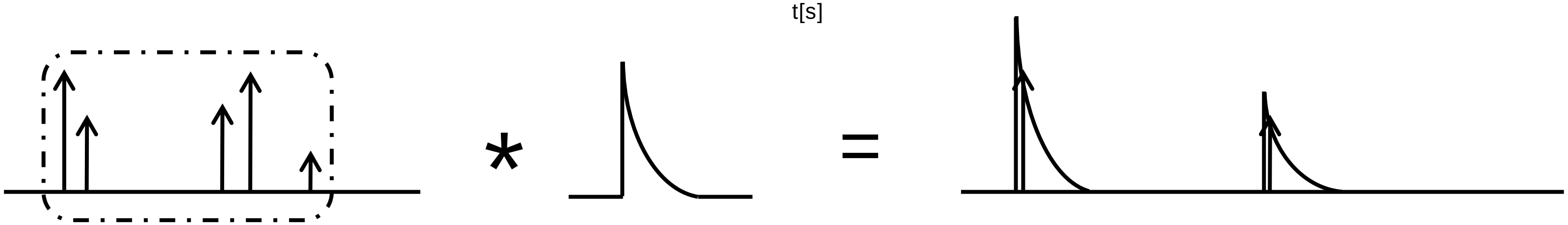
Simulations (ii)

- Real data
  - Simple case: search for the same spike shape
  - PR can be achieved
- Also tried
  - More spike shapes
  - More spikes at same time
  - Iterative: window (+) detect
  - Challenging

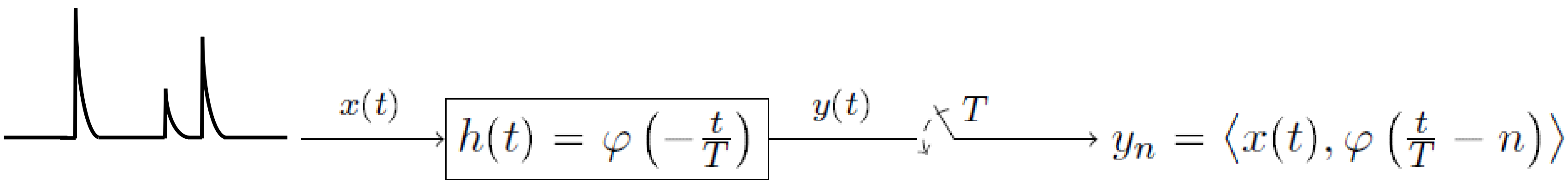


Modelling [Ca^{2+}] transients





Sampling the Calcium transients

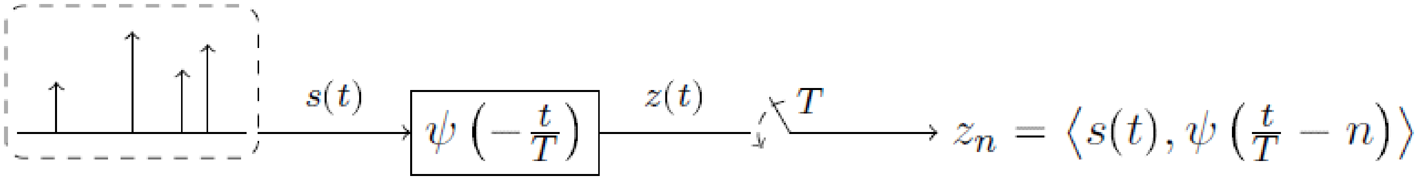


- Find equivalent sampling scheme
  - Rewrite the input
  - Weighted sample difference

$$
\begin{aligned}
x(t) &= \sum_{k=0}^{K-1} a_k e^{-\alpha(t-t_k)} u(t-t_k) \\
&= \sum_{k=0}^{K-1} a_k \delta(t-t_k) * e^{-\alpha t} u(t) \\
&= s(t) * \rho_\alpha(t)
\end{aligned}
$$

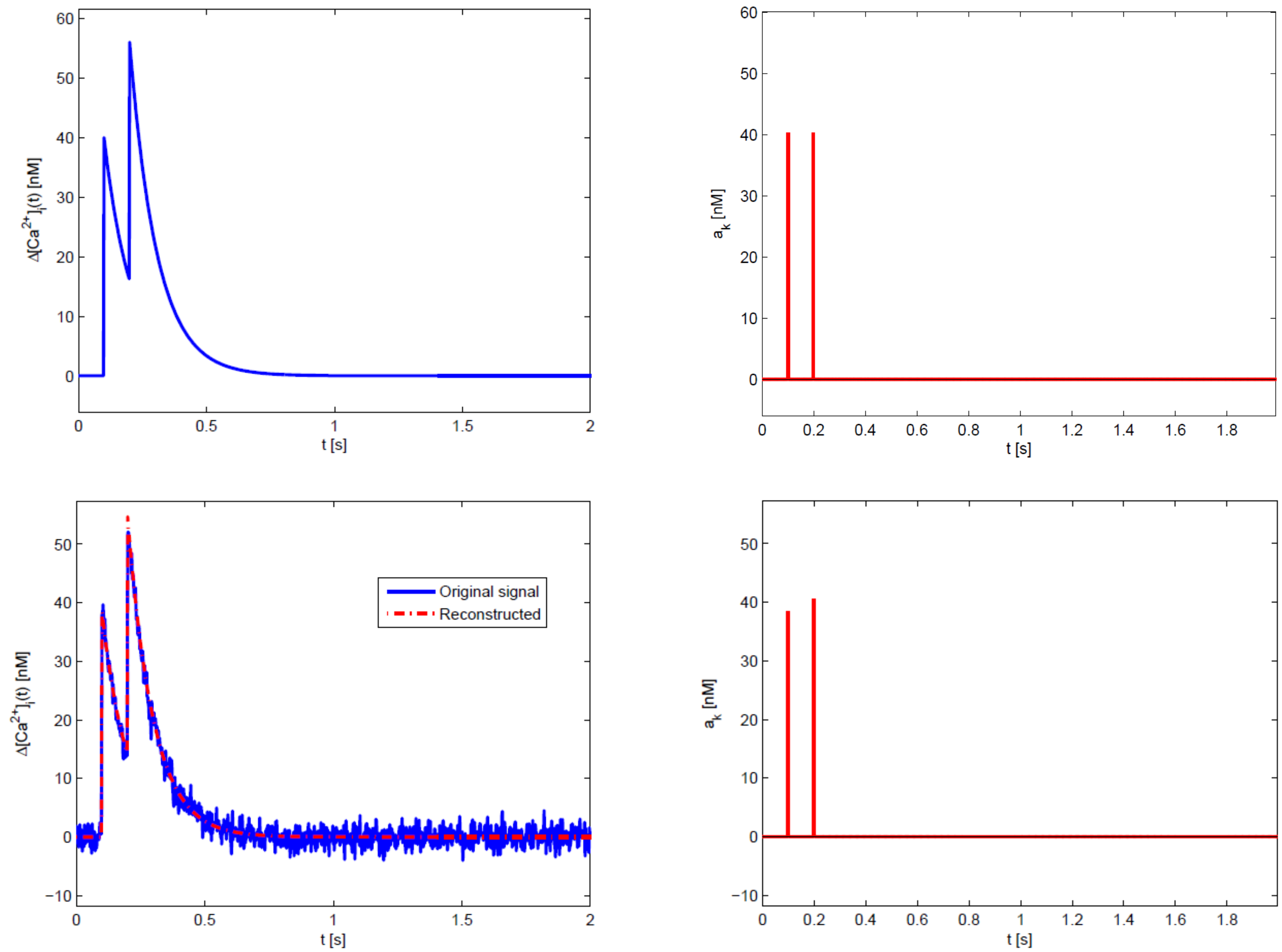
$$
\begin{aligned}
z_n &= y_n - e^{-\alpha T} y_{n-1} = \dots \\
&= \left\langle s(t), \psi\left(\frac{t}{T} - n\right)\right\rangle
\end{aligned}
$$

Equivalent sampling scheme

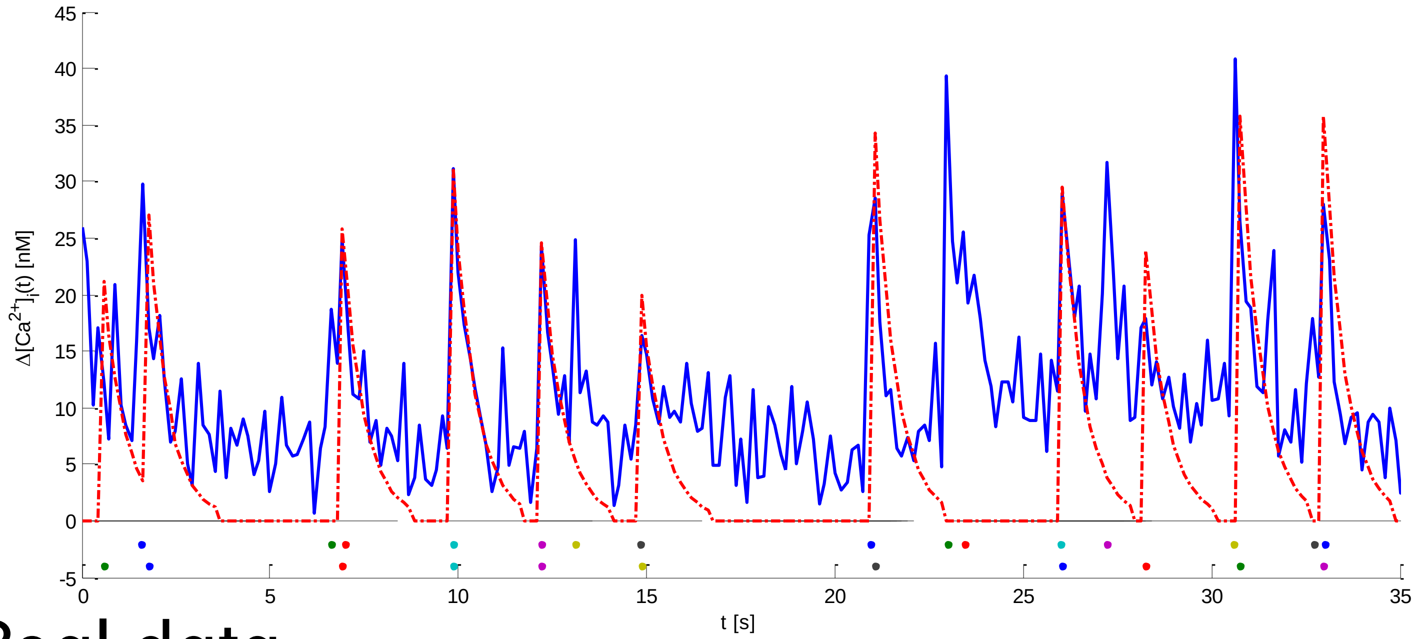


- "Basic" train of deltas scenario
- **Annihilating Filter Method** can be used
  - Sample the original signal
  - Calculate coefficients provided by equivalent scheme
  - Find locations and amplitudes of Diracs

Simulations (i)



Simulations (ii)



- Real data
  - Windowing: fixed size
  - Denoising: hard thresholding
  - Number of spikes: least squares

Conclusions

- Noisy FRI scenario
  - Modified TLS / Cadzow for coloured noise
  - Redesign kernels
  - Improvement (higher P better)
- Modelling neuronal signals
  - Ideally they are sparse
  - Goals: reduce sampling rate, spike detection, sorting
  - Real data
    - Different types of spikes
    - Noise: HT?

Future work

- Finite rate of innovation
  - Subspace denoising: alternative improvements
  - Other approaches?
  - Adaptive filtering
- Neuronal signals
  - Apply improved denoising + HT
  - Iterative retrieval
  - Different spike shapes / Superresolution?
- Compressed Sensing

Questions

- ***Part I*: Dealing with noise effectively in FRI**
  - *Overview*: the FRI sampling scheme
  - Modified TLS and Cadzow
  - Alternative exponential reproducing kernels
- ***Part II*: Sparse Characterization of Neuronal Signals through FRI theory**
  - Neurons & Neuronal activity
    - Action Potentials
    - Calcium Transients
  - Modelling Neuronal Signals